3

Univariate asymptotics

In this chapter we review some classical results on the asymptotics of univariate generating functions. Throughout, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ will be a univariate generating function for the sequence $\{a_n\}$, and for any complex function $g(z)$ analytic at the origin we write $[z^n]g(z)$ for the coefficient of z^n in the power series expansion of $g(z)$ at the origin.

3.1 An explicit formula for rational functions

For rational functions in one variable, it is possible to determine an exact formula for a_n when n is sufficiently large. For instance, when $\rho \neq 0$ the equality

$$[z^n]\frac{1}{(1 - z/\rho)^k} = \binom{n+k-1}{k-1}\rho^{-n} \tag{3.1}$$

holds for $k = 1$ as the left-hand side is a geometric series, and repeated differentiation proves inductively that it holds for all $k \in \mathbb{N}$. More generally, suppose $f(z) = p(z)/q(z)$ is any *rational function* that is analytic at $z = 0$. We assume, without loss of generality, that p and q are relatively prime polynomials with q having the distinct roots $\rho_1, \ldots, \rho_t \in \mathbb{C}$, and that $q(0) = 1$. For each $j \in \{1, \ldots, t\}$, let m_j denote the multiplicity of the root ρ_j and let $q_j(z) = q(z)/(1 - z/\rho_j)^{m_j}$.

Because the q_j have no common root, there exist polynomials $p_1, \ldots, p_t \in \mathbb{C}[z]$ such that the numerator p of f can be written as a linear combination $p(z) = \sum_{j=1}^{t} p_j(z)q_j(z)$. This yields a ***partial fraction decomposition***

$$f(z) = \frac{p(z)}{q(z)} = \sum_{j=1}^{t} \frac{p_j(z)q_j(z)}{q(z)} = h_0(z) + \sum_{j=1}^{t} \frac{h_j(z)}{(1 - z/\rho_j)^{m_j}},$$

where $h_0, \ldots, h_t \in \mathbb{C}[z]$ and for every $j \in \{1, \ldots, t\}$ the polynomial $h_j(z)$ has degree at most $m_j - 1$ and does not vanish at ρ_j. Further decomposing each term $h_j(z)/(1 - z/\rho_j)^{m_j}$ as a sum $\sum_{i=0}^{m_j} c_{ij}/(1 - z/\rho_j)^i$ for constants $c_{ij} \in \mathbb{C}$ and applying (3.1) then implies

$$a_n = \sum_{j=1}^{t} \sum_{i=0}^{m_j} c_{ij} \binom{n+i-1}{i-1} \rho_j^{-n} \quad \text{when } n > \deg(p_0). \tag{3.2}$$

In this way, a partial fraction decomposition of f results in an explicit expression for the coefficients in its power series expansion.

Proposition 3.1 (univariate rational coefficients)**.** *Suppose $f(z) = p(z)/q(z)$ is the ratio of coprime polynomials p and q, where q has the distinct roots $\rho_1, \ldots, \rho_t \in \mathbb{C}$ and $q(0) \neq 0$. Then there exist $N \in \mathbb{N}$ and polynomials*

$$P_1(n, x), \ldots, P_t(n, x) \in \mathbb{Q}[n, x]$$

such that

$$[z^n] f(z) = \sum_{k=1}^{t} P_k(n, \rho_k)\, \rho_k^{-n}$$

for all $n \geq N$. If the zero ρ_k of $q(z)$ has multiplicity m_k then as a function of n the polynomial $P_k(n, x)$ has degree $m_k - 1$ and leading term expansion

$$P_k(n, x) = n^{m_k - 1} (-1)^{m_k} \frac{m_k\, p(x)}{x^{m_k}\, q^{(m_k)}(x)} + O(n^{m_k - 2}). \tag{3.3}$$

If $q(z)$ has degree d then the polynomials $P_1, \ldots, P_t$ have degree at most d in x, and they can all be computed explicitly in polynomial time with respect to d.

Proof The stated decomposition and the degree of $P_k(n, x)$ as a polynomial in n follow from (3.2) after noting that the binomial coefficient $\binom{n+i-1}{i-1}$ is a polynomial of degree $i - 1$ in n. The fact that each $P_k(n, x)$ has rational coefficients, and a method to compute them, follows most easily from an analytic argument given in Lemma 3.6 below. Each ρ_j is an algebraic number of degree at most d, so the degree of $P_k(n, x)$ in x can be taken to be at most d. □

Proposition 3.1 has strong consequences for asymptotics. Most importantly, the roots of q each give a *contribution* to the asymptotics of a_n, with the roots of minimal modulus having the exponentially largest asymptotic contributions.

Remark 3.2. When $q(x)$ has a unique root ρ_1 of minimal modulus then

$$a_n = P_1(n, \rho_1)\, \rho_1^{-n} + O(\rho_\dagger^{-n}),$$

where $\rho_\dagger$ is any root of $q(z)$ with second smallest modulus. When ρ_1 has multiplicity one then $P_1(n, \rho_1)$ is constant, and the expression

$$a_n = -\frac{p(\rho_1)}{\rho_1 \, q'(\rho_1)} \rho_1^{-n} + O(\rho_\dagger^{-n})$$

gives an asymptotic expansion of a_n with exponentially small error term. When $q(x)$ has a unique root ρ_1 of minimal modulus with multiplicity larger than one then, because the coefficients of $P_1(n, x)$ get rather unwieldy to compute, it is common to give only the leading term (or the first few terms) in $P_1(n, x)$ and leave an error of polynomially smaller size. Algorithms to separate the roots of a univariate polynomial by modulus are discussed in [GS96] and [MS21].

If there are several roots of q with minimal modulus then only those with maximum multiplicity contribute to dominant asymptotic behavior. The existence of several roots of minimal modulus and maximum multiplicity means one must compute the terms in Proposition 3.1 coming from each, and potentially deal with cancellation in the powers of these roots. Because it can be very difficult to track algebraic relations between terms involving powers of algebraic numbers with the same modulus, there are (perhaps surprisingly) still open problems related to when such cancellation can occur. Thankfully, the very pathological cases where it is difficult to detect dominant asymptotic behavior do not arise for combinatorial examples; see [Mel21, Section 2.2] for further discussion of these issues.

Exercise 3.1. Let a_0 and a_1 be any real numbers and suppose $a_{n+1} = 10a_n - 25a_{n-1}$ for all integers $n \geq 1$. Explicitly determine the generating function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and use this to determine the asymptotic behavior of a_n as $n \to \infty$. Split the parameter space determined by a_0 and a_1 into regions depending on the different asymptotic behaviors of a_n.

3.2 Meromorphic asymptotics

Partial fraction decomposition gives a simple algebraic method to determine asymptotics for rational generating functions. In this section we introduce *analytic* techniques, allowing for a vast generalization from rational functions to functions that behave locally like rational functions. Our arguments make use of standard results about meromorphic functions and, although we make our presentation as self-contained as possible, the reader not familiar with this aspect of complex analysis can consult [Con78b] for further background.

Analytic methods require that the series $f(z)$ represents an analytic function at the origin. To that end, we now assume that the sequence $\{a_n\}$ behaves exponentially, meaning there exist $C_1, C_2 > 0$ such that $C_1^n < |a_n| < C_2^n$ for all sufficiently large n. Under this assumption, the open domain of convergence of $f(z)$ is a finite open disk around the origin, and the Cauchy Integral Formula implies

$$a_n = \frac{1}{2\pi i} \int_C f(z) \frac{dz}{z^{n+1}} \tag{3.4}$$

whenever C is a simple closed contour enclosing the origin in this disk. The domain of integration in a complex integral can be deformed without changing the value of the integral, as long as the deformation stays where the integrand is analytic. It is therefore crucial to understand where $f(z)$ is *not* analytic.

Definition 3.3 (singularities)**.** If $\mathcal{D}_1$ and $\mathcal{D}_2$ are domains (connected open subsets) of $\mathbb{C}$ and $g_1(z)$ and $g_2(z)$ are analytic functions that agree on $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \emptyset$ then we say g_2 is a ***direct analytic continuation*** of g_1 to $\mathcal{D}_2$. More generally, we say that $g_2(z)$ is an ***analytic continuation*** of $g_1(z)$ if there exists a sequence of direct analytic continuations on consecutively overlapping domains that begins with g_1 and end with g_2. If $f(z)$ can be analytically continued to the interior of a simple closed curve γ but cannot be analytically continued to a neighborhood of a point $\omega \in \gamma$ then we call ω a ***singularity*** of f.

