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RICE THEOREMS FOR D.R.E. SETS

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1. Introduction. Two of the basic theorems in the classification of index sets of classes of recursively enumerable (r.e.) sets are the following:

(i) The index set of a class C of r.e. sets is recursive if and only if C is empty or contains all r.e. sets; and

(ii) the index set of a class C or r.e. sets is recursively enumerable if and only if C is empty or consists of all r.e. sets which extend some element of a canonically enumerable class of finite sets.

The first theorem is due to Rice [**7**, p. 364, Corollary B]. The second was conjectured by Rice [**7**, p. 361] and proved independently by McNaughton, Shapiro, and Myhill [**6**]. (A proof of both (i) and (ii) is given in [**9**, p. 324, Theorem XIV].) In this paper we consider the corresponding classification problem in the case where C is a class of sets which are *differences of r.e. sets* (d.r.e. sets). The main results are the following:

(i) The index set of a class C of d.r.e. sets is recursively enumerable if and only if C is empty or contains all d.r.e. sets;

(ii) the index set of a class C of d.r.e. sets is d.r.e. if and only if C is empty or consists of all d.r.e. sets which extend a *single* finite set.

In addition, a complete classification is given for the index sets of classes C of d.r.e. sets which consist of all d.r.e. sets which extend some element of a finite class of finite sets; these turn out to have maximum 1-degree at alternate levels of the difference hierarchy generated by the r.e. sets (i.e., the levels Σ_{2n}^{-1} of the hierarchy developed in [**2**]).

2. Notation. The basic recursion-theoretic notation will be that of [**9**]. $\{W_x\}_{x\geqslant 0}$ denotes a standard enumeration of all r.e. sets. K denotes the complete r.e. set $\{x|x \in W_x\}$. N denotes the set of natural numbers and 2^N the class of all subsets of N. $\mathscr{E} \subseteq 2^N$ denotes the class of all r.e. sets and $\mathscr{D}$ the class of all d.r.e. sets. If $C \subseteq \mathscr{E}$, the index set of C is $\{x|W_x \in C\}$ and is denoted by θC. A set $A \subseteq N$ is called non-trivial if $A \neq \emptyset$ or N. $\langle x, y\rangle$ denotes a standard recursive pairing function $N \times N \rightarrow N$ with recursive inverse functions $\pi_1(z)$, $\pi_2(z)$, so that $z = \langle \pi_1(z), \pi_2(z)\rangle$ for all $z \in N$. If $A, B \subseteq N$, $\bar{A}$ denotes $N - A$, $|A|$ denotes the cardinality of A, and $A \times B = \{\langle x, y\rangle | x \in A \,\&\, y \in B\}$. $\leqq_1$ and $\leqq_T$ denote $1 - 1$ reducibility and Turing reducibility respectively, and $\equiv$, $\equiv_T$ denote the corresponding equivalences. If $C \subseteq \mathscr{D}$, $\bar{C}$ denotes $\mathscr{D} - C$. $K' = \{x|x \in W_x^K\}$ denotes the complete Σ_2^0 set. $\{D_u\}_{u\geqslant 0}$ denotes the canonical indexing of finite sets.

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3. Index sets of d.r.e. sets. A set $A \subseteq N$ is d.r.e. if and only if $A = B - C$ for some r.e. sets B, C. It is natural to associate with A the set $\{\langle x, y\rangle | A = W_x - W_y\}$. This yields an enumeration $\{V_x\}_{x \geqslant 0}$ of all d.r.e. sets, where V_x is defined by: $V_x = W_{\pi_1(x)} - W_{\pi_2(x)}$. It is easily verified that this enumeration satisfies the usual recursion-theoretic properties; more precisely, call a d.r.e. set A *d.r.e.-complete* if $B \leqq_1 A$ for every d.r.e. set B, and *d.r.e.-creative* if $\bar{A}$ is productive with respect to d.r.e. subsets. The following then holds:

