BULL. AUSTRAL. MATH. SOC. 35J25
VOL. 27 (1983), 1-30. (35D10)

ON THE BOUNDARY VALUES OF THE SOLUTIONS OF LINEAR ELLIPTIC EQUATIONS

J. Chabrowski and H.B. Thompson

The purpose of this article is to investigate the traces of weak solutions of a linear elliptic equation. In particular, we obtain a sufficient condition for a solution belonging to the Sobolev space $W^{1,2}_{\text{loc}}$ to have an L^2-trace on the boundary.

Introduction

This paper deals with L^2-behaviour near the boundary of weak solutions of the elliptic equation

$$(1)\qquad Lu = -\sum_{i,j=1}^{n} D_j\left(a_{ij}(x)D_iu\right) + \sum_{i=1}^{n} b_i(x)D_iu + c(x)u = f(x)$$

in a bounded domain Q. The problem we are concerned with originates in the theory of analytic functions. We say that the analytic function $f(z)$ defined on $(|z| < 1)$ has a limit in L^2 on the boundary, if there exists a function $\phi \in L^2(0, 2\pi)$ such that

$$\lim_{r\to 1-0} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)-\phi(\theta)\right|^2 d\theta = 0 .$$

Riesz [14] proved the following criterion: the analytic function $f(z)$ on $(|z| < 1)$ has a limit in L^2 on the boundary if and only if the function

Received 20 May 1982.

$$U(r) = \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \ , \quad 0 \le r < 1 \ ,$$

is bounded. Later on Littlewood and Paley [6] established the following theorem: the analytic function $f(z)$ on $(|z| < 1)$ has a limit in L^2 on the boundary if and only if the function

$$g(\theta) = \left[\int_0^1 (1-r)|f'(re^{i\theta})|^2 dr\right]^{\frac{1}{2}} \ , \quad 0 < \theta < 2\pi \ ,$$

belongs to $L^2(0, 2\pi)$. The results of Riesz and Littlewood and Paley were extended by many authors to harmonic functions and solutions of elliptic equations (for further historical material and bibliographic references see [10]). In particular Mikhailov in a series of the articles [7], [8], [9] and [10] extended the above results to solutions of the equation (1) under the assumption that $a_{ij} \in C^1(\bar{Q})$, $b_i \in C^1(\bar{Q})$ and $c \in C(\bar{Q})$. In this paper we establish Mikhailov's results under weaker assumptions, namely $b_i \in L^s(Q)$, $s > n$, and $c \in L^r(Q)$, $r > n/2$, and by different methods. The plan of the paper is as follows. Section 1 is devoted to preliminaries. In Sections 2 and 3 we discuss traces of solutions in $W^{1,2}_{\mathrm{loc}}(Q)$ of (1). Section 4 deals with energy estimates for solutions in $W^{1,2}_{\mathrm{loc}}(Q)$. Finally in Section 5 we briefly study the Dirichlet problem in the $W^{1,2}_{\mathrm{loc}}$-framework. Recall that a weak solution in $W^{1,2}(Q)$ of the equation (1) is a solution of the Dirichlet problem with the boundary condition $u = \phi$ on ∂Q if and only if $u - \phi_1 \in \mathring{W}^{1,2}(Q)$. In this definition it is assumed that the boundary data ϕ is a trace of some function ϕ_1 from $W^{1,2}(Q)$. Of course this assumption is rather restrictive, because not every function in $L^2(\partial Q)$ is the trace of some function belonging to $W^{1,2}(Q)$. In connection with the results obtained in Sections 2, 3 and 4 it makes sense to consider the Dirichlet problem with boundary data in $L^2(\partial Q)$. This possibility has already been noted by Nečas [12] and [13] (see Chapter 6) and by Mikhailov [9] and Guščin and

Mikhailov [3].

1. Preliminaries

Let $Q \subset R_n$ be a bounded domain with the boundary ∂Q of class C^2. In Q we consider the equation (1).

We make the following assumptions:

(A) there exists a positive constant γ such that

$$\gamma^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \gamma|\xi|^2$$

for all $x \in Q$ and $\xi \in R_n$; moreover the coefficients a_{ij} are measurable and of class C^1 in some neighborhood of ∂Q ;

(B) $b_i \in L^s(Q)$ $(i = 1, \ldots, n)$, $c \in L^r(Q)$, where $n < s \le \infty$, $n/2 < r \le \infty$;

(C) $\int_Q f(x)^2 r(x)^\theta dx < \infty$, where $2 \le \theta < 3$, $r(x) = \operatorname{dist}(x, \partial Q)$.

In this paper we use the notion of a generalized solution of (1) involving the Sobolev spaces $W^{1,2}_{\text{loc}}(Q)$, $W^{1,2}(Q)$ and $\overset{\circ}{W}{}^{1,2}(Q)$ (for the definitions of these spaces see [2] or [4]).

A function $u(x)$ is said to be a weak solution of the equation (1) if $u \in W^{1,2}_{\text{loc}}(Q)$ and u satisfies

$$(2) \qquad \int_Q \left[\sum_{i,j=1}^{n} a_{ij}(x)D_iuD_jv + \sum_{i=1}^{n} b_i(x)D_iu.v+c(x)uv\right]dx = \int_Q f(x)v\,dx$$

for every $v \in W^{1,2}(Q)$ with compact support in Q .

It follows from the regularity of the boundary ∂Q that there is a number $\delta_0 > 0$ such that for $\delta \in (0, \delta_0]$ the domain $Q_\delta = Q \cap \{x;\ \min_{y\in\partial Q} |x-y| > \delta\}$, with the boundary ∂Q_δ , possesses the following

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

property: to each $x_0 \in \partial Q$ there is a unique point $x_\delta(x_0) \in \partial Q_\delta$ such that $x_\delta(x_0) = x_0 - \delta\nu(x_0)$, where $\nu(x_0)$ is the outward normal to ∂Q at x_0. The inverse mapping of $x_0 \to x_\delta(x_0)$ is given by the formula $x_0 = x_\delta + \delta\nu_\delta(x_\delta)$, where $\nu_\delta(x_\delta)$ is the outward normal to ∂Q_δ at x_δ.

Let x_δ denote an arbitrary point of ∂Q_δ. For fixed $\delta \in (0, \delta_0]$ let

$$A_\varepsilon = \partial Q_\delta \cap \{x;\ |x-x_\delta| < \varepsilon\},$$

$$B_\varepsilon = \{x;\ x = \tilde{x}_\delta+\delta\nu_\delta(\tilde{x}_\delta),\ \tilde{x}_\delta \in A_\varepsilon\},$$

and

$$\frac{dS_\delta}{dS_0} = \lim_{\varepsilon\to 0}\frac{|A_\varepsilon|}{|B_\varepsilon|},$$

where $|A|$ denote the $n-1$ dimensional Hausdorff measure of a set A. Mikhailov [8] proved that there is a positive number γ_0 such that

$$\gamma_0^{-2} \le \frac{dS_\delta}{dS_0} \le \gamma_0^2 \tag{3}$$

and

$$\lim_{\delta\to 0}\frac{dS_\delta}{dS_0} = 1 \tag{4}$$

uniformly with respect to $x_\delta \in \partial Q_\delta$.

According to Lemma 1 in [2], p. 382, the distance $r(x)$ belongs to $C^2(\overline{Q}-Q_{\delta_0})$ if δ_0 is sufficiently small. Denote by $\rho(x)$ the extension of the function $r(x)$ into $\overline{Q}$ satisfying the following properties $\rho(x) = r(x)$ for $x \in \overline{Q} - Q_{\delta_0}$, $\rho \in C^2(\overline{Q})$, $\rho(x) \ge 3\delta_0/4$ in Q_{δ_0}, $\gamma_1^{-1}r(x) \le \rho(x) \le \gamma_1 r(x)$ in Q for some positive constant γ_1, $\partial Q_\delta = \{x;\ \rho(x) = \delta\}$ for $\delta \in (0, \delta_0]$ and finally $\partial Q = \{x;\ \rho(x) = 0\}$.

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

It follows from assumption (A) that the constant δ_0 can be chosen so small that $a_{ij} \in C^1\left(\overline{Q-Q_{\delta_0}}\right)$.