Example 3.4. If $f(z)$ is a rational function with coprime numerator and denominator then f has singularities at the roots of its denominator. Aside from division by zero, the most common types of singularities encountered in combinatorial applications include substitution of zero into an algebraic root or logarithm (see Section 3.4 below). ◁

One implication of the Cauchy Integral Formula is that the radius of convergence $0 < R < \infty$ of f equals the minimum modulus of a singularity of f, and this correspondence allows us to obtain a rough estimate of the growth of a_n. Since

$$|a_n| = \left| \frac{1}{2\pi i} \int_{|z|=R-\varepsilon} z^{-n-1} f(z)\, dz \right| \leq (R-\varepsilon)^{-n} \sup_{|z|=R-\varepsilon} |f(z)|,$$

we see that $\limsup_{n\to\infty} |a_n|^{1/n} \leq (R-\varepsilon)^{-1}$ for all $0 < \varepsilon < R$. Conversely, because there is a singularity of modulus R the power series for f does not converge for $|z| > R$, so for any $0 < \varepsilon < R$ we have $|a_n|^{1/n} \geq (R-\varepsilon)^{-1}$ infinitely often. Thus, the ***exponential growth rate*** $\limsup_{n\to\infty} |a_n|^{1/n}$ of a_n satisfies

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R} = \frac{1}{|\rho|}, \tag{3.5}$$

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where ρ is a singularity of $f(z)$ with minimal modulus. The exponential growth rate of a_n can be viewed as the coarsest measure of its asymptotic behavior.

Just as for rational functions, the singularities of $f(z)$ give contributions to the asymptotic behavior of a_n. For the rest of this section, we restrict to the case where f locally behaves like a rational function near the singularities that determine dominant asymptotics of a_n. The asymptotic contributions of more general types of singularities are discussed in Section 3.4 below.

Definition 3.5 (poles and meromorphic functions)**.** We say that f has a ***pole*** (or polar singularity) of order $\kappa \in \mathbb{Z}_{>0}$ at the point $z = \omega$ if $f(z)$ cannot be analytically continued to $z = \omega$ but $(z - \omega)^\kappa f(z)$ can, and κ is the smallest positive integer with this property. A pole of order one is usually called a ***simple pole***. If f is either analytic or has a pole at every point of a set $\mathcal{D} \subset \mathbb{C}$ then we say $f(z)$ is a ***meromorphic function*** on $\mathcal{D}$.

Suppose now that $f(z)$ is analytic on a closed disk $D = \{z \in \mathbb{C} : |z| \leq S\}$ for some $S > 0$, except at a nonempty collection $\{\rho_1, \ldots, \rho_t\}$ of poles of orders $\kappa_1, \ldots, \kappa_t$ which lie in the interior of D (by our running assumption that a_n grows exponentially, the ρ_j must be non-zero). If C_- is any positively oriented circle around the origin with radius less than $R = \min_j |\rho_j|$, and C_+ is the positively oriented circle around the origin with radius S, then the Cauchy residue theorem implies

$$\frac{1}{2\pi i} \int_{C_+} f(z) \frac{dz}{z^{n+1}} - \frac{1}{2\pi i} \int_{C_-} f(z) \frac{dz}{z^{n+1}} = \sum_{j=1}^{t} \operatorname*{Res}_{z=\rho_j} \left[z^{-n-1} f(z) \right]. \tag{3.6}$$

For readers unfamiliar with complex residues, the residue of a function $g(z)$ at a pole $z = \omega$ of order k can be defined by the explicit formula

$$\operatorname*{Res}_{z=\omega} g(z) = \frac{1}{(k-1)!} \lim_{z \to \omega} \left[\frac{d^{k-1}}{dz^{k-1}} \left((z-\omega)^k g(z) \right) \right], \tag{3.7}$$

which for a simple pole reduces to

$$\operatorname*{Res}_{z=\omega} g(z) = \lim_{z \to \omega} \left[(z-\omega) g(z) \right]. \tag{3.8}$$

It is a classic result in complex analysis that if $z = \omega$ is a pole of $f(z)$ of order k then for z in a neighborhood of ω we can write f as a ratio $f(z) = p(z)/q(z)$ of analytic functions with $p(\omega), q^{(k)}(\omega) \neq 0$ and $q(\omega) = \cdots = q^{(k-1)}(\omega) = 0$ (taking a series expansion of q at $z = \omega$ shows the converse is also true). We can use such a representation to compute the residue of f at $z = \omega$.

Lemma 3.6. *Under the assumptions of the previous paragraph,*

$$\operatorname*{Res}_{z=\omega} f(z) z^{-n-1} = \omega^{-n} P(n),$$

where $P(n)$ is a polynomial in n of degree $k-1$ with leading term expansion

$$P(n) = n^{k-1}\,(-1)^{k-1}\,\frac{k\,p(\omega)}{\omega^k\,q^{(k)}(\omega)} + O(n^{k-2}).$$

Proof The vanishing of the repeated derivatives of q implies $q(z) = \frac{q^{(k)}(\omega)}{k!}(z-\omega)^k + h(z)(z-\omega)^{k+1}$ for some function $h(z)$ analytic at $z=\omega$, so (3.7) yields

$$\operatorname*{Res}_{z=\omega} f(z)z^{-n-1} = \frac{1}{(k-1)!}\lim_{z\to\omega}\left[\frac{d^{k-1}}{dz^{k-1}}\left(\frac{p(z)}{\frac{q^{(k)}(\omega)}{k!}+(z-\omega)h(z)}z^{-n-1}\right)\right].$$

The product rule gives a finite sum expression for the repeated derivative in this limit, each term of which is a multiple of ω^{-n} times a polynomial in n. The leading term of this polynomial in n is contained only in the summand

$$\lim_{z\to\omega}\left[\frac{p(z)}{\frac{q^{(k)}(\omega)}{k!}+(z-\omega)h(z)}\cdot\frac{d^{k-1}}{dz^{k-1}}\left(z^{-n-1}\right)\right] = \frac{k!\,p(\omega)}{q^{(k)}(\omega)}\,\omega^{-n-k}\,(-n)^{k-1} + O(\omega^{-n-k}n^{k-2}),$$

and algebraic simplification gives the stated result. □

Exercise 3.2. Compute $P(n)$ in Lemma 3.6 when $f(z) = p(z)/q(z) = (2z-1)/(2-z-e^{1-z})$.

Combining Lemma 3.6 with (3.7) gives a generalization of Proposition 3.1 from rational functions to functions whose closest singularities to the origin are poles.

Proposition 3.7 (meromorphic coefficients)**.** *Suppose that $f(z)$ is analytic on a closed disk $D = \{z \in \mathbb{C} : |z| \le S\}$ for some $S > 0$, except at a nonempty collection $\{\rho_1,\dots,\rho_t\}$ of non-zero poles of orders $\kappa_1,\dots,\kappa_t$ lying in the interior of D. Then there exist polynomials $P_1(n),\dots,P_t(n)$ in n such that*

$$[z^n]f(z) = -\sum_{j=1}^{t} P_j(n)\,\rho_j^{-n} + O(S^{-n}),$$

where $P_j(n)$ has degree $\kappa_j - 1$. If $f(z) = p(z)/q(z)$ represents f as a ratio of analytic functions at $z = \rho_j$ with $p(\rho_j) \neq 0$ then the polynomial $P_j(n)$ is the polynomial $P(n)$ in Lemma 3.6 with $\omega = \rho_j$ and $k = \kappa_j$. The terms of $P_j(n)$ can be computed explicitly from the evaluations of the first $\kappa_j - 1$ derivatives of p and the first $2\kappa_j$ derivatives of q at $z = \rho_j$.