THEOREM 3.1. (a) $\{x | x \in V_x\}$ *is a d.r.e.-complete set.*

(b) *A set is d.r.e.-creative if and only if it is d.r.e.-complete.*

(c) (*Fixed point property*) *If f is any recursive function, there is a number e such that $V_e = V_{f(e)}$.*

Proof. (a) and (b) are proved in [**2**, p. 28, Theorem 1 and p. 35, Theorem 4]. (c) is a simple consequence of the Smullyan double recursion theorem [**9**, p. 190, Theorem $X(a)$], which implies that for any recursive function f there exist e_1, e_2 such that $W_{\pi_1 f(\langle e_1, e_2\rangle)} = W_{e_1}$ and $W_{\pi_2 f(\langle e_1, e_2\rangle)} = W_{e_2}$, so that

$$V_{\langle e_1, e_2\rangle} = W_{e_1} - W_{e_2} = W_{\pi_1 f(\langle e_1, e_2\rangle)} - W_{\pi_2 f(\langle e_1, e_2\rangle)} = V_{f(\langle e_1, e_2\rangle)}.$$

It follows from Theorem 3.1 that there are d.r.e. sets whose complements are not d.r.e., and which therefore are neither r.e. nor co-r.e. It has in fact been shown by Cooper [**1**] that there are d.r.e. sets which are not Turing-equivalent to any r.e. set. However, the familiar properties of the enumeration $\{V_x\}_{x \geqslant 0}$ suggest that index sets for classes of d.r.e. sets may have properties analogous to those of index sets of classes of r.e. sets, and it will be seen below that this is indeed the case, using the obvious definition:

Definition. If C is a class of d.r.e. sets, the *index set* of C is $\{x | V_x \in C\}$, and is denoted by δC.

For our purposes, it will be useful to have the following characterization of the complete d.r.e .set:

PROPOSITION 3.2. (a) $K \times \bar{K}$ *is a complete d.r.e. set.*

(b) $\overline{K \times \bar{K}} \nleqq_1 K \times \bar{K}$.

(*c*) *If A is d.r.e., then* $K \times \bar{K} \nleqq_1 \bar{A}$.

Proof. (a) It is evident that $K \times \bar{K} = (K \times N) - (K \times K)$ is d.r.e. To show completeness, let $A = W_a - W_b$ be any d.r.e. set. Define recursive functions $g(x)$, $h(x)$ as the indices of r.e. sets generated as follows:

$$W_{g(x)} = \begin{cases} N & \text{if } x \in W_a, \\ \emptyset & \text{otherwise;} \end{cases}$$

$$W_{h(x)} = \begin{cases} N & \text{if } x \in W_b, \\ \emptyset & \text{otherwise.} \end{cases}$$

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Let $f(x) = \langle g(x), h(x) \rangle$. Then f is $1 - 1$, and

$$\begin{aligned} x \in W_a - W_b &\Rightarrow W_{g(x)} = N \;\&\; W_{h(x)} = \emptyset \\ &\Rightarrow g(x) \in W_{g(x)} \;\&\; h(x) \notin W_{h(x)} \\ &\Rightarrow f(x) = \langle g(x), h(x) \rangle \in K \times \bar{K}; \end{aligned}$$

while

$$\begin{aligned} x \notin W_a - W_b &\Rightarrow W_{g(x)} = \emptyset \vee W_{h(x)} = N \\ &\Rightarrow g(x) \in \bar{K} \vee h(x) \in K \\ &\Rightarrow f(x) \notin K \times \bar{K}. \end{aligned}$$

So $A \leqq_1 K \times \bar{K}$ via f.

(b) now follows from (a) and Theorem 3.1(b), since $K \times \bar{K}$ is d.r.e.-creative and hence its complement cannot be d.r.e. ((b) also follows directly from the fact that, as shown in [**4**, p. 39, Theorem 1], $K \times \bar{K}$ is recursively isomorphic to an index set θA and hence cannot be $1 - 1$ reducible to its complement.)