We will use the surface integrals

$$M_1(\delta) = \int_{\partial Q} \left|u\left(x_\delta(x)\right)\right|^2 dS_x \quad \text{and} \quad M(\delta) = \int_{\partial Q_\delta} |u(x)|^2 dS_x ,$$

where $u \in W^{1,2}_{\text{loc}}(Q)$ and the values of $u\left(x_\delta(x)\right)$ on ∂Q and $u(x)$ on ∂Q_δ are understood in the sense of traces (see [4], Chapter 6). It follows from Lemma 4 in [1] that $M_1(\delta)$ and $M(\delta)$ are absolutely continuous on $[\delta_1, \delta_0]$ for every $0 < \delta_1 < \delta_0$. Moreover if $M(\delta)$ is bounded on $(0, \delta_0]$, then for every $0 \le \alpha < 1$ there is a positive constant C such that

(5)
$$\int_{Q_\delta} \frac{u(x)^2}{\left(\rho(x)-\delta\right)^\alpha}\, dx < C$$

for every $\delta \in (0, \delta_0/2]$ (for details see Lemma 5 in [1]).

The following result is a modification of Lemma 6 in [1].

LEMMA 1. *Suppose that* $u \in W^{1,2}_{\text{loc}}(Q)$ *and that* $\int_Q |Du(x)|^2 r(x)dx < \infty$. *Then if* $0 \le \mu < 1$ *and* $0 < \delta_1 \le \delta_0/2$ *we have, for* $\delta \in (0, \delta_1/2]$,

$$\int_{Q_\delta} \frac{u(x)^2}{\left(\rho(x)-\delta\right)^\mu}\, dx$$
$$\le K\left[\delta_1^{-\mu} \int_{Q_{\delta_1}} u(x)^2 dx + \delta_1^{1-\mu} \int_{\partial Q_{\delta_1}} u(x)^2 dS + \delta_1^{1-\mu} \int_{Q_\delta - Q_{\delta_1}} |Du(x)|^2 \left(\rho(x)-\delta\right) dx\right],$$

where K *is a positive constant independent of* δ_1 *and* δ.

Proof. Let $\delta \in (0, \delta_1/2]$ and put

$$\int_{Q_\delta} \frac{u^2}{(\rho-\delta)^\mu}\, dx = \int_{Q_\delta - Q_{\delta_1}} \frac{u^2}{(\rho-\delta)^\mu}\, dx + \int_{Q_{\delta_1}} \frac{u^2}{(\rho-\delta)^\mu}\, dx .$$

Since $\rho(x) \geq \delta_1$ on Q_{δ_1}, we have

$$\int_{Q_{\delta_1}} \frac{u^2}{(\rho-\delta)^\mu}\, dx \leq (2/\delta_1)^\mu \int_{Q_{\delta_1}} u^2 dx .$$

We now note that

$$\int_{Q_\delta - Q_{\delta_1}} \frac{u^2}{(\rho-\delta)^\mu}\, dx = \int_\delta^{\delta_1} (t-\delta)^{-\mu} \int_{\partial Q} u\big(x_t(x_0)\big)^2 \frac{dS_t}{dS_0}\, dS_0 dt$$

$$\leq \gamma_0^2 \int_\delta^{\delta_1} (t-\delta)^{-\mu} \int_{\partial Q} u\big(x_t(x_0)\big)^2 dS_0 dt .$$

As $\int_{\partial Q} u\big(x_t(x)\big)^2 dS_x$ is absolutely continuous on $[\delta, \delta_1]$, integrating by parts

$$\int_{Q_\delta - Q_{\delta_1}} \frac{u^2}{(\rho-\delta)^\mu}\, dx$$

$$\leq \frac{\gamma_0^2 \delta_1^{1-\mu}}{1-\mu} \int_{\partial Q} u\big(x_{\delta_1}(x)\big)^2 dS$$

$$+ \frac{2\gamma_0^2}{1-\mu} \int_\delta^{\delta_1} (t-\delta)^{1-\mu} \int_{\partial Q} \big|u\big(x_t(x_0)\big)\big|\,\big|Du\big(x_t(x_0)\big)\big| \left|\frac{\partial}{\partial t} x_t(x_0)\right| dS_0 dt$$

$$\leq \frac{\gamma_0^4 \delta_1^{1-\mu}}{1-\mu} \int_{\partial Q_{\delta_1}} u^2 dS + \frac{2\gamma_0^4}{1-\mu} \int_{Q_\delta - Q_{\delta_1}} |u(x)|\,|Du(x)|\,(\rho-\delta)^{1-\mu} dx$$

$$\leq \frac{\gamma_0^4 \delta_1^{1-\mu}}{1-\mu} \int_{\partial Q_{\delta_1}} u^2 dS + \frac{2\beta\gamma_0^4}{1-\mu} \int_{Q_\delta - Q_{\delta_1}} \frac{u^2}{(\rho-\delta)^\mu}\, dx$$

$$+ \frac{2\gamma_0^4 \delta_1^{1-\mu}}{\beta(1-\mu)} \int_{Q_\delta - Q_{\delta_1}} |Du|^2 (\rho-\delta)\, dx ,$$

where we have used Young's inequality in the final step. Now choosing

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

$2\gamma_0^{4\beta}/(1-\mu) = \frac{1}{2}$ the result follows.

2. Main result

We are now in a position to establish a criterion for the continuity of $M_1(\delta)$ and $M(\delta)$ on $[0, \delta_0]$ which plays an essential part in the ensuing treatment of the Dirichlet problem.

THEOREM 1. *Let u be a solution of* (1) *belonging to $W^{1,2}_{loc}(Q)$, then the following conditions are equivalent:*

I. $M(\delta)$ *is a bounded function on* $(0, \delta_0]$ *;*

II. $\int_Q |Du(x)|^2 r(x)dx < \infty$ *;*

III. $M_1(\delta)$ *is continuous on* $[0, \delta_0]$.

Proof. Let Φ be a smooth function on $\bar{Q}$ such that $\Phi \equiv 1$ on $Q - Q_{\delta_1/2}$, $\Phi = 0$ on Q_{δ_1} and $0 \le \Phi \le 1$ on $\bar{Q}$, where $0 < \delta_1 \le \delta_0/2$ is to be determined later. Put

$$v(x) = \begin{cases} u(x)\big(\rho(x)-\delta\big)\Phi(x)^2 & \text{for } x \in Q_\delta , \\ 0 & \text{for } x \in Q-Q_\delta , \end{cases}$$

where $0 < \delta \le \delta_0/2$. It is clear that v is an admissible test function in (2) and

$$(6)\quad \int_{Q_\delta} \sum_{i,j=1}^n a_{ij}D_iuD_ju(\rho-\delta)\Phi^2dx + \int_{Q_\delta} \sum_{i,j=1}^n a_{ij}D_iu.uD_j\rho\Phi^2dx$$

$$+ 2\int_{Q_\delta} \sum_{i,j=1}^n a_{ij}D_iu.u(\rho-\delta)\Phi D_j\Phi dx + \int_{Q_\delta} \sum_{i=1}^n b_iD_iu.u(\rho-\delta)\Phi^2dx$$

$$+ \int_{Q_\delta} cu^2(\rho-\delta)\Phi^2dx$$

$$= \int_{Q_\delta} fu(\rho-\delta)\Phi^2dx .$$

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

The proof of I ⇒ II. Denote the integrals on the left hand side of (6) by J_1, J_2, J_3, J_4 and J_5. It follows from assumption (A) that

$$J_1 \geq \gamma^{-1} \int_{Q_\delta} |Du|^2(\rho-\delta)\Phi^2 dx . \tag{7}$$

By Green's formula (see [11], p. 139) we have

$$J_2 = \tfrac{1}{2} \int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij} D_i u^2 D_j \rho \Phi^2 dx$$

$$= -\tfrac{1}{2} \int_{\partial Q_\delta} \sum_{i,j=1}^{n} a_{ij} D_i \rho D_j \rho u^2 \Phi^2 dS - \tfrac{1}{2} \int_{Q_\delta} \sum_{i,j=1}^{n} D_i\left(a_{ij} D_j \rho \Phi^2\right) u^2 dx .$$

Thus

$$|J_2| \leq \tfrac{1}{2} \int_{\partial Q_\delta} \left(\sum_{i,j=1}^{n} a_{ij} D_i \rho D_j \rho\right) u^2 \Phi^2 dS + C_1 \int_{Q_\delta} u^2 \Phi^2 dx + C_2 \int_{Q_\delta} u^2 |D_x \Phi| \Phi dx , \tag{8}$$

where

$$C_1 = \tfrac{1}{2} \sup_{Q-Q_{\delta_0}} \sum_{i,j=1}^{n} |D_i(a_{ij} D_j \rho)| , \quad C_2 = \sup_{Q} \sum_{i,j=1}^{n} |a_{ij} D_j \rho| .$$