Proof Because f is analytic inside and on C_-, the Cauchy integral formula

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implies $\frac{1}{2\pi}\int_{C_-} f(z)\frac{dz}{z^{n+1}} = a_n$, and (3.6) can be rearranged to give

$$a_n = -\sum_{j=1}^{t} \operatorname*{Res}_{z=\rho_j}\left[z^{-n-1}f(z)\right] + \frac{1}{2\pi i}\int_{C_+} f(z)\frac{dz}{z^{n+1}}.$$

Since $f(z)$ is analytic on the circle C_+, which is a compact set, the function $|f(z)|$ is bounded on C_+, and

$$\left|\frac{1}{2\pi i}\int_{C_+} f(z)\frac{dz}{z^{n+1}}\right| \leq \max_{z\in C_+}|f(z)|\cdot S^{-n} = O(S^{-n}).$$

The stated forms for the residues follow from Lemma 3.6. □

As was observed above for rational functions, if $f(z)$ satisfies the conditions of Proposition 3.7 and has a unique singularity closest to the origin then the contribution of this point determines dominant asymptotics of a_n, up to an exponentially smaller error. If $f(z)$ has multiple poles of minimal modulus, and several of them have maximum order, then we must consider cancellation between their asymptotic contributions.

Example 3.8 (surjection asymptotics)**.** As seen in Example 2.53, the number a_n of surjections from a set of size n has exponential generating function

$$f(z) = \frac{1}{2-e^z}.$$

This function is meromorphic in the entire complex plane, with poles at the solutions $\Xi = \{\log 2 + k2\pi i : k \in \mathbb{Z}\}$ to the equation $2 - e^z = 0$ in the complex plane. Writing $p(z) = 1$ and $q(z) = 2 - e^z$, we see that $\omega \in \Xi$ implies $p(\omega), q'(\omega) \neq 0$, so every element of Ξ is a simple pole. Because $\log 2$ is the unique element of Ξ with minimal modulus, Proposition 3.7 implies

$$\frac{a_n}{n!} \sim \frac{-p(\log 2)}{(\log 2)q'(\log 2)}\left(\frac{1}{\log 2}\right)^n = \frac{1}{2}\left(\frac{1}{\log 2}\right)^{n+1}.$$

In fact, because all singularities of $f(z)$ are poles, Proposition 3.7 allows us to obtain an asymptotic expansion of a_n to arbitrary accuracy. If $S > 0$ is not equal to the modulus of any element in Ξ, and Ξ_S denotes the elements of Ξ with modulus at most S, then

$$\frac{a_n}{n!} = \sum_{\omega\in\Xi_S}\frac{-p(\omega)}{\omega q'(\omega)}\omega^{-n} + O\left(S^{-n}\right) = \frac{1}{2}\sum_{\omega\in\Xi_S}\omega^{-n-1} + O\left(S^{-n}\right).$$

◁

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3.3 Darboux's method

The fact that the asymptotic contribution of a pole singularity ω is easy to compute using the theory of residues is partially a reflection of the fact that $f(z)$ is analytic in a *punctured disk* around ω, which is a disk centered at ω with the center removed. Unfortunately, this property no longer holds near singularities where $f(z)$ locally behaves like a complex logarithm or a non-integral power, due to the branch cuts required to define such functions. A singularity where branch cuts are required to discuss local behavior of $f(z)$ is called a *branch point*, and in this section we illustrate a classical method for computing asymptotics in the presence of a branch point coming from a non-integral power.

Our first general asymptotic result (3.5), which bounded the exponential growth of a_n, was achieved by pushing the domain of integration in the Cauchy integral to the boundary of the domain of convergence of $f(z)$. Crucially, even if $f(z)$ admits a branch point on the boundary of its domain of convergence, such a deformation can be performed without needing to cross a branch cut. Integrating a slight modification of $f(z)$ on the boundary of the domain of convergence leads to *Darboux's method* and *Darboux's Theorem*. Before describing Darboux's method we require two preliminary results, the first of which asymptotically bounds integrals of smooth functions.

Lemma 3.9. *Suppose a complex-valued function f is k times continuously differentiable on the circle γ of radius R for some integer $k \geq 0$. Then*

$$\int_\gamma z^{-n-1} f(z)\, dz = O\left(n^{-k} R^{-n}\right)$$

as $n \to \infty$.

Proof Replacing $f(z)$ by $f(z/R)$ we may assume without loss of generality that $R = 1$. Integrating by parts shows

$$\int_\gamma z^{-n-1} f(z)\, dz = \int_\gamma \frac{1}{n} z^{-n} f'(z)\, dz\,,$$

where the term involving $\frac{z^{-n}}{-n} f(z)$ vanishes because γ has no boundary, and induction on k implies

$$\int_\gamma z^{-n-1} f(z)\, dz = \frac{1}{k!\binom{n}{k}} \int_\gamma z^{k-n-1} f^{(k)}(z)\, dz\,.$$

Since $f^{(k)}$ is continuous, it is bounded on the unit circle. Thus, the last integral above is bounded independently of n and our result follows from the behavior $k!\binom{n}{k} \sim n^k$ when k is fixed and $n \to \infty$. □

Our second preliminary result concerns expansions of power functions.

Lemma 3.10. *For any $\alpha \in \mathbb{C}$ the series $(1-z)^\alpha$ has the power series expansion*

$$(1-z)^\alpha = \sum_{n\geq 0} (-1)^n \binom{\alpha}{n} z^n,$$

which converges for $|z| < 1$, where

$$\binom{\alpha}{n} = \frac{\prod_{j=1}^{n}(\alpha - j + 1)}{n!}.$$

Furthermore, if $\alpha \notin \mathbb{N}$ then there is a series expansion

$$\binom{\alpha}{n} = \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} \left(1 + \sum_{k=1}^{\infty} \frac{e_k}{n^k}\right)$$

as $n \to \infty$, where each e_k is a polynomial in α of degree $2k$ that can be computed explicitly.

Proof The series expansion of $(1-z)^\alpha$ is Newton's generalized binomial theorem. The series expansion for $\binom{\alpha}{n}$ follows from an asymptotic analysis of the Euler Gamma function, and can be found in [FS09, Theorem VI.1]. □

Darboux's method consists of decomposing a generating function of interest into the sum of an error term that can be bounded by Lemma 3.9 and a finite number of terms that can be asymptotically approximated with Lemma 3.10. The method dates back to nineteenth-century work of Darboux on complex functions with algebraic singularities.

Example 3.11. The techniques of Section 2.5 often give non-rational (or even non-algebraic) generating functions to which Darboux's method can be applied. For instance, if C denotes the class of even length cycles of length at least four, with exponential generating function

$$C(z) = \sum_{n\geq 2} \frac{z^{2n}}{(2n)!} = \frac{1}{2}\log\frac{1}{1-z^2} - \frac{z^2}{2},$$

then the class $\mathcal{P}$ of permutations with disjoint cycles of even length at least four has exponential generating function

$$P(z) = e^{C(z)} = \frac{e^{-z^2/2}}{\sqrt{1-z^2}}.$$

Since P is a function of z^2, we make the substitution $t = z^2$ and analyze $f(t) =$

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$P(\sqrt{t}) = e^{-t/2}/\sqrt{1-t}$. The Taylor series expansion $e^{-t/2} = e^{-1/2} + \frac{e^{-1/2}}{2}(1-t) + O((1-t)^2)$ at $t = 1$ proves that we can write

$$f(t) = e^{-1/2}(1-t)^{-1/2} + \frac{e^{-1/2}}{2}\sqrt{1-t} + \psi(t)$$

for some C^1 function $\psi(t)$. Lemma 3.9 then implies

$$[t^n]f(t) = e^{-1/2}[t^n](1-t)^{-1/2} + \frac{e^{-1/2}}{2}[t^n](1-t)^{1/2} + o\left(n^{-1}\right),$$

so Lemma 3.10 shows that the counting sequence p_n of $\mathcal{P}$ satisfies

$$\frac{p_{2n}}{(2n)!} = [t^n]f(t) \sim \frac{e^{-1/2}}{\Gamma(-\alpha)}n^{-1/2} = \frac{e^{-1/2}}{\sqrt{\pi n}}.$$

◁

Exercise 3.3. Find real constants $C \neq 0$ and β such that the generating function $f(t)$ in Example 3.11 satisfies

$$[t^n]\, f(t) = \frac{e^{-1/2}}{\sqrt{\pi n}} + (C + o(1))n^{\beta}.$$

As seen in Example 3.11, it is common that a generating function is an analytic function multiplied by a pure power. Applying Darboux's method in this context gives the following result.