For (c), assume A is d.r.e. Then by (a), $A \leqq_1 K \times \bar{K}$. If $K \times \bar{K} \leqq_1 \bar{A}$, then $\overline{K \times \bar{K}} \leqq_1 A$ which implies $\overline{K \times \bar{K}} \leqq_1 K \times \bar{K}$, contradicting (b); hence $K \times \bar{K} \nleqq_1 \bar{A}$.

4. The first Rice theorem for d.r.e. sets. An examination of the proof of Rice's first theorem [**9**, p. 324, Theorem XIV(a)] shows that it can be given the following more precise form:

THEOREM R-1. *If C is a class of r.e. sets such that θC is non-trivial, then $K \leqq_1 \theta C$ or $K \leqq_1 \overline{\theta C}$.*

COROLLARY. *If C is a class of r.e. sets such that θC is r.e., then either θC is trivial or $\theta C \equiv K$.*

The analogous result for classes of d.r.e. sets is the following:

THEOREM 4.1. *If C is a class of d.r.e. sets such that δC is non-trivial, then $K \times \bar{K} \leqq_1 \delta C$ or $\bar{K} \times K \leqq_1 \overline{\delta C}$.*

Proof. The theorem will be an immediate consequence of the following lemma:

LEMMA 4.2. *Let C be a class of d.r.e. sets. If there are d.r.e. sets X, Y such that X is finite, $X \subseteq Y$, $X \in C$ and $Y \notin C$, then $K \times \bar{K} \leqq_1 \overline{\delta C}$.*

Proof. Assume there exist d.r.e. sets X, Y satisfying the hypothesis. Then $X = D_u$ for some u, while $Y = W_a - W_b$ for some a, b. Since $X \in C$ but $Y \notin C$, it follows that $X \subsetneq Y$ so that $D_u \subsetneq W_a - W_b$. Hence $D_u \subseteq W_a$ and $W_b \subseteq W_a - D_u$, and we can define recursive functions $g(x)$, $h(x)$ as the

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indices of r.e. sets generated as follows:

$$W_{g(x)} = \begin{cases} D_u, & \text{if } \pi_1(x) \notin K \text{ and } \pi_2(x) \notin K, \\ W_a, & \text{otherwise;} \end{cases}$$

$$W_{h(x)} = \begin{cases} W_b, & \text{if } \pi_2(x) \notin K, \\ W_a - D_u, & \text{otherwise.} \end{cases}$$

Let $f(x) = \langle g(x), h(x) \rangle$. Then f is $1 - 1$ and

$$\begin{aligned} x \in K \times \bar{K} &\Rightarrow \pi_1(x) \in K \;\&\; \pi_2(x) \notin K \\ &\Rightarrow W_{g(x)} = W_a \;\&\; W_{h(x)} = W_b \\ &\Rightarrow V_{f(x)} = W_{g(x)} - W_{h(x)} = W_a - W_b = Y \notin C; \end{aligned}$$

while

$$\begin{aligned} x \notin K \times \bar{K} &\Rightarrow \pi_1(x) \notin K \;\vee\; \pi_2(x) \in K \\ &\Rightarrow (W_{g(x)} = D_u \;\&\; W_{h(x)} = W_b) \\ &\qquad\qquad \vee\; (W_{g(x)} = W_a \;\&\; W_{h(x)} = W_a - D_u) \\ &\Rightarrow V_{f(x)} = W_{g(x)} - W_{h(x)} = D_u \in C. \end{aligned}$$

So $x \in K \times \bar{K} \Leftrightarrow f(x) \in \overline{\delta C}$, and $K \times \bar{K} \leqq_1 \overline{\delta C}$ via f.

Theorem 4.1 now follows since if δC is non-trivial, then $\delta C \neq \emptyset$ and $\overline{\delta C} \neq \emptyset$. Suppose $\emptyset \in C$; then for some $Y \supseteq \emptyset$, $Y \notin C$ and by Lemma 4.2, $K \times \bar{K} \leqq_1 \overline{\delta C}$. If on the other hand $\emptyset \notin C$, then $\emptyset \in \bar{C}$ and, by symmetry, $K \times \bar{K} \leqq_1 \overline{\delta \bar{C}} = \delta C$.