It follows from Young's inequality that

$$|J_3| \leq \frac{\gamma^{-1}}{8} \int_{Q_\delta} |Du|^2(\rho-\delta)\Phi^2 dx + C_3 \int_{Q_\delta} u^2(\rho-\delta)|D_x \Phi|^2 dx , \tag{9}$$

where

$$C_3 = 8\gamma \sup_{Q} \sum_{i,j=1}^{n} |a_{ij}| .$$

To estimate J_4 and J_5 let us first suppose that $n > 3$. In view of Hölder's inequality we obtain

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

$$|J_4| \le \Big\|\sum_{i=1}^{n} b_i\Big\|_{L^s(Q)} \|(\rho-\delta)^{-\varepsilon}\|_{L^{s_1}(Q_\delta - Q_{\delta_1})}$$
$$\times \|u(\rho-\delta)^{\frac12+\varepsilon}\Phi\|_{L^{2^*}(Q_\delta)} \Big\|D_x u(\rho-\delta)^{\frac12}\Phi\Big\|_{L^2(Q_\delta)} ,$$

where

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n} , \quad \frac{1}{s} + \frac{1}{s_1} = \frac{1}{n} ,$$

and $0 < \varepsilon < 1$ is chosen so that $\varepsilon s_1 < 1$. In view of (5),

$$\int_{Q_\delta} \frac{|u(x)|^2}{(\rho(x)-\delta)^\alpha}\, dx \le C , \quad 0 < \alpha < 1 ,$$

for $\delta \in (0, \delta_0/2)$ and consequently $u(\rho-\delta)^{\frac12+\varepsilon} \in \overset{\circ}{W}{}^{1,2}(Q_\delta)$. By Sobolev's inequality ([2], p. 148) we have

$$\|u(\rho-\delta)^{\frac12+\varepsilon}\Phi\|_{L^{2^*}(Q_\delta)}$$
$$\le S\Big[\Big\|D_x u(\rho-\delta)^{\frac12+\varepsilon}\Phi\Big\|_{L^2(Q_\delta)} + (\tfrac12+\varepsilon)\Big\|u|D_x\rho|(\rho-\delta)^{\varepsilon-\frac12}\Phi\Big\|_{L^2(Q_\delta)} + \Big\|u(\rho-\delta)^{\frac12+\varepsilon}D_x\Phi\Big\|_{L^2(Q_\delta)}\Big]$$

for some positive constant S independent of δ . Therefore there is a positive constant $\tilde{S}$ such that

$$|J_4| \le \tilde{S}\Big\|\sum_{i=1}^{n} b_i\Big\|_{L^s(Q)} \|(\rho-\delta)^{-\varepsilon}\|_{L^{s_1}(Q_\delta - Q_{\delta_1})}$$
$$\times \Big[\int_{Q_\delta} |D_x u|^2(\rho-\delta)\Phi^2 dx + \Big\|D_x u(\rho-\delta)^{\frac12}\Phi\Big\|_{L^2(Q_\delta)} \|u(\rho-\delta)^{\varepsilon-\frac12}\Phi\|_{L^2(Q_\delta)}$$
$$+ \Big\|D_x u(\rho-\delta)^{\frac12}\Phi\Big\|_{L^2(Q_\delta)} \Big\|u(\rho-\delta)^{\frac12}D_x\Phi\Big\|_{L^2(Q_\delta)}\Big] .$$

Note that there is a constant $A > 0$ independent of δ such that

$$\|(\rho-\delta)^{-\varepsilon}\Phi^2\|_{L^{s_1}(Q_\delta-Q_{\delta_1})} \le A\left[\int_\delta^{\delta_1}(t-\delta)^{-\varepsilon s_1}dt\right]^{1/s_1}$$

$$= A\left(1-\varepsilon s_1\right)^{-1/s_1}\delta_1^{\left(1-\varepsilon s_1\right)/s_1} .$$

Choose δ_1 so that

$$A\tilde{S}\left(1-\varepsilon s_1\right)^{-1/s_1}\delta_1^{\left(1-\varepsilon s_1\right)/s_1}\left\|\sum_{i=1}^{n} b_i\right\|_{L^s(Q)} \le \frac{\gamma^{-1}}{8} .$$

Consequently by Young's inequality we obtain

$$(10)\quad |J_4| \le \frac{3\gamma^{-1}}{8}\int_{Q_\delta}|D_x u|^2(\rho-\delta)\Phi^2 dx + \frac{\gamma^{-1}}{8}\int_{Q_\delta}u^2(\rho-\delta)^{2\varepsilon-1}dx$$

$$+ \frac{\gamma^{-1}}{8}\int_{Q_\delta}u^2(\rho-\delta)|D_x\Phi|^2 dx$$

Similarly

$$|J_5| \le \|c\|_{L^r(Q)}\|(\rho-\delta)^{-2\varepsilon_1}\|_{L^{r_1}(Q_\delta-Q_{\delta_1})}\left\|u(\rho-\delta)^{\frac{1}{2}+\varepsilon_1}\Phi\right\|^2_{L^{2^*}(Q_\delta)} ,$$

where

$$\frac{1}{r}+\frac{1}{r_1}=\frac{2}{n} , \quad \varepsilon_1 r_1 < 1 .$$

Then in view of the Sobolev inequality there is a positive constant S_1 such that

$$|J_5| \le S_1\|c\|_{L^r(Q)}\|(\rho-\delta)^{-2\varepsilon_1}\|_{L^{r_1}(Q_\delta-Q_{\delta_1})}$$

$$\times\left[\int_{Q_\delta}u^2(\rho-\delta)^{-1+2\varepsilon_1}dx + \int_{Q_\delta}|Du|^2(\rho-\delta)\Phi^2 dx + \int_{Q_\delta}u^2(\rho-\delta)|D_x\Phi|^2 dx\right]$$

Finally taking δ_1 so small that

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

$$S_1 \|c\|_{L^r(Q)} \|(\rho-\delta)^{-2\varepsilon_1}\|_{L^{r_1}(Q_\delta - Q_{\delta_1})} \le \frac{\gamma^{-1}}{8}$$

we arrive at the inequality

$$(11) \quad |J_5| \le \frac{\gamma^{-1}}{8} \int_{Q_\delta} |D_x u|^2 (\rho-\delta)\Phi^2 dx + \frac{\gamma^{-1}}{8} \int_{Q_\delta} u^2 (\rho-\delta)^{-1+2\varepsilon_1} dx + \frac{\gamma^{-1}}{8} \int_{Q_\delta} u^2 (\rho-\delta)|D_x\Phi|^2 dx \; .$$

The remaining case $n = 2$ can now be obtained by using the fact that $W^{1,2}(Q)$ $(n = 2)$ is continuously imbedded in $L^q(Q)$ for all $1 \le q < \infty$ (see [4], p. 287).

By Young's inequality we obtain

$$(12) \quad \left|\int_{Q_\delta} fu(\rho-\delta)\Phi^2 dx\right| \le \tfrac{1}{2} \int_{Q_\delta} f^2 (\rho-\delta)^\theta dx + \tfrac{1}{2} \int_{Q_\delta} u^2 (\rho-\delta)^{-\alpha} dx \; ,$$

where $\alpha = \theta - 1 < 1$. Combining the identity (6) and the estimates (7), (8), (9), (10), (11) and (12) we deduce that there is a positive constant C_4 such that

$$\int_{Q_\delta} |Du|^2 (\rho-\delta)\Phi^2 dx \le C_4 \Big[M(\delta) + \int_{Q_\delta} u^2 dx + \int_{Q_\delta} u^2 (\rho-\delta)|D_x\Phi|^2 dx + \int_{Q_\delta} u^2 (\rho-\delta)^{-\beta} dx + \int_{Q_\delta} f^2 (\rho-\delta)^\theta dx \Big] \; ,$$

for all $\delta \in (0, \delta_1/2]$, where $\beta = \max(\alpha, 1-2\varepsilon_1, 1-2\varepsilon)$. The Monotone Convergence Theorem implies that

$$\int_{Q-Q_{\delta_1/2}} |Du(x)|^2 r(x) dx < \infty \; .$$

Since $u \in W^{1,2}_{\mathrm{loc}}(Q)$, it follows that I $\Rightarrow$ II.

