Canad. Math. Bull. Vol. **49** (2), 2006 pp. 281–284

Correction to a Theorem on Total Positivity

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Abstract. A well-known theorem states that if $f(z)$ generates a PF_r sequence then $1/f(-z)$ generates a PF_r sequence. We give two counterexamples which show that this is not true, and give a correct version of the theorem. In the infinite limit the result is sound: if $f(z)$ generates a PF sequence then $1/f(-z)$ generates a PF sequence.

1 The Bad News

Theorem 1.2 in Chapter 8 of Karlin's book [2] implies the following:

Theorem A *Let $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be power series with real coefficients such that $g(z) = 1/f(-z)$. For any positive integer r, the Toeplitz matrix of f is totally positive up to order r if and only if the Toeplitz matrix of g is totally positive up to order r.*

The bad news is that Theorem A is false. In the limit $r \to \infty$ the result is sound, and appears in work by Schoenberg *et al.* in the early 1950s [1, 4, 5]. We consider the possible source of error at the end of this section, but first let us review the definitions.

For a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the *Toeplitz matrix of f* is the infinite matrix $T[f]$, indexed by pairs of integers, with entries

$$T[f]_{ij} := \begin{cases} a_{j-i} & \text{if } j-i \geq 0, \\ 0 & \text{if } j-i < 0. \end{cases}$$

An infinite matrix M is *totally positive up to order r* when every minor of M of order at most r is nonnegative. This condition is abbreviated TP_r. If M is TP_r for all r then M is *totally positive*, abbreviated TP.

The matrix $T[f]$ is TP_1 if and only if the coefficients of $f(z)$ are nonnegative. If $T[f]$ is TP_2 then the sequence of coefficients $a_0, a_1, \ldots$ *has no internal zeros, i.e.,* if $0 \leq h < i < j$ and $a_h a_j \neq 0$, then $a_i \neq 0$. Also, if $T[f]$ is TP_2 then the sequence of coefficients $a_0, a_1, \ldots$ is *logarithmically concave*, *i.e.*, if $j \geq 1$ then $a_j^2 \geq a_{j-1}a_{j+1}$. Nonnegativity of the remaining 2-by-2 minors of $T[f]$ follows from these two conditions. That is, the Toeplitz matrix $T[f]$ is TP_2 if and only if the sequence of coefficients $a_0, a_1, \ldots$ is nonnegative, has no internal zeros, and is logarithmically concave.

Received by the editors July 14, 2004.
Research supported by the Natural Sciences and Engineering Research Council of Canada under operating grant OGP0105392.
AMS subject classification: 15A48, 15A45, 15A57, 05E05.
Keywords: Total positivity, Toeplitz matrix, Pólya frequency sequence, skew Schur function.

Our first counterexample is the polynomial $f(z) = 1 + 4z + 3z^2 + z^3$. By the preceding paragraph, one sees easily that $T[f]$ is TP_2. Elementary calculation with linear recurrence relations yields

$$g(z) = \frac{1}{1 - 4z + 3z^2 - z^3} = 1 + 4z + 13z^2 + 41z^3 + 129z^4 + 406z^5 + \cdots .$$

Since $129^2 - 41 \cdot 406 = -5 < 0$, the Toeplitz matrix $T[g]$ is evidently not TP_2. Theorem A is false. With hindsight, one notices that the coefficients of $f(z) = 1 + z + 2z^2$ are nonnegative, but that

$$g(z) = \frac{1}{1 - z + 2z^2} = 1 + z - z^2 - 3z^3 - z^4 + 5z^5 + \cdots$$

has negative coefficients. Thus, $T[f]$ is TP_1 while $T[g]$ is not TP_1.

The approach of Schoenberg *et al.* [1, 4, 5] to the $r \to \infty$ limit of Theorem A proceeds via Jacobi's theorem on complementary minors of inverse matrices. Assume that $T[f]$ is TP, and let M be a k-by-k submatrix of $T[g]$. Then M is contained in a suitably large n-by-n principal submatrix B of $T[g]$ supported on consecutive rows and columns. Let A be the corresponding principal submatrix of $T[f]$. Multiplying every row and column of B with even index by -1, we obtain a matrix B' such that $AB' = I$. Both A and B' have determinant one. Let N be the $(n-k)$-by-$(n-k)$ submatrix of A supported on rows and columns complementary to those supporting M in B. Application of Jacobi's theorem and careful accounting for signs shows that $\det(M) = \det(N)$. Since $T[f]$ is assumed to be TP, this shows that $T[g]$ is TP.