Theorem 3.12 (Darboux's Theorem)**.** *Suppose that $f(z) = (1 - z/R)^{\alpha}\psi(z)$ for some $R > 0$, where $\alpha \notin \mathbb{N}$ and ψ is analytic on the closed disk $|z| \leq R$ and satisfies $\psi(R) \neq 0$. If the expansion of ψ about R is $\psi(z) = \sum_{n=0}^{\infty} b_n(R-z)^n$ then the power series coefficients $\{a_n\}$ of f have an asymptotic expansion*

$$a_n \approx R^{-n}\sum_{k=0}^{\infty} c_k n^{-\alpha-1-k},$$

where the coefficient c_k is an explicit linear combination of $b_0, \ldots, b_k$. In particular,

$$a_n \sim \frac{\psi(R)}{\Gamma(-\alpha)}n^{-\alpha-1}R^{-n}.$$

Proof Again, by rescaling our variable we assume without loss of generality that $R = 1$. Lemma 3.10 implies that we can expand $\binom{\alpha}{n}$ into a series in decreasing powers $n^{-\alpha-1-k}$ with explicit coefficients, making it possible to convert an asymptotic series of the form $a_n \approx \sum_{k=0}^{\infty} c'_k(-1)^n\binom{\alpha+k}{n}$ into a series $a_n \approx \sum_{k=0}^{\infty} c_k n^{-\alpha-1-k}$ with $c_0 = c'_0/\Gamma(-\alpha)$. Thus, to prove our claimed result it is sufficient to show that a_n can be expressed as a series in $b_k(-1)^n\binom{\alpha+k}{n}$.

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Let m be a positive integer greater than $\mathrm{Re}\{-\alpha\}$ and let ψ_m be the Taylor series remainder such that

$$\psi(z) - \sum_{k=0}^{m} b_k(1-z)^k = (1-z)^{m+1}\psi_m(z)\,.$$

Multiplying by $(1-z)^\alpha$ yields

$$f(z) - \sum_{k=0}^{m} b_k(1-z)^{\alpha+k} = (1-z)^{\alpha+m+1}\psi_m(z)$$

on the open unit disk, and taking the coefficient of z^n on both sides implies

$$a_n - \sum_{k=0}^{m-1} b_k(-1)^n\binom{\alpha+k}{n} + O(n^{-\alpha-m-1}) = [z^n](1-z)^{\alpha+m+1}\psi_m(z)\,. \tag{3.9}$$

By assumption $\alpha + m + 1 \geq 0$, so the function $(1-z)^{\alpha+m+1}\psi_m$ is $\lfloor \alpha + m + 1 \rfloor$ times continuously differentiable on the unit circle and Lemma 3.9 implies the right-hand side of (3.9) is $O(n^{-\alpha-m})$. Since this argument works for any m sufficiently large, this proves the desired series for a_n exists. □

Example 3.13 (2-regular graphs: an algebraic singularity)**.** Let

$$f(z) = e^{-z/2-z^2/4}/\sqrt{1-z}$$

be the exponential generating function for the number a_n of 2-regular graphs that was derived in Example 2.52 of Chapter 2. Applying Darboux's Theorem with $R = 1$, $\alpha = -1/2$, and $\psi = \exp(-z/2 - z^2/4)$ gives

$$\frac{a_n}{n!} \sim \frac{\psi(1)}{\Gamma(-\alpha)} n^{-1/2} = \frac{e^{-3/4}}{\sqrt{\pi n}}\,.$$

◁

Exercise 3.4. Use Darboux's Theorem to compute an asymptotic estimate for the coefficients of the generating function $f(z) = \dfrac{1}{1-4z+z^2}$.

3.4 Transfer theorems

Our proof of Darboux's Theorem uses analyticity of $\psi(z) = f(z)/(R-z)^\alpha$ beyond the disk of radius R only to provide a series development of f at $z = R$. By making stronger use of analytic properties, and using a sharper estimate than Lemma 3.9 to bound error terms, it is possible to do better. There are various results along these lines, our favorite being the *transfer theorems* of Flajolet and Odlyzko [FO90]. The idea of this approach is to establish an estimate of the form $a_n = O(n^{-\alpha-1})$ for the coefficients of *any* power series $f(z)$ that is analytic in neighborhood of the unit disk in a *slit plane*, except at $z = 1$ where $f(z) = O((1-z)^\alpha)$.

Remark. To simplify our notation in this section we state our results for functions with singularities at $z = 1$. As noted above, this loses no generality since $[z^n]f(z/R) = R^{-n}[z^n]f(z)$ for any non-zero constant R and analytic function f.

The transfer theorem method is also flexible enough to extend beyond powers to other branch singularities. Let **alg-log** be the class of functions that are a product of a power of $1-z$, a power of $z^{-1}\log(1/(1-z))$, and a power of $\log\left[z^{-1}\log(1/(1-z))\right]$. We begin with a description of asymptotics for all functions in the class **alg-log**, then discuss asymptotics of functions which locally behave as if they are in **alg-log** near their singularities.

Proposition 3.14. *Let $\alpha, \gamma, \delta \in \mathbb{C} \setminus \mathbb{N}$ and let*

$$f(z) = (1-z)^\alpha \left(\frac{1}{z}\log\frac{1}{1-z}\right)^\gamma \left(\frac{1}{z}\log\left(\frac{1}{z}\log\frac{1}{1-z}\right)\right)^\delta .$$

Then the power series coefficients $\{a_n\}$ of f satisfy

$$a_n \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}(\log n)^\gamma (\log\log n)^\delta .$$

Proof See [FO90, Theorem 3B]. □

Remark. When at least one of α, γ, or δ is a nonnegative integer, different formulae can hold. For example, when $\gamma \notin \mathbb{N}$ but $\delta = 0$ and $\alpha \in \mathbb{N}$ the coincidence of α with a nonnegative integer decreases the exponent of the logarithm by one, giving the estimate

$$a_n \sim Cn^{-\alpha-1}(\log n)^{\gamma-1} . \tag{3.10}$$

For any $R > 0$ and $\varepsilon \in (0, \pi/2)$, the ***Δ-domain*** (or ***Camembert-shaped region***) defined by R and ε is

$$\Delta(R, \varepsilon) = \{z \in \mathbb{C} : |z| < R + \varepsilon,\ z \neq R,\ |\arg(z - R)| \geq \pi/2 - \varepsilon\} ,$$

pictured in Figure 3.1.

Theorem 3.15 (Transfer Theorem)**.** *Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be analytic in a Δ-domain $\Delta(1, \varepsilon)$. If $g(z) = \sum_{n=0}^\infty b_n z^n$ is in* **alg-log** *then the following statements hold.*

(i) If $f(z) = O(g(z))$ as $z \to 1$ then $a_n = O(b_n)$ as $n \to \infty$.

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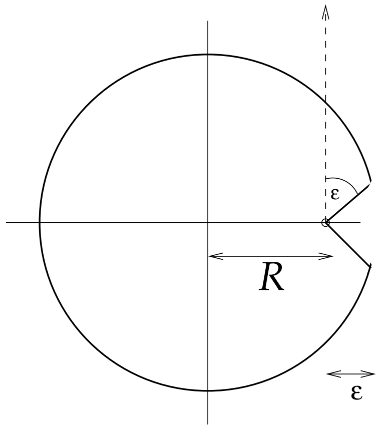

Figure 3.1 A Δ-domain.

(ii) If $f(z) = o(g(z))$ as $z \to 1$ then $a_n = o(b_n)$ as $n \to \infty$.
(iii) If $f(z) \sim g(z)$ as $z \to 1$ then $a_n \sim b_n$ as $n \to \infty$.

Theorem 3.15 with $g(z) = C(1 - z)^\alpha$ strengthens Theorem 3.12. So as not to devote too much space to computation, we only prove Theorem 3.15 for the subset of **alg-log** given by powers $(1 - z)^\alpha$.