To prove II $\Rightarrow$ III note that Lemma 1 implies

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

$$\int_{Q_\delta} \frac{u(x)^2}{(\rho(x)-\delta)^\mu}\,dx \le C\ , \quad 0 \le \mu < 1\ ,$$

for $\delta \in [0, \delta_0/2]$, where C is independent of δ . First we prove that $M(\delta)$ is continuous at $\delta = 0$. Indeed from the first part of the proof

$$\tfrac{1}{2}\int_{\partial Q_\delta} \sum_{i,j=1}^{n} a_{ij}D_i\rho D_j\rho u^2\Phi^2 dS = \int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij}D_i uD_j u(\rho-\delta)\Phi^2 dx$$

$$- \tfrac{1}{2}\int_{Q_\delta} \sum_{i,j=1}^{n} D_i\left(a_{ij}D_j\rho\Phi^2\right)u^2 dx + 2\int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij}D_i uu(\rho-\delta)\Phi D_j\Phi dx$$

$$+ \int_{Q_\delta} \sum_{i=1}^{n} b_i D_i uu(\rho-\delta)\Phi^2 dx + \int_{Q_\delta} cu^2(\rho-\delta)\Phi^2 dx - \int_{Q_\delta} fu(\rho-\delta)\Phi^2 dx\ .$$

Thus

$$\lim_{\delta\to 0}\int_{\partial Q_\delta} u^2 \sum_{i,j=1}^{n} a_{ij}D_i\rho D_j\rho dS_x$$

exists, by the Dominated Convergence Theorem. Since

$$\gamma^{-1} \le \sum_{i,j=1}^{n} a_{ij}D_i\rho D_i\rho \le \gamma$$

is continuous on $\overline{Q - Q_{\delta_0}}$ it follows that $M(\delta)$ is continuous at $\delta = 0$.

That II $\Rightarrow$ III follows from the relationship

$$M(\delta) - M_1(\delta) = \int_{\partial Q} u\left(x_\delta(x)\right)^2\left[\frac{dS_\delta}{dS_0} - 1\right]dS\ ,$$

since $dS_\delta/dS_0 \to 1$ uniformly as $\delta \to 0$.

Finally III $\Rightarrow$ I follows from the proof II $\Rightarrow$ III.

3. Traces in $L^2(\partial Q)$

Our next objective is to prove that u has a trace on ∂Q in $L^2(\partial Q)$; that is, $u(x_\delta)$ converges in $L^2(\partial Q)$ as $\partial \to 0$. To do this we

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

first show that $u(x_\delta)$ converges weakly in $L^2(\partial Q)$ to some function ζ (Theorems 2 and 3) and then show norm $u(x_\delta)$ converges to norm ζ. The result then follows by uniform convexity.

THEOREM 2. *Let* $u \in W^{1,2}_{\text{loc}}(Q)$ *be a solution of* (1). *Assume that one of the conditions* I, II *or* III *holds. Then there is a sequence* $\delta_\nu \to 0$ *as* $\nu \to \infty$ *and a function* $\zeta \in L^2(\partial Q)$ *such that*

$$\lim_{\nu\to\infty} \int_{\partial Q} u\left(x_{\delta_\nu}(x)\right) g(x) dS_x = \int_{\partial Q} \zeta(x) g(x) dS_x$$

for each $g \in L^2(\partial Q)$.

THEOREM 3. *Let* $u \in W^{1,2}_{\text{loc}}(Q)$ *be a solution of* (1). *If one of the conditions* I, II *or* III *holds, then the function*

$$G(\delta) = \int_{\partial Q} u\left(x_\delta(x)\right) \Psi(x) dS_x$$

is continuous on $[0, \delta_0]$ *for every* Ψ *in* $L^2(\partial Q)$.

Proof. It is clear that $G(\delta)$ is absolutely continuous on $[\delta_1, \delta_0]$ for any $\delta_1 < \delta_0$, hence it remains to prove the continuity at $\delta = 0$. Note that

$$g(x) = \sum_{i,j=1}^{n} a_{ij}(x) D_i \rho D_j \rho$$

is uniformly continuous on $\overline{Q - Q_{\delta_0}}$ and $\gamma^{-1} \le g(x) \le \gamma$ on $\overline{Q}$. On the other hand $M_1(\delta)$ is bounded and the elements of $C^1(\overline{Q})$ vanishing on Q_{δ_0} are dense in $L^2(\partial Q)$, so it suffices to show that

$$\overline{G}(\delta) = \int_{\partial Q_\delta} u\Psi \sum_{i,j=1}^{n} a_{ij} D_i \rho D_j \rho dS_x$$

is continuous for each $\Psi \in C^1(\overline{Q})$ vanishing on Q_{δ_0}.

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

Taking

$$v = \begin{cases} \Psi(\rho-\delta) & \text{for } x \in Q_\delta , \\ 0 & \text{for } x \in Q-Q_\delta , \end{cases}$$

in (2) as a test function, we obtain

$$\overline{G}(\delta) = \int_{Q_\delta} \Big[\sum_{i,j=1}^{n} a_{ij} D_i u D_j \Psi(\rho-\delta) - \sum_{i,j=1}^{n} D_i \left(a_{ij} D_j \rho \Psi\right) u + \sum_{i=1}^{n} b_i D_i u \Psi(\rho-\delta) + cu\Psi(\rho-\delta) - f\Psi(\rho-\delta) \Big] dx .$$

By an argument similar to that used in the proof of Theorem 1 one can easily show that the integrand on the right hand side is in $L^1(Q)$ and the result follows.

In order to prove the convergence of the norm we use the following function and technical lemmas.

For $\delta \in (0, \delta_0]$ we define the mapping $x^\delta : \overline{Q} \to \overline{Q}_\delta$ by

$$x^\delta(x) = \begin{cases} x & \text{for } x \in Q_\delta , \\ x_\delta + \frac{1}{2}\left(x - x_\delta\right) & \text{for } x \in Q-Q_\delta . \end{cases}$$

Thus $x^\delta(x) = x$ for each $x \in Q_\delta$ and $x^\delta(x) = x_{\delta/2}(x)$ for each $x \in \partial Q$. Moreover $\rho\left(x^\delta\right) \geq \delta/2$ and x^δ is uniformly Lipschitz continuous. Note that if $u \in W^{1,2}_{\mathrm{loc}}(Q)$ then $u\left(x^\delta\right) \in W^{1,2}(Q)$.

The proofs of the following lemmas can be found in [1].

LEMMA 2. *Let* $h \in L^1(Q)$, *then*

$$\lim_{\delta \to 0} \int_{Q-Q_\delta} h\left(x^\delta(x)\right) dx = 0 .$$

LEMMA 3. *If* $\rho^{\mu/2} f \in L^2(Q)$, $0 \leq \mu < 1$, $g \in L^2(Q)$ *and* $\int_{\partial Q_\delta} g(x)^2 dS_x$ *is bounded on* $(0, \delta_0]$, *then*

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

$$\lim_{\delta\to 0}\int_{Q-Q_\delta} g\left(x^\delta(x)\right)f(x)dx = 0 .$$

LEMMA 4. *If* f *is a non-negative function in* $L^1(Q)$ *and if* $\int_{\partial Q_\delta} f(x)dS_x$ *is bounded on* $[0, \delta_0]$ *, then*

$$\int_{Q-Q_\delta} \frac{f\left(x^\delta(x)\right)}{\rho(x)^\mu}dx \le \frac{\gamma_0^4\delta^{1-\mu}}{1-\mu}\sup_{[0,\delta_0]}\int_{\partial Q_t} f(x)dS_x ,$$

where $0 \le \mu < 1$.

LEMMA 5. *Let* $g \in L^2(Q)$, $\rho^{\frac{1}{2}}f \in L^2(Q)$ *and suppose that* $\int_{\partial Q_\delta} |g(x)|^2 dS_x$ *is bounded on* $[0, \delta_0]$ *, then*

$$\lim_{\delta\to 0}\int_{Q-Q_\delta} f\left(x^\delta(x)\right)g(x)dx = 0 .$$

LEMMA 6. *If* $\rho^{\frac{1}{2}}f$ *and* $\rho^{\frac{1}{2}}g$ *belong to* $L^2(Q)$ *, then*

$$\lim_{\delta\to 0}\int_{Q-Q_\delta} f\left(x^\delta(x)\right)g(x)\rho(x)dx = 0 .$$

Let $L_1^2 = L^2\left(\partial Q,\ dS_x\right)$ with inner product (norm) denoted by $\langle\cdot,\ \cdot\rangle_1$ $\left(\|\cdot\|_1\right)$ and $L_2^2 = L^2\left(\partial Q,\ gdS_x\right)$ with inner product (norm) denoted by $\langle\cdot,\ \cdot\rangle_2$ $\left(\|\cdot\|_2\right)$, where

$$g(x) = \sum_{i,j=1}^n a_{ij}(x)D_i\rho(x)D_j\rho(x) .$$

Now we are in a position to prove the main result of this section.