Corollary 4.3. *If C is a class of d.r.e. sets such that δC is d.r.e., then either δC is trivial or $\delta C \equiv K \times \bar{K}$.*

Proof. Suppose δC is d.r.e.; then by Proposition 3.2, $\delta C \leqq_1 K \times \bar{K}$ and $K \times \bar{K} \leqq_1 \overline{\delta C}$. If δC is non-trivial, then by Theorem 4.1, $K \times \bar{K} \leqq_1 \delta C$ or $K \times \bar{K} \leqq_1 \overline{\delta C}$; since the latter cannot hold, it follows that $\delta C \equiv K \times \bar{K}$.

Corollary 4.4. *The index set of a class C of d.r.e. sets is recursively enumerable if and only if C is empty or contains all d.r.e. sets.*

Proof. If C is empty or contains all d.r.e. sets, then $\delta C = \emptyset$ or N and is hence r.e. Conversely, suppose δC is r.e.; it follows that δC is d.r.e., so by Corollary 4.3, either δC is trivial or $\delta C \equiv K \times \bar{K}$. But if $\delta C \equiv K \times \bar{K}$ then δC is not r.e., since $K \times \bar{K}$ r.e. $\Rightarrow K \times \bar{K} \leqq_1 K \Rightarrow \bar{K} \leqq_1 K$ which is impossible. So δC is r.e. only if δC is trivial.

5. The second Rice theorem for d.r.e. sets. In [**7**, p. 359] a class C of r.e. sets is called *completely recursively enumerable* (c.r.e.) if θC is r.e. A precise statement of what we shall call the "second Rice theorem" is then the following:

Theorem R-2. *A class C of r.e. sets is c.r.e. if and only if C is empty or there is a recursive function f such that $C = \{W_x | (\exists u)(D_{f(u)} \subseteq W_x)\}$.*

By analogy, call a class C of d.r.e. sets *completely d.r.e.* (c.d.r.e.) if δC is d.r.e. The following analogue to the second Rice theorem is then obtained:

THEOREM 5.1. *A class C of d.r.e. sets is c.d.r.e. if and only if C is empty or there is a single finite set D_u such that $C = \{V_x | D_u \subseteq V_x\}$.*

Proof. If C is empty then $\delta C = \emptyset$ is d.r.e. Suppose there is a finite set D_u such that $C = \{V_x | D_u \subseteq V_x\}$. Then $\delta C = \{x | D_u \subseteq W_{\pi_1(x)} - W_{\pi_2(x)}\} = \{x | D_u \subseteq W_{\pi_1(x)}\} - \{x | D_u \cap W_{\pi_2(x)} \neq \emptyset\}$ which is evidently d.r.e. Hence in either case C is c.d.r.e. The converse will follow from a sequence of lemmas.

LEMMA 5.2. *Le W_a be an infinite r.e. set. Then there is a recursive function g such that for all $x \in N$,*

$$W_x \text{ infinite} \Rightarrow W_{g(x)} = W_a,$$

$$W_x \text{ finite} \Rightarrow W_{g(x)} \text{ is a finite subset of } W_a.$$

Proof. This is left to the reader.

LEMMA 5.3. *Let C be a class of d.r.e. sets. If there is a d.r.e. set X such that $X \in C$ but no finite subset of X is in C, then $\overline{K'} \leqq_1 \delta C$.*

Proof. Suppose $X \in C$ is an infinite d.r.e. set none of whose finite subsets are in C. Let $X = W_a - W_b$; clearly W_a must be infinite. Let g be as in Lemma 5.2, and define a recursive function $f(x)$ by $f(x) = \langle g(x), b \rangle$. Then for all $x \in N$,

$$W_x \text{ infinite} \Rightarrow V_{f(x)} = W_{g(x)} - W_b = W_a - W_b = X,$$

$$W_x \text{ finite} \Rightarrow V_{f(x)} \text{ is a finite subset of } W_a - W_b = X \rightarrow V_{f(x)} \notin C.$$

Hence if $D = \{W_x | W_x \text{ is infinite}\}$, then $\theta D \leqq_1 \delta C$ via f. But it is well-known that $\theta D \equiv \overline{K'}$; it follows that $\overline{K'} \leqq_1 \delta C$.