Proof for $g(z) = (1 - z)^\alpha$. We need only prove the first two statements in the theorem, as the third follows as an immediate consequence. Cauchy's integral formula implies a_n can be expressed as a sum of integrals

$$\begin{aligned} a_n = &\frac{1}{2\pi i}\int_{\gamma_1} f(z)z^{-n-1}dz + \frac{1}{2\pi i}\int_{\gamma_2} f(z)z^{-n-1}dz \\ &+ \frac{1}{2\pi i}\int_{\gamma_3} f(z)z^{-n-1}dz + \frac{1}{2\pi i}\int_{\gamma_4} f(z)z^{-n-1}dz \end{aligned}$$

defined by two parameters ξ and η, where

- γ_1 is the circular arc parametrized by $1 + n^{-1}e^{-it}$ for $\xi \le t \le 2\pi - \xi$,
- γ_2 is the line segment between $1 + n^{-1}e^{i\xi}$ and the number β of modulus $1 + \eta$ and $\arg(\beta - 1) = \xi$,
- γ_3 is the arc on the circle of radius $1 + \eta$ running between β and $\overline{\beta}$ the long way, and
- γ_4 is the conjugate of γ_2, oriented oppositely.

Our argument works with any $0 < \eta < \varepsilon$ and any $0 < \xi < \pi/2$ large enough so that the curves are contained in $\Delta(1, \varepsilon)$; see Figure 3.2 for an illustration.

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Suppose first that $f(z) = O((1-z)^\alpha)$ near $z = 1$, so that for some $K > 0$ the inequality $|f(z)| \le K|1-z|^\alpha$ holds everywhere on the curves.

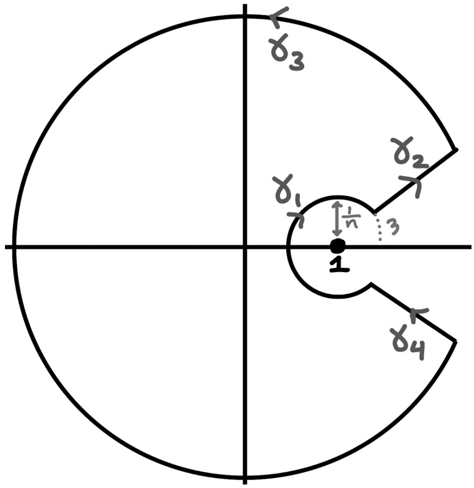

Figure 3.2 The contour γ.

On γ_1 the modulus of f is at most $Kn^{-\alpha}$ and the modulus of z^{-n-1} is at most $(1-n^{-1})^{-n-1} \le 2e \le 6$ so, since the length of the curve is less than $2\pi n^{-1}$, the Cauchy integral over γ_1 has size at most $6Kn^{-\alpha-1}$.

On γ_3 the z^{-n-1} factor reduces the modulus of the integrand to at most $C(\eta)(1+\eta)^{-n}$, where $C(\eta)$ grows at most polynomially with η. Thus, the Cauchy integral over γ_3 is $O(n^{-N})$ for any $N \in \mathbb{N}$.

By symmetry, it remains only to bound the integral over γ_2. Set $\omega = e^{i\xi}$ and parametrize the integral as $z = 1 + (\omega/n)t$ for t from 1 to En, where $E = |\beta - 1|$. We have $|f(z)| \le K|z-1|^\alpha = K(t/n)^\alpha$ and $|z|^{-n-1} = \left|1 + \frac{\omega t}{n}\right|^{-n-1}$, so

$$\left|\int_{\gamma_2} f(z) z^{-n-1} dz\right| \le \int_{\gamma_2} |f(z)||z^{-n-1}| dz \le \int_1^{En} K\left(\frac{t}{n}\right)^\alpha \left|1 + \frac{\omega t}{n}\right|^{-n-1} \frac{dt}{n}$$

$$\le Kn^{-\alpha-1} \int_1^\infty t^\alpha \left|1 + \frac{\omega t}{n}\right|^{-n-1} dt. \tag{3.11}$$

The inequality $|1 + \omega t/n| \ge 1 + \mathrm{Re}\{\omega t/n\} = 1 + (t/n)\cos(\xi)$ implies an upper bound of

$$\int_1^\infty t^\alpha \left(1 + \frac{t\cos(\xi)}{n}\right)^{-n-1} dt$$

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for the integral in (3.11), which can be relaxed to

$$J_n = \int_1^\infty t^\alpha \left(1 + \frac{t\cos(\xi)}{n}\right)^{-n} dt$$

because $\cos(\xi) > 0$. The integrand of J_n monotonically decreases as n increases, and is finite for any positive n larger than the real part of α, so the decreasing limit is

$$J = \lim_{n\to\infty} J_n = \int_1^\infty t^\alpha e^{-t\cos(\xi)}\, dt\,,$$

which is finite as $0 < \xi < \pi/2$. We have now bounded all four integrals by multiples of $n^{-\alpha-1}$, so the proof of statement *(i)* in the theorem is complete.

The proof of statement *(ii)* is contained in this argument too. When $|f(z)| \leq Kg(z)$ then the integral over γ_1 is bounded above by $6Kn^{-\alpha-1}$, the integral over γ_3 is $o(n^{-\alpha-1})$, and the integrals over γ_2 and γ_4 are bounded by $JKn^{-\alpha-1}$. Furthermore, the contributions to each of these four integrals from parts of γ at distance greater than any fixed $\delta > 0$ from 1 are $o(n^{-\alpha-1})$. If $f(z) = o(g(z))$ at $z = 1$ then for any $\varepsilon > 0$ there is a δ such that $|f(z)| \leq \varepsilon|g(z)|$ when $|1 - z| \leq \delta$. It follows that $a_n \leq (2J + 6 + o(1))\varepsilon n^{-\alpha-1}$. This is true for every $\varepsilon > 0$, whence $a_n = o(n^{-\alpha-1})$. □

Example 3.16 (Catalan asymptotics)**.** Let $a_n = \frac{1}{n+1}\binom{2n}{n}$ be the nth Catalan number, whose generating function

$$f(z) = \sum_{n=0}^\infty a_n z^n = \frac{1 - \sqrt{1-4z}}{2z} = \frac{1 - 2\sqrt{\frac{1}{4} - z}}{2z}$$

was described in Example 2.14 of Chapter 2. The function $f(z)$ has an algebraic singularity at $z = 1/4$, near which the asymptotic expansion for f begins

$$f(z) = 2 - 4\sqrt{\frac{1}{4} - z} + 8\left(\frac{1}{4} - z\right) - 16\left(\frac{1}{4} - z\right)^{3/2} + O\left(\left(\frac{1}{4} - z\right)^2\right).$$

Note that $f(z)/\sqrt{1/4 - z}$ is not analytic in any disk of radius $1/4 + \varepsilon$, since both integral and half-integral powers appear in f, but f is analytic in a Δ-domain. Since the integral powers of $(1 - z)$ do not contribute to asymptotic behavior as

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they are polynomials, Theorem 3.15 thus gives an expansion

$$a_n = -4 \cdot \underbrace{4^n \left(-\frac{1}{4n^{\frac{3}{2}}\sqrt{\pi}} - \frac{3}{32n^{\frac{5}{2}}\sqrt{\pi}} + O\left(n^{-\frac{7}{2}}\right) \right)}_{[z^n](1/4-z)^{1/2}} - 16 \cdot \underbrace{4^n \left(\frac{-3}{32n^{\frac{5}{2}}} + O\left(n^{-\frac{7}{2}}\right) \right)}_{[z^n](1/4-z)^{3/2}}$$

$$+ \underbrace{O\left(4^n n^{-\frac{7}{2}}\right)}_{[z^n]O((1/4-z)^2)}$$

$$= 4^n \left(n^{-\frac{3}{2}} \frac{1}{\sqrt{\pi}} - n^{-\frac{5}{2}} \frac{9}{8\sqrt{\pi}} + O\left(n^{-\frac{7}{2}}\right) \right).$$

◁

Exercise 3.5 (common subexpression problem)**.** Flajolet and Odlyzko [FO90] quote the generating function

$$f(z) = \frac{1}{2z} \sum_{p \geq 0} \frac{1}{p+1} \binom{2p}{p} \left[\sqrt{1 - 4z + 4z^{p+1}} - \sqrt{1-4z} \right]$$

involved in the representation of trees by directed acyclic graphs.

(a) Show that the minimal modulus singularity occurs at $z = 1/4$, around which

$$f(z) \sim \frac{c}{\sqrt{(1-4z)\log(1-4z)^{-1}}}.$$

(b) Compute the asymptotic behavior of the coefficients of f (you can check your answer against [FO90, (6.7b)]).