THEOREM 4. *Let* $u \in W_{\text{loc}}^{1,2}(Q)$ *be a solution of* (1) *such that one of the conditions* I, II *or* III *holds. Then there is a function* ζ *belonging to* $L^2(\partial Q)$ *such that* $u\left(x_\delta\right)$ *converges to* ζ *in* L_1^2 .

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

Proof. Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent it suffices to show that there is $\zeta \in L_2^2$ such that $\lim_{\delta\to 0} u(x_\delta) = \zeta$ in L_2^2. By Theorem 2 and 3 there is $\zeta \in L_2^2$ such that $\lim_{\delta\to 0} u(x_\delta) = \zeta$ weakly in L_2^2. Since L_2^2 is uniformly convex it suffices to show that $\lim_{\delta\to 0} \|u(x_\delta)\|_2 = \|\zeta\|_2$. Let $\Psi \in W^{1,2}(Q)$ and $\Psi \equiv 0$ on Q_{δ_0}, set

$$F(\Psi(x)) = \sum_{i,j=1}^{n} a_{ij} D_i u D_j \Psi\rho - \sum_{i,j=1}^{n} D_i(a_{ij} D_j \rho\Psi)u + \sum_{i=1}^{n} b_i D_i u\Psi\rho + cu\Psi\rho - fu\Psi\rho .$$

As in the proof of Theorem 1 we find that

$$\langle \zeta, \Psi \rangle_2 = \int_Q F(\Psi(x))dx$$

for all $\Psi \in C^1(\bar{Q})$ such that $\Psi \equiv 0$ on Q_{δ_0} and hence for all $\Psi \in W^{1,2}(Q)$ with $\Psi \equiv 0$ on Q_{δ_0}. Let Φ be as defined in the proof of Theorem 1. Since $u(x^\delta)\Phi(x)^2 \in W^{1,2}(Q)$ and $u(x^\delta)\Phi(x)^2 = 0$ on Q_{δ_0} we have

$$\langle \zeta, u(x^\delta) \rangle_2 = \int_Q F(u(x^\delta(x)))dx = \int_{Q-Q_\delta} F(u(x^\delta(x)))dx + \int_{Q_\delta} F(u(x)\Phi(x)^2)dx ,$$

for $\delta \le \delta_1/2$, since $x^\delta(x) = x$ on Q_δ and $\Phi \equiv 1$ on $Q - Q_{\delta_1/2}$. We show that

$$\lim_{\delta\to 0} \int_{Q-Q_\delta} F(u(x^\delta(x)))dx = 0$$

and that

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

$$\lim_{\delta\to 0}\int_{Q_\delta} F\left(u(x)\Phi(x)^2\right)dx = \lim_{\delta\to 0}\left\|u\left(x_\delta\right)\right\|_2^2 ,$$

so that

$$\|\zeta\|_2^2 = \lim_{\delta\to 0}\langle \zeta,\ u\left(x^\delta\right)\rangle = \lim_{\delta\to 0}\left\|u\left(x^\delta\right)\right\|_2^2$$

since $x^\delta(x) = x_{\delta/2}(x)$ on ∂Q . Setting

$$v(x) = \begin{cases} u(x)\left(\rho(x)-\delta\right)\Phi(x)^2 & \text{for } x \in Q_\delta ,\\ \\ 0 & \text{for } x \in Q-Q_\delta , \end{cases}$$

it follows from (6) that

$$\lim_{\delta\to 0}\int_{Q_\delta} F\left(u\Phi^2\right)dx = \lim_{\delta\to 0}\int_{Q_\delta}\left[\sum_{i,j=1}^{n} a_{ij}D_iuD_ju(\rho-\delta)\Phi^2\right.$$

$$+ 2\sum_{i,j=1}^{n} a_{ij}D_iuu\Phi D_j\Phi(\rho-\delta) - \sum_{i,j=1}^{n} D_i\left(a_{ij}D_j\rho\Phi^2u\right)u$$

$$\left.+ \sum_{i=1}^{n} b_iD_iuu(\rho-\delta)\Phi^2+cu^2(\rho-\delta)\Phi^2-fu(\rho-\delta)\Phi^2\right]dx$$

$$= -\lim_{\delta\to 0}\int_{Q_\delta}\sum_{i,j=1}^{n} D_i\left(a_{ij}D_j\rho\Phi^2u^2\right)dx$$

$$= \lim_{\delta\to 0}\int_{\partial Q_\delta} u^2 g\,dS_x$$

$$= \lim_{\delta\to 0}\int_{\partial Q} u\left(x_\delta\right)^2 g(x)\,dS_x + \lim_{\delta\to 0}\int_{\partial Q} u\left(x_\delta\right)^2\left[g\left(x_\delta\right)-g(x)\right]dS_x$$

$$= \lim_{\delta\to 0}\int_{\partial Q} u\left(x_\delta\right)^2 g(x)\,dS_x$$

$$= \lim_{\delta\to 0}\left\|u\left(x_\delta\right)\right\|_2^2 .$$

Now it remains to prove that $\lim_{\delta\to 0}\int_{Q-Q_\delta} F\left(u\left(x^\delta(x)\right)\right)dx = 0$. Note that

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

$$|F(u(x^\delta(x)))|$$
$$\le C[|Du(x)||Du(x^\delta(x))|\rho(x)+|u(x)||u(x^\delta(x))|+|Du(x^\delta(x))||u(x)|]$$
$$+\sum_{i=1}^{n}|b_i(x)||Du(x)||u(x^\delta(x))|\rho(x) + |c(x)||u(x)||u(x^\delta(x))|\rho(x)$$
$$+|f(x)||u(x^\delta(x))|\rho(x) ,$$

for $x \in Q-Q_\delta$, where C is a positive constant independent of δ . Since $\rho^{(\theta/2)-1}(\rho f) \in L^2$, $0 \le \theta/2 - 1 < \frac{1}{2}$ and $\int_{\partial Q_\delta} u^2 dS_x$ is bounded, by Lemma 3,

$$\lim_{\delta\to 0}\int_{Q-Q_\delta}|f(x)||u(x^\delta)|\rho(x)dx = 0 .$$

It follows from Lemma 2 and Hölder's inequality that

$$\lim_{\delta\to 0}\int_{Q-Q_\delta}|u(x)||u(x^\delta)|\rho(x)dx = 0 .$$

In view of Lemmas 5 and 6,

$$\lim_{\delta\to 0}\int_{Q-Q_\delta}|Du(x^\delta(x))||u(x)|dx = 0 ,$$

$$\lim_{\delta\to 0}\int_{Q-Q_\delta}|Du(x)||Du(x^\delta(x))|\rho(x)dx = 0 .$$

To estimate the remaining terms we restrict ourselves to the case $n > 2$. By Hölder's inequality

$$\int_{Q-Q_\delta}\sum_{i=1}^{n}|b_i(x)||Du(x)||u(x^\delta(x))|\rho(x)dx$$
$$\le \left\|\sum_{i=1}^{n} b_i\right\|_{L^s(Q-Q_\delta)}\|Du\rho^{\frac{1}{2}}\|_{L^2(Q-Q_\delta)}\|\rho^{-\varepsilon}\|_{L^{s_1}(Q-Q_\delta)}\|u(x^\delta(x))\rho^{\frac{1}{2}+\varepsilon}\|_{L^{2*}(Q-Q_\delta)} ,$$

where

$$\frac{1}{s}+\frac{1}{s_1}=\frac{1}{n} , \quad \frac{1}{2*}=\frac{1}{2}-\frac{1}{n} , \quad \varepsilon s_1 < 1 .$$

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

Now the Sobolev inequality implies that

$$\|u(x^\delta(x))\rho^{\frac{1}{2}+\varepsilon}\|_{L^{2*}} \le S\left[\|Du(x^\delta(x))\rho^{\frac{1}{2}+\varepsilon}\|_{L^2(Q-Q_\delta)} + \|u(x^\delta(x))\rho^{-\frac{1}{2}+\varepsilon}\|_{L^2(Q-Q_\delta)}\right]$$

and consequently, by Lemmas 4 and 6,

$$\lim_{\delta\to 0}\int_{Q-Q_\delta}\sum_{i=1}^{n}|b_i(x)||Du(x)||u(x^\delta(x))|\rho(x)dx .$$

Similarly one can prove that

$$\lim_{\delta\to 0}\int_{Q-Q_\delta}|c(x)||u(x)||u(x^\delta(x))|\rho(x)dx = 0 ,$$

and this completes the proof.