LEMMA 5.4. *Let C be a class of d.r.e. sets. Suppose there exist finite sets D_u, $D_v \in C$* such that *$D_u \cap D_v \notin C$. Then $K \times \bar{K} \leqq_1 \overline{\delta C}$.*

Proof. Assume the hypothesis. Define recursive functions $g(x)$, $h(x)$ as follows:

$$W_{g(x)} = \begin{cases} D_u, & \text{if } \pi_1(x) \notin K \vee \pi_2(x) \notin K, \\ D_u \cup D_v, & \text{otherwise;} \end{cases}$$

$$W_{h(x)} = \begin{cases} \emptyset, & \text{if } \pi_1(x) \notin K, \\ D_u - D_v & \text{otherwise.} \end{cases}$$

Let $f(x) = \langle g(x), h(x) \rangle$. Then

$$\begin{aligned} x \in K \times \bar{K} &\Rightarrow W_{g(x)} = D_u \,\&\, W_{h(x)} = D_u - D_v \\ &\Rightarrow V_{f(x)} = W_{g(x)} - W_{h(x)} = D_u \cap D_v \notin C, \end{aligned}$$

while

$$\begin{aligned} x \notin K \times \bar{K} &\Rightarrow \pi_1(x) \notin K \ \vee \ \pi_2(x) \in K \\ &\Rightarrow (W_{g(x)} = D_u \,\&\, W_{h(x)} = \emptyset) \\ &\qquad\qquad\qquad \vee \ (W_{g(x)} = D_u \cup D_v \,\&\, W_{h(x)} = D_u - D_v) \\ &\Rightarrow V_{f(x)} = W_{g(x)} - W_{h(x)} = D_u \in C \ \vee \ V_{f(x)} = D_v \in C. \end{aligned}$$

So $x \in K \times \bar{K} \Leftrightarrow f(x) \in \overline{\delta C}$ and $K \times \bar{K} \leqq_1 \overline{\delta C}$ via f.

End of proof of Theorem 5.1. Assume C is c.d.r.e. and non-empty. We will show that there exists a finite set D_u such that $C = \{V_x | D_u \subseteq V_x\}$. Since C is c.d.r.e., δC is d.r.e. and hence by Proposition 3.2, $\delta C \leqq_1 K \times \bar{K}$ and $K \times \bar{K} \nleqq_1 \overline{\delta C}$. Let X be any set in C. If C contains no finite subset of X, then by Lemma 5.3, $\overline{K'} \leqq_1 \delta C$; but this implies $\overline{K'} \leqq_1 K \times \bar{K}$, which is impossible since $K \times \bar{K} \leqq_T K <_T K'$. So $X \in C$ only if some finite subset of X is in C; it follows that $\{v | D_v \in C\} \neq \emptyset$. Let D_u be a finite set in C of minimum cardinality. We claim $C = \{V_x | D_u \subseteq V_x\}$. Suppose $V_x \notin C$ for some $V_x \supseteq D_u$. Then since $D_u \in C$, it follows by Lemma 4.2 that $K \times \bar{K} \leqq_1 \overline{\delta C}$, which is a contradiction; hence $\{V_x | V_x \supseteq D_u\} \subseteq C$. Now suppose $V_x \in C$ but $D_u \nsubseteq V_x$. It was shown above that $V_x \in C$ only if $V_x \supseteq D_v$ for some $D_v \in C$, and $D_u \nsubseteq V_x$ implies $D_u \nsubseteq D_v$. Then $D_u \cap D_v \subsetneq D_u$ which implies $|D_u \cap D_v| < |D_u|$ and hence $D_u \cap D_v \notin C$ since D_u has minimum cardinality. But then by Lemma 5.4, $K \times \bar{K} \leqq_1 \overline{\delta C}$ which again gives a contradiction. Hence $V_x \in C \Rightarrow D_u \subseteq V_x$ and $C = \{V_x | D_u \subseteq V_x\}$.