Example 3.17 (branching random walk: logarithmic singularity)**.** For an example including a logarithmic term, recall from Example 2.13 the implicit equation

$$\phi(z) = [(1-p)z + p\phi(\phi(z))]^2.$$

This characterizes the probability generating function for the number X of particles to reach the origin in a binary branching nearest-neighbor random walk with absorption at the origin. Aldous (see [AB05, Theorem 29] and [Ald98, Theorem 6]) showed that there is a critical value $p = p_*$ satisfying $16p_*(1 - p_*) = 1$, such that if $p > p_*$ then X is sometimes infinite, while if $p < p_*$ then X is never infinite. At the critical value X is always finite, and it is of interest to know the likelihood of large values of X.

Below, we show that

$$\phi(z) = 1 - \frac{1-z}{4p} - (c + o(1))\frac{1-z}{\log(1/(1-z))}, \tag{3.12}$$

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where $c = \log(1/(4p))/(4p)$ and the statement holds for $z \in [0, 1]$ (the interesting situation is when $z \to 1$). If we knew this for all z in a Δ-domain, we could use (3.10) to conclude $a_n \sim cn^{-2}(\log n)^{-2}$, so that X has a first moment but not a "1 + log" moment. Here we establish (3.12) on the unit interval, although it is probably true in a Δ-domain and this is left to the interested reader. Just knowing (3.12), we can deduce information on the partial sums $\sum_{k=0}^{n} a_k$ via a *Tauberian theorem* of Hardy and Littlewood, and perhaps asymptotic information on a_n itself (see [FS09, Sec. VI.11]).

To show (3.12), fix $0 < z_0 < 1$ and consider the iterates $z_n = \phi^{(-n)}(z_0)$ of the inverse of ϕ. The function ϕ is convex on $[0, 1]$ with $\phi(0) > 0$, $\phi(1) = 1$, and one other fixed point k with $p_* < k < 1$. Because $\phi(x) < x$ on $(k, 1)$, if we iterate ϕ on any point in $(c, 1)$ it converges downward to c. Likewise, if we iterate the inverse function ϕ^{-1} starting with any point in $(c, 1)$, it converges upwards to 1, so $z_n \uparrow 1$. The recursion for ϕ gives

$$z_n = ((1-p)z_{n+1} + pz_{n-1})^2,$$

and changing variables to $y_n = 1 - z_n$ implies

$$\begin{aligned} y_n &= 1 - ((1-p)(1-y_{n+1}) + p(1-y_{n-1}))^2 \\ &= 1 - (1 - ((1-p)y_{n+1} - py_{n-1}))^2. \end{aligned}$$

Solving for y_{n+1} gives

$$y_{n+1} = \frac{1 - \sqrt{1-y_n} - py_{n-1}}{1-p}.$$

Setting $x_n = y_n/(4p)^n$ and using $16p(1-p) = 1$ results in

$$x_{n+1} = 2x_n - x_{n-1} + O(y_n)^2.$$

Verifying first that y_n is small, we then approximately solve the linear recurrence for x_n to obtain $x_n \sim An + B$, for some constants A, B, whence $y_n \sim (4p)^n(An + B)$. We may write this as

$$y_{n+1} = 4py_n + (1 + o(1))\frac{y_{n+1}}{n+1} = 4py_n + (1 + o(1))\frac{y_{n+1}}{\log y_{n+1} / \log(4p)}.$$

Let $z = 1 - y_{n+1}$ so $\phi(z) = 1 - y_n$. We then have

$$1 - \phi(z) = \frac{1-z}{4p} - (1 + o(1))\frac{1-z}{4p}\frac{\log(4p)}{\log(1-z)}$$

for all real $z \uparrow 1$, proving (3.12). ◁

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3.5 The saddle point method

One of the crowning achievements of complex analysis is the development of techniques to evaluate integrals through clever deformations of their contours of integration. Much of this work can be grouped together under the umbrella of the ***saddle point method***, aimed at discovering the best deformation for an asymptotic analysis. Unlike the techniques discussed above, saddle point methods do not require an integrand to have singularities, and it is common to use a saddle point analysis in situations where transfer theorems cannot be utilized (in fact, the presence of singularities can complicate the saddle point method). In this section we give a short overview of univariate saddle point techniques, with further development of the univariate case covered in Chapter 4 and multivariate generalizations discussed in Chapter 5.

The heart of the saddle point method is the following statement: *when the modulus of an integrand falls steeply on either side of its maximum, most of the contribution to the integral comes from a small interval about the maximum.* If the descent is steep enough, multiplying the integrand by the length of the interval where the modulus is sufficiently near its maximum (or doing something slightly more fancy) gives an accurate estimate. Most contours, however, cannot be used for this purpose: such an estimate cannot hold if the contour can be deformed so as to decrease the maximum modulus of the integrand, since then the integral would be less than the claimed estimate.

Let γ be a contour and let $I = \log f(z) - (n+1)\log z$ be the logarithm of the Cauchy integrand in (3.4). Fixing $z_0 \in \gamma$, we write $\mathrm{Re}\{I'\}$ and $\mathrm{Im}\{I'\}$ for the real and imaginary parts of the derivative at z_0 of I restricted to the curve γ. If z_0 maximizes the modulus of the Cauchy integrand on γ then $\mathrm{Re}\{I'\} = 0$, however it is not usually true that $\mathrm{Im}\{I'\} = 0$. In fact, the Cauchy–Riemann equations imply that $\mathrm{Im}\{I'\}$ equals the real part of the derivative at z_0 of I along any curve *perpendicular* to γ at z_0. Thus, when $\mathrm{Im}\{I'\} \neq 0$ the curve γ may be locally perturbed, fixing the endpoints but pushing the center in the direction of increasing $\mathrm{Re}\{I\}$, thereby decreasing the maximum modulus of the Cauchy integrand on the contour. In other words, if the modulus of the integrand is maximized on γ at z_0, and this maximum cannot be reduced by perturbing γ, then I' must vanish at z_0. The univariate saddle point method thus consists of the following steps:

(i) locate the zeros of I', which form a discrete set of points,

(ii) see whether the contour of integration can be deformed so as to minimize $\mathrm{Re}\{I\}$ at such a point,

(iii) estimate the integral via a Taylor series expansion of the integrand.

In Chapter 4 we will see that for integrals of the form

$$\int A(z)\exp(-\lambda\phi(z))$$

with parameter λ going to infinity, including the Cauchy integral, one can get away with approximating the critical point $z_0(\lambda)$ by the critical point z_0 for ϕ, ignoring A and removing the dependence of z_0 on λ. This approximation is often good enough to provide an asymptotic expansion of the integral, but here we consider cases where we can deal with $z_0(\lambda)$ directly. For the second step above not to fail, either f must be entire or the saddle point where I' vanishes must have smaller modulus than the singularities of f. In practice this is often satisfied, and this classic method is widely applicable. For instance, the seminal paper [Hay56] defines a broad class of functions, called *admissible functions*, for which the saddle point method works and can be automated.

Examples of saddle point integrals

Because we go into great detail on saddle point integrals in Chapter 4, here we simply present two examples illustrating the theory. At their heart, these examples rely on the estimate

$$\int_\gamma A(z)\exp(-\lambda\phi(z))\,dz \sim A(z_0)\sqrt{\frac{2\pi}{\phi''(z_0)\lambda}}\exp(-\lambda\phi(z_0)), \tag{3.13}$$

where A and ϕ are smooth functions with $\mathrm{Re}\{\phi\}$ minimized in the interior of γ at a point z_0 where ϕ'' does not vanish. The approximation (3.13) follows from Theorem 4.1, however we compute it directly in our first example to illustrate why it is true.