4. The energy estimate

Consider the elliptic equation of the form

$$(1') \quad Lu + \lambda u = -\sum_{i,j=1}^{n} D_j\big(a_{ij}(x)D_i u\big) + \sum_{i=1}^{n} b_i(x)D_i u + \big(c(x)+\lambda\big)u = f(x)$$

in Q , where λ is a real parameter.

The results of Section 3 suggest the following concept of the Dirichlet problem.

Let $\phi \in L^2(\partial Q)$. A weak solution $u \in W^{1,2}_{\text{loc}}(Q)$ of (1') (or (1)) is a solution of the Dirichlet problem with the boundary condition

$$(13) \qquad u(x) = \phi(x) \quad \text{on} \quad \partial Q$$

if

$$\lim_{\delta\to 0}\int_{\partial Q}\left[u(x_\delta(x))-\phi(x)\right]^2 dS_x = 0 .$$

Under stronger assumptions the above form of the Dirichlet problem has already been considered by Nečas [12], [13] (Chapter 6), Guščin and Mikhailov [3] and Mikhailov [9]. The relation to the Dirichlet problem in $W^{1,2}(Q)$ will be discussed in the next section.

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

We now establish the following energy estimate.

THEOREM 5. *Let* $u \in W^{1,2}_{loc}(Q)$ *be a solution of the Dirichlet problem* (1'), (13). *Then there exist positive constants* d, λ_0 *and* C *independent of* u , *such that*

$$(14)\quad \int_Q |Du(x)|^2 r(x)dx + \sup_{0\le\delta\le d} M(\delta) + \int_{Q_d} u(x)^2 dx \le C\left(\int_{\partial Q} \phi(x)^2 dS + \int_Q f(x)^2 r(x)^\theta dx\right)$$

for $\lambda \ge \lambda_0$.

Proof. Note that if $u \in W^{1,2}_{loc}(Q)$ is a solution of the Dirichlet problem (1'), (13), then the conditions I, II, III hold. Let v be the test function introduced in the proof of Theorem 1. Thus we have

$$(15)\quad \tfrac{1}{2}\int_{\partial Q_\delta} \sum_{i,j=1}^{n} a_{ij}D_i\rho D_j\rho u^2\Phi^2 dS$$
$$= \int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij}D_iuD_ju(\rho-\delta)\Phi^2 dx - \tfrac{1}{2}\int_{Q_\delta} \sum_{i,j=1}^{n} D_i\left(a_{ij}D_j\rho\Phi^2\right)u^2 dx$$
$$+ 2\int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij}D_iu\cdot u(\rho-\delta)\Phi D_j\Phi dx + \int_{Q_\delta} \sum_{i=1}^{n} b_iD_iu\cdot u(\rho-\delta)\Phi^2 dx$$
$$+ \int_{Q_\delta} (c+\lambda)u^2(\rho-\delta)\Phi^2 dx - \int_{Q_\delta} fu(\rho-\delta)\Phi^2 dx .$$

Since we may assume that $\delta_1 \le 1$, it is clear from the proof of I ⇒ II (Theorem 1) that

$$(16)\quad \int_{\partial Q_\delta} u^2 dS_x \le L_1\left[\int_{Q_\delta} |Du|^2(\rho-\delta)\Phi^2 dx + \int_Q u^2\Phi^2 dx\right.$$
$$+ \int_Q u^2|D\Phi|^2 dx + \int_{Q_\delta} u^2(\rho-\delta)|D\Phi|^2 dx + \lambda\int_{Q_\delta} u^2(\rho-\delta)\Phi^2 dx$$
$$\left.+ \int_{Q_\delta} u^2(\rho-\delta)^{-\beta}\Phi^2 dx + \int_{Q_\delta} f^2(\rho-\delta)^\theta\Phi^2 dx\right] ,$$

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

for all $\delta \in (0, \delta_0/2)$, where L_1 is a positive constant independent of δ, $\beta = \min(\alpha, 1-2\varepsilon, 1-2\varepsilon_1)$, $\alpha = 1 - \theta$; the constants $0 < \varepsilon < 1$ and $0 < \varepsilon_1 < 1$ were introduced in the proof of I $\Rightarrow$ II (Theorem 1). On the other hand it follows from (15) and assumption (A) that

$$\begin{aligned}(17)\quad \gamma^{-1}\int_{Q_\delta} |Du|^2(\rho-\delta)\Phi^2 dx + \lambda \int_{Q_\delta} u^2(\rho-\delta)\Phi^2 dx \\ \le \tfrac{1}{2}\int_{\partial Q_\delta} \sum_{i,j=1}^{n} a_{ij} D_i\rho D_j\rho u^2\Phi^2 dS + \tfrac{1}{2}\int_{Q} \sum_{i,j=1}^{n} D_i\left(a_{ij} D_j\rho\Phi^2\right) u^2 dx \\ - \int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij} D_i u \cdot u(\rho-\delta)\Phi D_j\Phi dx - \int_{Q_\delta} \sum_{i=1}^{n} b_i D_i u \cdot u(\rho-\delta)\Phi^2 dx \\ - \int_{Q_\delta} cu^2(\rho-\delta)\Phi^2 dx + \int_{Q_\delta} fu(\rho-\delta)\Phi^2 dx .\end{aligned}$$

Denote the fourth and fifth integrals on the right hand side of the last inequality by I_1 and I_2. As in the proof of Theorem 1 we have

$$\begin{aligned}(18)\quad |I_1| \le L_2\delta_1^{(1-\varepsilon s_1)/s_1}\Big[\int_{Q_\delta} |Du|^2(\rho-\delta)\Phi^2 dx + \int_{Q_\delta} u^2(\rho-\delta)^{2\varepsilon-1}\Phi^2 dx \\ + \int_{Q_\delta} u^2(\rho-\delta)|D\Phi|^2 dx\Big] ,\end{aligned}$$

$$\begin{aligned}(19)\quad |I_2| \le L_3\delta_1^{(1-2\varepsilon_1\delta_1)/r_1}\Big[\int_{Q_\delta} |Du|^2(\rho-\delta)\Phi^2 dx + \int_{Q_\delta} u^2(\rho-\delta)^{2\varepsilon_1-1}\Phi^2 dx \\ + \int_{Q_\delta} u^2(\rho-\delta)|D\Phi|^2 dx\Big] .\end{aligned}$$

Choose δ_1 such that

$$L_2\delta_1^{(1-2\varepsilon s_1)/s_1} + L_3\delta_1^{(1-2\varepsilon_1 s_1)/s_1} \le \frac{\gamma^{-1}}{4} .$$

Consequently we deduce from (17), (18) and (19) that

$$(20)\quad \int_{Q_\delta} |Du|^2(\rho-\delta)\Phi^2dx + \lambda\int_{Q_\delta} u^2(\rho-\delta)\Phi^2dx$$
$$\le L_4\Big[M(\delta) + \int_Q u^2\Phi^2dx + \int_{Q_\delta} u^2(\rho-\delta)|D\Phi|^2dx + \int_Q u^2|D\Phi|^2dx$$
$$+ \int_{Q_\delta} u^2(\rho-\delta)^{-\beta}\Phi^2dx + \int_{Q_\delta} f^2(\rho-\delta)^{\theta}\Phi^2dx\Big]$$

for all $\delta \in (0, \delta_1/2]$, where L_4 is a positive constant. Letting δ tend to zero we therefore obtain

$$(21)\quad \int_Q |Du|^2\rho(x)\Phi^2dx + \lambda\int_Q u^2\rho^2\Phi^2dx$$
$$\le L_4\Big[\int_{\partial Q}\phi(x)^2dS + \int_Q u^2\Phi^2dx + \int_Q u^2\rho|D\Phi|^2dx + \int_Q u^2|D\Phi|^2dx$$
$$+ \int_Q u^2\rho^{-\beta}\Phi^2dx + \int_Q f^2\rho^{\theta}\Phi^2dx\Big].$$