We note that Theorems R-2 and 5.1 can be given a topological interpretation by means of the "inclusion topology" on 2^N [**9**, p. 217, Ex. 11-35], as follows:

Definition. Let $A \subseteq 2^N$. Then
(a) A is a *basic open class* $\Leftrightarrow A = \{X | D_u \subseteq X\}$ for some D_u.
(b) A is an *open class* $\Leftrightarrow A$ is a union of basic open classes.

It follows that if we let A_u denote $\{X | D_u \subseteq X\}$, then A is an open class $\Leftrightarrow A = \bigcup_{u \in Z} A_u$ for some set $Z \subseteq N$. Clearly A is a basic open class if and only if Z can be chosen to be a singleton. A is an *r.e. open class* if and only if Z can be chosen to be r.e. If $A \subseteq \mathscr{E}$ ($A \subseteq \mathscr{D}$) we call A a basic open class or an open class if $A = B \cap \mathscr{E}$ ($B \cap \mathscr{D}$) where B is a basic open class or an open class in 2^N, respectively, and similarly for r.e. open classes. Theorems R-2 and 5.1 then take the following form:

THEOREM R-2 (restated). *Let $C \subseteq \mathscr{E}$. Then δC is recursively enumerable if and only if C is an r.e. open class.*

THEOREM 5.1 (restated). *Let $C \subseteq \mathscr{D}$. Then δC is d.r.e. if and only if C is a basic open class.*

The following question suggests itself: What can be said about δC if $C =$

$\bigcup_{u \in Z} A_u$ when Z is finite but $|Z| > 1$? This question is answered in the next section.

6. Index sets of finite unions of basic open classes of d.r.e. sets. Following [**8**, p. 306] we introduce the notion of the "core" of a class of d.r.e. sets.

Definition. Let C be a class of d.r.e. sets, $Z \subseteq N$. The set Z is a *core* of C if
(a) $u \in Z \Rightarrow D_u \in C$;
(b) if $u \in Z$, $D_v \in C$ and $D_v \subseteq D_u$ then $D_v = D_u$.

Z is thus a core of C if it is the set of canonical indices of "minimal" finite sets in C. Since these "minimal" finite sets are evidently uniquely determined by C and since canonical indices are also unique, the core of C is uniquely determined. Evidently C is a finite union of basic open classes if and only if C is an open class with finite core. Hence the problem of classifying the index sets of classes of d.r.e. sets which are finite unions of basic open classes reduces to that of classifying the index sets of open classes with finite core. This classification will require some notation for the highest 1-degrees of sets of form $\bigcup_{i=1}^{n} (R_{2i-1} - R_{2i})$ where $R_1, \ldots, R_{2n}$ are r.e. sets. The existence of these maximum 1-degrees was proved in [**2**, p. 33, Theorem 2] (where the class of such sets is denoted by Σ_{2n}^{-1}), and in [**5**] a descriptive notation was introduced for these 1-degrees. In the following, if $\mathbf{a}$ is a 1-degree and $X \subseteq N$ we shall use $X \leqq_1 \mathbf{a}$ ($\mathbf{a} \leqq_1 X$) to mean that $X \leqq_1 Y$ ($Y \leqq_1 X$) for some $Y \in \mathbf{a}$.

PROPOSITION 6.1. *Let* $\mathbf{a}$, $\mathbf{b}$ *be any* 1*-degrees. Then*
(a) *there is a* 1*-degree which is maximum for sets* $X \cap Y$ *where* $X \leqq_1 \mathbf{a}$, $Y \leqq_1 \mathbf{b}$;
(b) *there is a* 1*-degree which is maximum for sets* $X \cup Y$ *where* $X \leqq_1 \mathbf{a}$, $Y \leqq_1 \mathbf{b}$.