Example 3.18 (ordered-set partitions: an isolated essential singularity)**.** Example 2.51 implies that the exponential generating function for the number a_n of ordered-set partitions of $[n]$ is

$$f(z) = \exp\left(\frac{z}{1-z}\right).$$

Our goal here is to prove the estimate

$$a_n \sim n!\sqrt{\frac{1}{4\pi e}}n^{-3/4}\exp\left(2\sqrt{n}\right),$$

starting with the Cauchy integral expression

$$\frac{a_n}{n!} = \frac{1}{2\pi i}\int_{|z|=\varepsilon}\exp\left(\frac{z}{1-z}\right)z^{-n-1}\,dz$$

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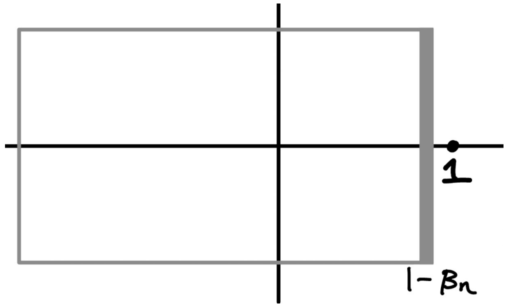

Figure 3.3 The circle $|z| = 1 - \beta_n$ can be deformed to a rectangle with right edge (bold) on the line $x = 1 - \beta_n$ and other edges arbitrarily far from the origin.

that holds for any $0 < \varepsilon < 1$ (in fact, we will select ε to vary with n, with $\varepsilon \to 0$ as $n \to \infty$).

Following the outline of the saddle point method above, we let

$$I(z) = I_n(z) = -(n+1)\log z + \frac{z}{1-z}$$

be the logarithm of the integrand and begin by computing the points where the derivative

$$I'(z) = \frac{-n-1}{z} + \frac{1}{(1-z)^2}$$

vanishes. The closest solution of $I'(z) = 0$ to the origin is $1 - \beta_n$ where

$$\beta_n = n^{-1/2} - \frac{1}{2}n^{-1} + O\left(n^{-3/2}\right), \tag{3.14}$$

and we thus take the Cauchy contour of integration to be the circle of radius $\varepsilon = 1 - \beta_n$ (which is less than one for all n sufficiently large). Because the only singularities of the Cauchy integrand lie at the origin and the point $z = 1$, without changing the value of the Cauchy integral we can deform this circle to a rectangle with right edge on the line $x = 1 - \beta_n$ and all other points arbitrarily far from the origin (see Figure 3.3). When $|z| \geq 2$ the modulus of the Cauchy integrand is upper bounded by $\exp\left(\frac{|z|}{|z|-1}\right)|z|^{-n-1} \leq e^2|z|^{-n-1}$, meaning we can take the left, top, and bottom edges of the rectangle in Figure 3.3 to infinity and use the change of variables $z = 1 - \beta_n + it$ to obtain

$$\frac{a_n}{n!} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \exp(I(1 - \beta_n + it))\,(i\,dt) \tag{3.15}$$

for n sufficiently large, fulfilling the second step of the saddle point method.

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The final step is to prove an approximation of the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(I(1-\beta_n+it))\,dt \sim \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[I(1-\beta_n)+\frac{1}{2}I''(1-\beta_n)(it)^2\right]dt, \tag{3.16}$$

where $I(1-\beta_n+it)$ is replaced by its second-degree Taylor approximation. This is very useful because, as a Gaussian integral, the right-hand side of (3.16) can be computed exactly,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[I(1-\beta_n)+\frac{1}{2}I''(1-\beta_n)(it)^2\right]dt = \sqrt{\frac{1}{2\pi I''(1-\beta_n)}}\exp(I(1-\beta_n))\,. \tag{3.17}$$

The estimate (3.16) – comparable to (3.13) with $\lambda = n+1$ and $\phi(t) = -I(1-\beta_n+it)$ – is verified through several integral bounds. The approximation (3.14) for β_n implies

$$I''(1-\beta_n) = \frac{n+1}{(1-\beta_n)^2} + \frac{2}{\beta_n^3} = (2+o(1))n^{3/2}, \tag{3.18}$$

so that the right-hand side of (3.17) is

$$\begin{aligned}
&\sqrt{\frac{1}{2\pi I''(1-\beta_n)}}\exp(I(1-\beta_n))\\
&\sim \sqrt{\frac{1}{4\pi n^{3/2}}}\exp\left(-(n+1)\log(1-\beta_n)-1+\frac{1}{\beta_n}\right)\\
&\sim \sqrt{\frac{1}{4\pi n^{3/2}}}\exp\left(-(n+1)\left(-n^{-1/2}+O\left(n^{-3/2}\right)\right)-1+n^{1/2}+\frac{1}{2}+O\left(n^{-1/2}\right)\right)\\
&\sim \sqrt{\frac{1}{4\pi e}}n^{-3/4}\exp\left(2\sqrt{n}\right).
\end{aligned}$$

Our claimed asymptotic result for a_n thus holds as long as we can show the left and right sides of (3.16) are equal up to an error that is $o\left(n^{-3/4}\exp\left(2\sqrt{n}\right)\right)$. The approximation (3.18) suggests that the main contributions to the integrals in (3.16) come from the region where $t^2n^{3/2}$ is not too small, meaning $|t|$ is roughly $n^{-3/4}$ or smaller. Accordingly, we pick a cutoff $L = 2n^{-3/4}\log n$ a little greater than that and break our integrals into the two parts $|t| \le L$ and $|t| > L$. Up to the cutoff the two integrals are close, and past the cutoff they are both

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small. More precisely, define

$$M_1 = \int_{|t|\geq L} \left| \exp\left[I(1-\beta_n) + \frac{1}{2} I''(1-\beta_n)(it)^2 \right] \right| dt$$

$$M_2 = \int_{|t|\geq n^{-1/2}} \left| \exp[I(1-\beta_n+it)] \right| dt$$

$$M_3 = \int_{n^{-1/2}>|t|\geq L} \left| \exp[I(1-\beta_n+it)] \right| dt$$

$$M_4 = \int_{|t|<L} \left| \exp\left[I(1-\beta_n) + \frac{1}{2} I''(1-\beta_n)(it)^2 \right] - \exp\left[I(1-\beta_n+it) \right] \right| dt$$

so that

- M_1 is the integral on the right-hand side of (3.16) beyond L,
- the sum of M_2 and M_3 bounds the integral on the left-hand side of (3.16) beyond L,
- M_4 bounds the difference between the left- and right-hand sides of (3.16) on $[-L, L]$,
- and the modulus of the difference between the left and right sides of (3.16) is bounded by the sum $M_1 + M_2 + M_3 + M_4$.

Letting $M = \exp[I(1-\beta_n)]$, we prove M_1, M_2, and M_3 have upper bounds of the form $M \cdot \exp(-c(\log n)^2)$ for some $c > 0$, and that $M_4 = o(Mn^{-3/4})$. These bounds all lie in $o\left(n^{-3/4}\exp\left(2\sqrt{n}\right)\right)$, completing our derivation of asymptotics for this example.

Bound on M_1 We bound M_1 with a standard Gaussian tail estimate. For any $a, C > 0$

$$\begin{aligned}\int_{|t|\geq C} e^{-at^2} dt = 2\int_{t\geq C} e^{-at^2} dt &= 2e^{-aC^2} \int_{t\geq 0} e^{-at^2-2aCt} dt \\ &\leq 2e^{-aC^2} \int_{t\geq 0} e^{-at^2} dt \\ &= \sqrt{\pi/a}\, e^{-aC^2},\end{aligned}$$

so the growth rates of $I''(1-\beta_n)$ and L give the asserted upper bound on M_1 for any $c < 8$.

Bound on M_2 To bound M_2, observe first that if $|t| \geq n^{-1/2}$ then the exponent $-t^2/(\beta_n^3 + \beta_n t^2)$ decreases to $-\beta_n^{-1} \sim -\sqrt{n}$. This is small, but integrating it over

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the unbounded region $[n^{-1/2}, \infty]$ requires us to be careful. In particular, we use the upper bound

$$\frac{|\exp(I(1-\beta_n+it))|}{\exp(I(1-\beta_n))} \leq \frac{|1-\beta_n|^n}{|1-\beta_n+it|^n} \exp\left(\mathrm{Re}\left\{\frac{1-\beta_n+it}{\beta_n-it} - \frac{1-\beta_n}{\beta_n}\right\}\right)$$
$$\leq (1+t^2)^{-n/2} \exp\left(-(1+o(1))n^{1/2}\right),$$

where we can bound the power of n in the first line by $(1+t^2)^{-n/2}$ since $1-\beta_n < 1$ and $|x/(x+it)|$ is increasing in $x \geq 0$. Integrating the factor $(1+t^2)^{-n/2}$ as t ranges from $n^{-1/2}$ to infinity gives a term of size $o(1)$, so

$$\frac{M_2}{M} \leq \exp\left(-(1+o(1))n^{1/2}\right) = o\left(\exp\left[-c(\log n)^2\right]\right)$$

for any $c > 0$.