Now let Ψ be a smooth function on $\bar{Q}$ such that $\Psi = 1$ on Q_{δ_2}, $\Psi = 0$ on $Q - Q_{\delta_3}$, $0 \le \Psi \le 1$ on $\bar{Q}$, where $0 < \delta_3 < \delta_2 \le \delta_0$. Taking $v = u\Psi^2$ as a test function by a standard argument, we deduce that

$$(22)\quad \int_Q |Du|^2\Psi^2dx + \lambda\int_Q u^2\Psi^2dx$$
$$\le L_5\Big[\int_Q u^2\Psi^2dx + \int_Q u^2|D\Psi|^2dx + \int_Q f^2\Psi^2dx\Big]$$

Since

$$\int_Q |Du|^2\Psi^2dx \ge \int_{Q_{\delta_2}} |Du|^2dx \ge \frac{1}{\delta^*}\int_{Q_{\delta_2}} |Du|^2\rho(x)dx\ ,$$

where $\delta^* = \sup_{\bar{Q}}\rho(x)$, by choosing $\delta_2 = \delta_1$ and using (21) and (22) we obtain

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

$$(23)\quad \int_Q |Du|^2\rho(x)dx + \lambda\int_Q u^2\rho dx$$
$$\le L_6\Big[\int_{\partial Q}\phi(x)^2dx + \int_Q u^2dx + \int_Q u^2|D\Psi|^2dx + \int_Q u^2|D\Phi|^2dx$$
$$+ \int_Q u^2\rho^{-\beta}dx + \int_Q f^2\rho^\theta dx + \int_Q f^2\Psi^2dx\Big] .$$

Thus combining (16) and (23) we obtain

$$(24)\quad \int_{\partial Q_\delta} u^2dS_x$$
$$\le L_7\Big[\int_{\partial Q}\phi^2dx + \int_Q u^2dx + \int_Q u^2|D\Phi|^2dx + \int_Q u^2|D\Psi|^2dx$$
$$+ \int_{Q_\delta} u^2(\rho-\delta)^{-\beta}dx + \int_Q u^2\rho^{-\beta}dx + \int_Q f^2\rho^\theta dx + \int_Q f^2\Psi^2dx\Big]$$

for all $\delta \in [0, \delta_1/2]$, where L_7 is a positive constant. To proceed further we derive from Lemma 1 that

$$(25)\quad \int_{Q_\delta} u^2(\rho-\delta)^{-\beta}dx$$
$$\le K\Big[\delta_1^{-\beta}\int_{Q_{\delta_1}} u^2dx+\delta_1^{1-\beta}\int_{\partial Q_{\delta_1}} u^2dS+\delta_1^{1-\beta}\int_{Q_\delta-Q_{\delta_1}} |Du|^2(\rho-\delta)dx\Big] .$$

On the other hand it follows from Theorem 1 in [11], p. 138, that

$$(26)\quad \int_{\partial Q_{\delta_1}} u^2dS_x \le \tilde{C}\|u\|^2_{W^{1,2}(Q_{\delta_1})} ,$$

where $\tilde{C}$ is an absolute constant independent of δ_1 . Combining (22) and (26) we obtain

$$(27)\quad \int_{\partial Q_{\delta_1}} u^2dS_x \le \tilde{C}_1\Big[\int_Q u^2dx + \int_Q u^2|D_x\Psi|^2dx + \int_Q f^2\Psi^2dx\Big] .$$

Choosing δ_1 such that $L_7K\delta_1^{1-\beta} < 1$, the estimates (23), (24), (25) and (27) imply

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

$$(28)\quad \int_Q |Du|^2\rho dx + \lambda \int_Q u^2\rho dx + M(\delta)$$
$$\le L_8\left[\int_{\partial Q} \Phi^2 dx + \int_Q u^2 dx + \int_Q u^2|D\Phi|^2 dx + \int_Q u^2|D\Psi|^2 dx + \int_Q u^2\rho^{-\beta}dx + \int_Q f^2\rho^{\theta}dx + \int_Q f^2\Psi^2 dx\right]$$

for $\delta \in (0, \delta_1/2]$, where L_8 is a positive constant. We can now easily complete the proof of the theorem. Indeed, note that

$$(29)\qquad \int_Q u^2 dx \le d \sup_{0\le\delta\le d} M(\delta) + \int_{Q_d} u^2 dx ,$$

and

$$(30)\qquad \int_Q u^2\rho^{-\beta}dx \le \frac{d^{1-\beta}}{1-\beta} \sup_{0\le\delta\le d} M(\delta) + \int_{Q_d} u^2\rho^{-\beta}dx$$

for $d \le \delta_0$. Taking d sufficiently small and λ sufficiently large, say $\lambda \ge \lambda_0$, the result follows from (28), (29) and (30).

We mention here that Guščin and Mikhailov [3], [9], also proved the energy estimate. The proof presented here is entirely different.

5. Application to the Dirichlet problem

In this section we study the solvability of the Dirichlet problem with boundary data in $L^2(\partial Q)$. The definition of the Dirichlet problem in $W^{1,2}_{\mathrm{loc}}(Q)$ of (1'), (13) has already been introduced in the previous section.

Recall that a function $u \in W^{1,2}(Q)$ is a solution of the Dirichlet problem (1), (13) [or (1'), (13)] if u is a weak solution of (1) [or (1')] and $u - \phi \in \overset{\circ}{W}{}^{1,2}(Q)$.

In this definition it is assumed that boundary data ϕ can be extended to an element of $W^{1,2}(Q)$ which is also denoted by ϕ . As we pointed out in the Introduction this is a restrictive assumption because not every function in $L^2(\partial Q)$ is the trace of an element in $W^{1,2}(Q)$.

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

Let us introduce the Hilbert space $\tilde{W}^{1,2}(Q)$ of all functions $u(x)$ in $W^{1,2}_{\mathrm{loc}}(Q)$ such that

$$\|u\|^2_{\tilde{W}^{1,2}(Q)} = \int_Q u(x)^2 dx + \int_Q |Du(x)|^2 r(x) dx < \infty .$$

It is evident that

$$(31) \quad \|u\|^2_{\tilde{W}^{1,2}} \le \max(1,\ d)\left[\int_Q |Du(x)|^2 r(x) dx + \int_{Q_d} u(x)^2 dx + \sup_{0\le\delta\le d} M(\delta)\right] .$$

THEOREM 6. *Let* $\lambda \ge \lambda_0$. *Then for every* $\phi \in L^2(\partial Q)$ *there is a unique solution of the Dirichlet problem* (1'), (13) *in* $W^{1,2}_{\mathrm{loc}}(Q)$.

Proof. Let $\{\phi_m\}$ be a sequence of functions in $L^1(\partial Q)$ converging in $L^2(\partial Q)$ to the function ϕ .

Define

$$f_m(x) = \begin{cases} f(x) & \text{for } x \in Q_{1/m} , \\ 0 & \text{for } x \in Q - Q_{1/m} , \end{cases}$$

for m such that $1/m \le \delta_0$. Let u_m be a solution of the Dirichlet problem

$$Lu + \lambda u = f_m \quad \text{in} \quad Q ,$$

$$u = \phi_m \quad \text{on} \quad \partial Q ,$$

in $W^{1,2}(Q)$ (see [*15*], [*16*] or [5]). Here we may assume that λ_0 is sufficiently large that the theorems on the existence of solutions in $W^{1,2}(Q)$ are applicable. It is obvious that inequality (14) is valid for u_m and

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

$$\int_Q |Du_p-Du_q|^2 r(x)dx + \sup_{0\le\delta\le d} \int_{\partial Q_\delta} (u_p-u_q)^2 dS_x + \int_{Q_d} (u_p-u_q)^2 dx$$
$$\le C\left(\int_Q (f_p-f_q)^2 r^\theta dx + \int_{\partial Q} (\phi_p-\phi_q)^2 dS_x\right) .$$

It follows from (31) that $\lim_{m\to\infty} u_m = u$ exists in $\tilde{W}^{1,2}(Q)$. As in [9] one can show that u satisfies the boundary condition (13) in $L^2(\partial Q)$.

THEOREM 7. *Suppose that* $\int_Q f(x)^2 r(x)^2 dx < \infty$ *and let* $\phi \in L^2(\partial Q)$. *If there is a function* ϕ_1 *in* $W^{1,2}(Q)$ *such that* $\phi_1 = \phi$ *on* ∂Q *(in the sense of trace), then a solution* $u(x)$ *in* $W^{1,2}_{loc}(Q)$ *of the Dirichlet problem* (1), (13) *is a solution in* $W^{1,2}(Q)$ *of the same problem.*

The proof is essentially the same as that of Lemma 3 in [9] and therefore is omitted.