This argument breaks down if $T[f]$ is merely assumed to be TP_r since the value of n required above can be strictly larger than $r + k$, in which case we lose control over the sign of the $(n-k)$-by-$(n-k)$ minor $\det(N)$ of $T[f]$. This seems to be the problem in [2].

2 The Good News

The good news is that Theorem A can be fixed.

To do this we need a few facts about symmetric functions; see Macdonald [3] for details. The ring Λ of symmetric functions consists of all formal power series of bounded degree in independent commuting indeterminates $x_1, x_2, \dots$ that are invariant under all permutations of the indeterminates. In particular, for $n \geq 1$ the *n-th elementary symmetric function* is

$$e_n := \sum_{1 \leq i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \dots x_{i_n}$$

and the *n-th complete symmetric function* is

$$h_n := \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \dots x_{i_n}.$$

https://doi.org/10.4153/CMB-2006-029-0 Published online by Cambridge University Press

Let $E(t) := 1 + \sum_{n=1}^{\infty} e_n t^n$ and $H(t) := 1 + \sum_{n=1}^{\infty} h_n t^n$ be the generating series for these sequences. Formally, $e_1, e_2, \dots$ and $h_1, h_2, \dots$ can be regarded as indeterminates that are algebraically independent over the field $\mathbb{Q}$ of rational numbers, except for the single relation $E(t) = H(-t)^{-1}$. By means of this relation one can determine each e_n as a polyomial in the h_n's, and conversely. The indeterminates $\{h_n\}$ remain algebraically independent over $\mathbb{Q}$, as do the indeterminates $\{e_n\}$. The ring Λ is a polynomial ring with coefficients in $\mathbb{Q}$ over either set of indeterminates $\{h_n\}$ or $\{e_n\}$.

Since the indeterminates $\{h_n\}$ are algebraically independent and generate Λ, a homomorphism $\varphi\colon \Lambda \to R$ from Λ to another ring R is determined by its sequence of values $\{\varphi(h_n)\}$. For our application we only need this fact when $R = \mathbb{R}$ is the real field. A real power series $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ determines such a homomorphism $\varphi_f\colon \Lambda \to \mathbb{R}$ by $\varphi_f(h_n) := a_n$. Notice that if $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ is such that $g(z) = 1/f(-z)$ then $\varphi_f(e_n) = b_n$ and $\varphi_g(e_n) = a_n$.

The set of all integer partitions, partially ordered by inclusion of Ferrers diagrams, is called *Young's lattice* and denoted by $\mathbb{Y}$. For $\mu \le \lambda$ in $\mathbb{Y}$ there is a symmetric function $s_{\lambda/\mu}$ called a *skew Schur function*. Every skew Schur function can be indexed by a pair of partitions in $\mathbb{Y}$ such that:

(i) $\mu \le \lambda$,
(ii) μ has strictly fewer parts than λ,
(iii) the largest part of μ is strictly smaller than the largest part of λ.

We will denote this relation by $\mu \prec \lambda$ in $\mathbb{Y}$. The formulae we need are the Jacobi–Trudy formula and its dual form:

$$s_{\lambda'/\mu'} = \det(e_{\lambda_i - i + j - \mu_j}) \quad \text{and} \quad s_{\lambda/\mu} = \det(h_{\lambda_i - i + j - \mu_j}).$$

The order of these determinants is the number of parts of λ, and if j exceeds the number of parts of μ, then $\mu_j := 0$. The notation λ' denotes the partition conjugate to λ. If $f(z)$ and $g(z)$ are real power series such that $g(z) = 1/f(-z)$, then

$$\varphi_f(s_{\lambda/\mu}) = \varphi_g(s_{\lambda'/\mu'}) \quad \text{and} \quad \varphi_g(s_{\lambda/\mu}) = \varphi_f(s_{\lambda'/\mu'}).$$

Consider the submatrix M of $T[f]$ supported on rows $\{i_1 < i_2 < \cdots < i_r\}$ and columns $\{j_1 < j_2 < \cdots < j_r\}$. If $j_k < i_k$ for any $1 \le k \le r$ then $\det(M) = 0$, so we may assume that $j_k \ge i_k$ for all $1 \le k \le r$. If $j_1 = i_1$ or $j_r = i_r$ then $\det(M)$ reduces by Laplace expansion to a smaller minor of $T[f]$. Thus we may assume as well that $j_1 > i_1$ and $j_r > i_r$. A minor satisfying all these conditions is called an *essential minor* of $T[f]$. It is clear that $T[f]$ is TP_r if and only if every essential minor of $T[f]$ of order at most r is nonnegative.