Proof. This follows from [**5**, Proposition 2.9].

This justifies the following operation on 1-degrees;

Definition. Let $\mathbf{a}$, $\mathbf{b}$ be any 1-degrees.
(a) $\mathbf{a} \wedge \mathbf{b}$ denotes the 1-degree described in Proposition 6.1(a).
(b) $\mathbf{a} \vee \mathbf{b}$ denotes the 1-degree described in Proposition 6.2(b).
(c) For all $n \geqq 1$, $\mathbf{a}\cdot n$ is defined inductively by $\mathbf{a}\cdot 1 = \mathbf{a}$, $\mathbf{a}\cdot(n+1) = (\mathbf{a}\cdot n) \vee \mathbf{a}$.

The following is then obtained:

PROPOSITION 6.2. *Let* $\mathbf{e}_1$, $\mathbf{a}_1$ *denote the* 1*-degrees of the sets* K, $\bar{K}$ *respectively. Then if* $n \geqq 1$,
(a) $(\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot n$ *is the maximum* 1*-degree for sets of form* $\bigcup_{i=1}^{n} (R_{2i-1} - R_{2i})$ *where* $R_1, \ldots, R_{2n}$ *are arbitrary r.e. sets*;

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(b) $(\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot n$ *is the maximum* 1*-degree for sets of form* $\bigcup_{i=1}^n (R_{2i-1} - R_{2i})$ *where* $R_1, \ldots, R_{2n}$ *are arbitrary r.e. sets satisfying* $R_1 \supseteq R_2 \supseteq \ldots \supseteq R_{2n}$;
(c) $(\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot n < (\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot(n+1)$ *for all* $n \geqq 1$.

Proof. (a) We use induction on n. For $n = 1$, note that $\mathbf{e}_1 \wedge \mathbf{a}_1$ is the maximum 1-degree for sets $X \cap Y$ where $X \leqq_1 K$ and $Y \leqq_1 \bar{K}$; but it is easy to verify that for all sets $S \subseteq N$, $S = X - Y$ where $X \leqq_1 K$ and $Y \leqq_1 \bar{K} \Leftrightarrow S = R_1 - R_2$ where R_1, R_2 are r.e. sets. Now assume that (a) holds for n. For $n + 1$, note that $(\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot(n+1) = (\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot n \vee (\mathbf{e}_1 \wedge \mathbf{a}_1)$ is the maximum 1-degree for sets $X \cup Y$ where $X \leqq_1 (\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot n$ and $Y \leqq_1 \mathbf{e}_1 \wedge \mathbf{a}_1$. By the induction hypothesis, there are r.e. sets $K_1, K_2, \ldots, K_{2n+2}$ such that $\bigcup_{i=1}^n (K_{2i-1} - K_{2i}) \equiv (\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot n$ and $K_{2n-1} - K_{2n} \equiv \mathbf{e}_1 \wedge \mathbf{a}_1$, and the following holds:

$$X \leqq_1 (\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot n \ \&\ Y \leqq_1 \mathbf{e}_1 \wedge \mathbf{a}_1 \Leftrightarrow X \leqq_1 \bigcup_{i=1}^{n} (K_{2i-1} - K_{2i}) \ \&$$

$$Y \leqq_1 K_{2n+1} - K_{2n+2} \Leftrightarrow \text{there exist } r.e. \text{ sets } R_1, R_2, \ldots, R_{2n+2} \text{ such that}$$

$$X = \bigcup_{i=1}^{n} (R_{2i-1} - R_{2i}) \ \&\ Y = R_{2n+1} - R_{2n+2}.$$

Hence $(\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot(n+1)$ is the maximum 1-degree for sets $X \cup Y$ for which there exist r.e. sets $R_1, R_2. \ldots, R_{2n+2}$ such that $X \cup Y = \bigcup_{i=1}^{n+1} (R_{2i-1} - R_{2i})$, which completes the induction.