Bound on M_3 To bound M_3 we pull out the factor of M, obtaining

$$M_3 \leq M \int_{L<|t|<n^{-1/2}} \exp\left[\mathrm{Re}\{I(1-\beta_n+it) - I(1-\beta_n)\}\right] dt.$$

The real part of $-(n+1)\log(1-\beta_n+it)$ is maximized at $t = 0$, whence

$$\mathrm{Re}\{I(1-\beta_n+it) - I(1-\beta_n)\} \leq \mathrm{Re}\left\{\frac{1-\beta_n+it}{\beta_n-it} - \frac{1-\beta_n}{\beta_n}\right\},$$

and

$$\begin{aligned} M_3 &\leq M \int_{L<|t|<n^{-1/2}} \exp\left(\mathrm{Re}\left\{\frac{1-\beta_n+it}{\beta_n-it} - \frac{1-\beta_n}{\beta_n}\right\}\right) dt \\ &= M \int_{L<|t|<n^{-1/2}} \exp\left(\frac{-t^2}{\beta_n^3+\beta_n t^2}\right) \\ &\leq M \int_{L<|t|<n^{-1/2}} \exp\left(-\frac{t^2}{2\beta_n^3}\right) dt \end{aligned}$$

because the β_n^3 term is the greatest term in the denominator when $t < n^{-1/2}$. The behavior $\beta_n \sim n^{-1/2}$ and $L = 2n^{-3/4}\log n$ proves the desired upper bound on M_3 for any constant $c < 2$.

Bound on M_4 Finally, for M_4 we use the Taylor approximation

$$\left|I(1-\beta_n+it) - I(1-\beta_n) + \frac{1}{2}t^2 I''(1-\beta_n)\right| \leq \frac{t^3}{6}\sup_{|s|\leq L}|I'''(1-\beta_n+s)|.$$

Differentiating $I'(z) = -(n+1)/z + 1/(1-z)^2$ twice we find that $I'''(z) \sim 6/(1-z)^4$ near $z = 1$, and hence that the right-hand side is bounded by $(k+o(1))t^3n^2 =$

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$(k + o(1))n^{-1/4} \log^3 n$ for some $k > 0$. Because the integrand on the right-hand side of (3.16) is everywhere positive, this implies the existence of $c > 0$ such that $M_4 \le cn^{-1/4} \log^3 n$ times the value of M, as desired. ◃

Remark. The approach of Example 3.18 yields a full asymptotic development of a_n with minor modifications.

Our second example simply assumes the approximation (3.13), greatly reducing the amount of work.

Example 3.19 (involutions: an entire function)**.** Let $f(z) = \exp(z+z^2/2)$ be the exponential generating function for the number a_n of involutions in the permutations group S_n, as discussed in Example 2.49. This is an entire function, and we apply a saddle point analysis. Let

$$I(z) = \log(f(z)z^{-n-1}) = z + \frac{z^2}{2} - (n+1)\log z\,.$$

Setting the derivative of I equal to zero gives the quadratic $z^2 + z - (n+1) = 0$ with roots $-\frac{1}{2} \pm \sqrt{n + \frac{5}{4}}$. The series coefficients a_n of f are positive, whereas $\exp(I(z))$ alternates in sign near the negative root, meaning a_n cannot be approximated by the integrand near the negative root.

We thus let γ be the positively oriented circle around the origin through $z_0 = \sqrt{n + \frac{5}{4}} - \frac{1}{2}$. The real part of I on γ is maximized at z_0, so the estimate (3.13) implies

$$[z^n]f(z) = \frac{1}{2\pi i}\int_\gamma \exp(I(z_n))\,dz \sim \exp(I(z_0))\sqrt{\frac{1}{2\pi\, I''(z_0)}}\,.$$

From the approximations

$$\begin{aligned} z_0 &= n^{1/2} - \frac{1}{2} + \frac{5}{8}n^{-1/2} + O(n^{-3/2}) \\ \frac{z_0^2}{2} &= \frac{1}{2}n - \frac{1}{2}n^{1/2} + \frac{3}{4} + O(n^{-1/2}) \\ \log(z_0) &= \frac{1}{2}\log n - \frac{1}{2}n^{-1/2} + \frac{1}{2}n^{-1} + O(n^{-3/2}) \end{aligned}$$

it follows that

$$I(z_0) = -\frac{1}{2}n\log n + \frac{1}{2}n + n^{1/2} - \frac{1}{2}\log n - \frac{1}{4} + O(n^{-1/2})$$

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and $I''(z_0) = 2 + o(1)$. Thus,

$$\begin{aligned} a_n &\sim n!\ \exp(I(z_0))\sqrt{\frac{1}{2\pi I''(z_0)}} \\ &= n!\ \exp\left(-\frac{1}{2}n\log n + \frac{1}{2}n + n^{1/2} - \frac{1}{2}\log n - \frac{1}{4} + O(n^{-1/2})\right) 2^{-1/2} \\ &\sim n^{n/2} e^{\sqrt{n}-n/2}\frac{1}{\sqrt{2\sqrt{e}}}, \end{aligned}$$

where the final line follows from Stirling's approximation for $n!$. ◁

Notes

One of the earliest and most well-known uses of a modern generating function analysis to obtain asymptotics was Hardy and Ramanujan's derivation of asymptotics for the number of partitions of an integer [HR00a]. Their original argument used a *Tauberian theorem* and the behavior of the generating function $f(s)$ as $s \uparrow 1$ through real values, though later work such as [HR00b] used a *circle method* obtained by integrating over a circle near the boundary of the domain of convergence. Saddle point methods are even more classical, dating back centuries. As mentioned in the chapter, [Hay56] was an influential work in developing the modern general theory.

The exposition in this chapter does not follow any one source, though it owes a debt to Chapter 11 of [Hen91] and to the beautiful paper [FO90]. A nice reference book for univariate asymptotics is the exemplary text [FS09].

Additional exercises

Exercise 3.6. The explicit leading term formulae in Lemmas 3.6 and Proposition 3.7 are only useful when the numerator of the meromorphic generating function is non-zero at the pole in question. Extend these two results to capture vanishing numerators and find the leading asymptotic term for the series coefficients of $f(z) = (1 - z)/(2 - z - e^{1-z})$ as $n \to \infty$.

Exercise 3.7. (set partition asymptotics) Use the exponential generating function $f(z) = \exp(e^z - 1)$ for the number a_n of set partitions of $[n]$ from Example 2.51 to derive the estimate

$$a_n = (\log n + O(1))^n .$$

Exercise 3.8. (Exercise 2.18 continued) Using the fact that the series coefficients a_n of the generating function f in Exercise 2.18 are positive, prove that its smallest positive singularity has the least modulus of any singularity of f. Approximate this singularity and then estimate the logarithmic exponential growth rate $\limsup_{n\to\infty} n^{-1} \log a_n$. Prove that this limsup is equal to the liminf, so the limit exists.

Exercise 3.9. Sometimes, even when f is given explicitly, it can be tricky to compute the minimal modulus of the singularities of f in order to obtain the exponential coefficient behavior using (3.5). The power series coefficients of the function

$$f(z) = \frac{\arctan\sqrt{2e^{-z}-1}}{\sqrt{2e^{-z}-1}}$$

were shown by H. Wilf to yield rational approximations to π. An asymptotic analysis was provided by [War10]; do the first step by finding the radius of convergence of the power series for f at zero.

Exercise 3.10. Suppose $P(x)$ is a polynomial of degree k with leading coefficient $a_k \neq 0$. What does the saddle point method tell you about the asymptotics of the Maclaurin coefficients a_n of $e^{P(x)}$? Specifically, can you identify an exponent β such that $\lim_{n\to\infty} n^{-\beta} \log |a_n|$ is finite?

Exercise 3.11. (Open Problem) Is the generating function ϕ from Example 3.17 analytic in a Δ-domain?

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