Every essential minor of $T[f]$ has the form $\varphi_f(s_{\lambda/\mu}) = \det(a_{\lambda_i - i + j - \mu_j})$ for some $\mu \prec \lambda$ in $\mathbb{Y}$. To see this, let $\det(M)$ be an essential minor of $T[f]$ supported on rows $\{i_1 < i_2 < \cdots < i_r\}$ and columns $\{j_1 < j_2 < \cdots < j_r\}$. For each $1 \le k \le r$ let $\lambda_k := j_r - i_k + k - r$. The inequalities $\lambda_1 \ge \cdots \ge \lambda_r > 0$ are easily seen, so that λ is an integer partition with r parts. For each $1 \le k \le r$ let $\mu_k := j_r - j_k + k - r$. One can check that μ is an integer partition with at most $r - 1$ parts, that $\mu \prec \lambda$ in $\mathbb{Y}$, and that $\det(M) = \det(a_{\lambda_i - i + j - \mu_j})$. This construction can be reversed, so that every

$\varphi_f(s_{\lambda/\mu})$ is an essential minor of $T[f]$. In this way the skew Schur functions can be regarded as *generic essential Toeplitz minors*.

The order of the minor $\varphi_f(s_{\lambda/\mu})$ of $T[f]$ is the number of parts of λ. This implies the following: (i) the Toeplitz matrix $T[f]$ is TP_r if and only if $\varphi_f(s_{\lambda/\mu}) \geq 0$ for all $\mu \prec \lambda$ in $\mathcal{Y}$ for which λ has at most r parts. Similarly, (ii) the Toeplitz matrix $T[g]$ is TP_r if and only if $\varphi_g(s_{\lambda/\mu}) \geq 0$ for all $\mu \prec \lambda$ in $\mathcal{Y}$ for which λ has at most r parts. If $g(z) = 1/f(-z)$ then, since $\varphi_g(s_{\lambda/\mu}) = \varphi_f(s_{\lambda'/\mu'})$, condition (ii) is equivalent to: (iii) the Toeplitz matrix $T[g]$ is TP_r if and only if $\varphi_f(s_{\lambda'/\mu'}) \geq 0$ for all $\mu \prec \lambda$ in $\mathcal{Y}$ for which λ has at most r parts. Or, in other words, (iv) the Toeplitz matrix $T[g]$ is TP_r if and only if $\varphi_f(s_{\lambda/\mu}) \geq 0$ for all $\mu \prec \lambda$ in $\mathcal{Y}$ for which λ has largest part at most r. Comparing (i) and (iv) we see that the two conditions in Theorem A are closely related, but not equivalent.

Interpreting $\varphi_f(s_{\lambda/\mu})$ as a minor of $T[f]$, bounding the number of parts of λ corresponds to bounding the order of the minor. What corresponds to bounding the largest part of λ? For the submatrix M of $T[f]$ supported on rows $\{i_1 < i_2 < \cdots < i_r\}$ and columns $\{j_1 < j_2 < \cdots < j_r\}$, define the *level* of M to be $\ell := j_r - i_1 + 1 - r$. The level of a minor of $T[f]$ is the level of the submatrix of which it is the determinant. The Toeplitz matrix $T[f]$ is *totally positive up to level* ℓ when every minor of $T[f]$ of level at most ℓ is nonnegative. This condition is abbreviated TP'_ℓ. If $T[f]$ is TP'_ℓ for all ℓ then $T[f]$ is totally positive, TP.

Theorem B *Let $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be power series with real coefficients such that $g(z) = 1/f(-z)$. For any positive integer r, $T[f]$ is totally positive up to* level r *if and only if $T[g]$ is totally positive up to* order r.

Notice that in the limit as $r \to \infty$ we get the equivalence: $T[f]$ is TP if and only if $T[g]$ is TP. This is the most important consequence of Theorem A in the literature, and it is a huge relief that it survives.

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