Theorem 7 implies that every solution in $W^{1,2}_{loc}(Q)$ of the problem

$$(32) \qquad \begin{cases} Lu = 0 & \text{on } Q , \\ u = 0 & \text{in } \partial Q , \end{cases}$$

or of the problem

$$(32^*) \qquad \begin{cases} L^*u = 0 & \text{on } \partial Q , \\ u = 0 & \text{in } \partial Q , \end{cases}$$

is a solution in $W^{1,2}(Q)$. Consequently the problem (32) or (32*) can have only a finite number of linearly independent solutions (see [5], p. 156, or [15], [16]).

It is known that for the Dirichlet problem (1), (13) to have a solution $W^{1,2}(Q)$ (we assume that the boundary data is the trace of an element in $W^{1,2}(Q)$), it is necessary and sufficient that f and ϕ be related as follows:

$$(33) \qquad \int_{\partial Q} \frac{\partial u_j^*}{\partial N} \phi dS_x = \int_Q u_j^* f dx \quad (j = 1, \ldots, p) \ ,$$

where $\{u_j^*\}$ is a maximal linearly independent system of solutions of the problem (32*) and $\partial/\partial N$ denote the conormal derivative of the elliptic operator (1). If $a_{ij}, b_i \in C^1(Q)$ then $u_j^* \in W^{2,2}(Q)$ and Mikhailov [9] was able to show that the Dirichlet problem (1), (13) in $W^{1,2}_{\text{loc}}(Q)$ has a solution if and only if (33) holds. Nevertheless we can show in our situation that the Dirichlet problems in $W^{1,2}(Q)$ and $W^{1,2}_{\text{loc}}(Q)$ have the same eigenvalues.

Fix λ, and consider the Dirichlet problem

$$(34) \qquad Lu + (\lambda_0+\lambda)u = f \quad \text{on} \quad Q \ ,$$

$$(35) \qquad u = \phi \quad \text{in} \quad \partial Q \ ,$$

where $\phi \in L^2(\partial Q)$ and $\int_Q f(x)^2 r(x)^2 dx < \infty$.

In view of Theorem 6 the Dirichlet problem (34), (35) has a unique solution u_0 in $W^{1,2}_{\text{loc}}(Q)$ provided λ_0 is sufficiently large.

THEOREM 8. *Let* $\phi \in L^2(\partial Q)$, $\int_Q f(x)^2 r(x)^2 dx < \infty$. *Then the Dirichlet problem* (1'), (13) *has a solution in* $W^{1,2}_{\text{loc}}(Q)$ *if and only if* $\lambda_0 u_0$ *is orthogonal to every eigenfunction of the problem*

$$(36) \qquad L^*u + \lambda u = 0 \quad \text{in} \quad Q \ , \quad u = 0 \quad \text{on} \quad \partial Q \ .$$

Proof. Suppose that u is a solution of the problem (1'), (13) in $W^{1,2}_{\text{loc}}(Q)$. Put

$$(37) \qquad w = u - u_0 \ .$$

Then w is a solution in $W^{1,2}_{\text{loc}}(Q)$ of the problem

$$(38) \qquad Lw + \lambda w = \lambda_0 u_0 \quad \text{in} \quad Q \ , \quad w = 0 \quad \text{on} \quad \partial Q \ .$$

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

It follows from Theorem 7 that $w \in \overset{\circ}{W}^{1,2}(Q)$ and consequently $\lambda_0 u_0$ is orthogonal to every eigenfunction of the problem (36).

Conversely if $\lambda_0 u_0$ is orthogonal to every eigenfunction of the problem (36) then the problem (38) has a unique solution in $W^{1,2}(Q)$ and the solution of the Dirichlet problem (1'), (13) is given by the formula (37).

Observe that the solution of the Dirichlet problem in $W^{1,2}_{\text{loc}}(Q)$ belongs to $\tilde{W}^{1,2}(Q)$.

THEOREM 9. *Suppose that* $f \in L^q(Q)$ *for some* $q > n/2$ *and* c *is bounded away from zero on* Q . *Then for every* $\phi \in C(\partial Q)$ *there exists a unique solution in* $\tilde{W}^{1,2}(\bar{Q}) \cap C(\bar{Q})$ *of the Dirichlet problem* (1), (13).

Proof. It follows from Theorem 3.8 in [15] that the Dirichlet problem $Lu = 0$ in Q , $u = 0$ on ∂Q has only the trivial solution in $\overset{\circ}{W}^{1,2}(Q)$. Consequently by Theorem 8 there exists a unique solution $u \in \tilde{W}^{1,2}(Q)$ of the problem (1), (13). It remains only to show that $u \in C(\bar{Q})$. Let $\{\phi_m\}$ be a sequence in $C^1(\bar{Q})$ converging uniformly to ϕ on ∂Q . By virtue of Theorem 14.1 in [5] the solution of the Dirichlet problem

$$Lu + \lambda u = f \quad \text{in} \quad Q \ ,$$

$$u = \phi_m \quad \text{on} \quad \partial Q \ ,$$

belongs to $W^{1,2}(Q) \cap C(\bar{Q})$. By Theorem 3.8 in [15],

$$\sup_Q |u_p(x)-u_q(x)| \le \sup_Q |\phi_p(x)-\phi_q(x)| \to 0$$

as $p, q \to \infty$ and the result follows.

This theorem is similar to, but slightly sharper than, Theorem 8.30 in [2] (p. 196).

https://doi.org/10.1017/S0004972700011461 Published online by Cambridge University Press

References

[1] J. Chabrowski and B. Thompson, "On traces of solutions of a semi-linear partial differential equation of elliptic type", *Ann. Polon. Math.* (to appear).

[2] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order* (Die Grundlehren der mathematischen Wissenschaften, 224. Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[3] А.К. Гущин, В.П. Михайлов [A.K. Guščin, V.P. Mikaĭlov], "О граничных значениях в L_p, $p > 1$, решений эллиптических уравнений" [Boundary values in L_p $(p > 1)$ of solutions of elliptic equations], *Mat. Sb.* 108 (150) (1979), 3-21.

[4] Alois Kufner, Oldřich John and Svatopluk Fučík, *Function spaces* (Noordhoff, Leyden; Academia, Prague, 1977).

[5] Olga A. Ladyzhenskaya and Nina N. Ural'tseva, *Linear and quasilinear elliptic equations* (Mathematics in Science and Engineering, 46. Academic Press, New York and London, 1968).

[6] J.E. Littlewood and R.E.A.C. Paley, "Theorems on Fourier series and power series (II)", *Proc. London Math. Soc.* (2) 42 (1937), 52-89.

[7] В.П. Михайлов [V.P. Mikaĭlov], "О граничных значениях решений эллиптических уравнений второго порядка" [The boundary values of the solutions of second order elliptic equations], *Mat. Sb.* 100 (142) (1976), 5-13.

[8] В.П. Михайлов [V.P. Mikaĭlov], "О граничных значениях решений эллиптических уравнений в областях с глалкой границей" [Boundary values of the solutions of elliptic equations in domains with a smooth boundary], *Mat. Sb.* 101 (143) (1976), 163-188.

[9] В.П. Михайлов [V.P. Mikaĭlov], "О задаче Дирихле для эллиптического уравнения второго порядка" [The Dirichlet problem for a second order elliptic equation], *Differential'nye Uravnenija* 12 (1976), 1877-1891.

[10] V.P. Mikhailov, "On the boundary values of the solutions of elliptic equations", *Appl. Math. Optim.* 6 (1980), 193-199.

[11] В.П. Михайлов [V.P. Mikaĭlov], *Дифференциальные уравнения в частных производных* [*Partial differential equations*] (Izdat. "Nauka", Moscow, 1976).

[12] Jindřich Nečas, "On the regularity of second-order elliptic partial differential equations with unbounded Dirichlet integral", *Arch. Rational Mech. Anal.* 9 (1962), 134-144.

[13] Jindřich Nečas, *Les méthodes directes en théorie des équations elliptiques* (Masson, Paris; Academia, Prague; 1967).

[14] Friedrich Riesz, "Über die Randwerte einer analytischen Funktion", *Math. Z.* 18 (1923), 87-95.

[15] Guido Stampacchia, "Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus", *Ann. Inst. Fourier (Grenoble)* 15 (1965), 189-258.

[16] Guido Stampacchia, *Équations elliptiques du second ordre à coefficients discontinus* (Séminaire de Mathématiques Supérieures, 16; Été, 1965. Les Presses de l'Université de Montréal, Montreal, Quebec, 1966).

Department of Mathematics,
University of Queensland,
St Lucia,
Queensland 4067,
Australia.