(b) now follows from (a), using the fact that, as shown in [**2**, p. 29, Proposition 1] a set S has form $\bigcup_{i=1}^n (R_{2i-1} - R_{2i})$ for r.e. sets $R_1, \ldots, R_{2n} \Leftrightarrow S$ has form $\bigcup_{i=1}^n (R_{2i-1}' - R_{2i}')$ for r.e. sets $R_1', \ldots, R_{2n}'$ satisfying $R_1' \subseteq \ldots \supseteq R_{2n}'$.

(c) then follows from (b) and [**2**, § 1].

It was shown in [**4**, p. 39, Theorem 1] and, independently in [**2**, p. 41, Theorem 6] that the 1-degrees $(\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot n$ for $n \geqq 1$ are exactly the 1-degrees of index sets θC where C is a non-empty finite class of finite sets such that $\emptyset \notin C$. It will be shown below that these 1-degrees are also exactly the 1-degrees of index sets δC of non-trivial open classes of d.r.e. sets with finite core. First we require some machinery for eliminating "redundant" information from the core.

Definition. Let C be an open class of d.r.e. sets with finite core Z. The sequence $\langle D_{u_1}, \ldots, D_{u_n}\rangle$ $(n \geqq 2)$ will be called *regular for* C if

(a) $u_i \in Z$ for $1 \leqq i \leqq n$;

(b) $(\forall k)_{1 \leqq k \leqq n}(\forall u \in Z)\left[D_u \nsubseteq \left(\bigcup_{1 \leqq i \leqq k} D_{u_i}\right) \cap \left(\bigcup_{k < i \leqq n} D_{u_i}\right)\right]$.

If such a sequence exists, we say C has a *regular sequence of length* n. Note that if C has a regular sequence of length $n \geqq 2$, then $\emptyset \notin C$ and that if $\langle D_{u_1}, \ldots, D_{u_n} \rangle$ is regular for C, then $D_{u_i} \neq D_{u_j}$ for $i \neq j$. We remark that it is necessary to consider *sequences* $\langle D_{u_1}, \ldots, D_{u_n} \rangle$ rather than just *sets* $\{D_{u_1}, \ldots, D_{u_n}\}$. For example, if $D_{u_1} = \{1, 2, 3\}$, $D_{u_2} = \{1, 3, 4\}$ and $D_{u_3} = \{2, 3, 5\}$ and $Z = \{u_1, u_2, u_3\}$, then it can be checked that $\langle D_{u_2}, D_{u_1}, D_{u_3} \rangle$ is regular for C but $\langle D_{u_1}, D_{u_2}, D_{u_3} \rangle$ is not (since $D_{u_1} \subseteq D_{u_1} \cap (D_{u_2} \cup D_{u_3})$).

LEMMA 6.3. *If C is an open class of d.r.e. sets with core Z and $|Z| \geqq 2$, then C has a regular sequence of length* 2.

Proof. Let $|Z| \geqq 2$, and suppose $u_1, u_2 \in Z$, $u_1 \neq u_2$. If $\langle D_{u_1}, D_{u_2} \rangle$ is not regular for C, then for some $u \in Z$, $D_u \subseteq D_{u_1} \cap D_{u_2}$; but then $D_u \subseteq D_{u_1}$ and $D_u \subseteq D_{u_2}$ which implies $D_u = D_{u_1} = D_{u_2}$ since $u_1, u_2 \in$ core C. But then $u_1 = u_2$, contrary to assumption. Hence $\langle D_{u_1}, D_{u_2} \rangle$ must be a regular sequence for C of length 2.

LEMMA 6.4. *Let C be an open class of d.r.e. sets which has a regular sequence of length $n \geqq 2$. Then* $(\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot n \leqq_1 \delta C$.

Proof. Assume C is an open class with core Z, and let $\langle D_{u_1}, \ldots, D_{u_n} \rangle$ $(n \geqq 2)$ be a regular sequence for C. As noted above, this implies $\emptyset \notin C$. By Proposition 6.2(b) it suffices to show that if $R_1, R_2, \ldots, R_{2n}$ are arbitrary r.e. sets satisfying $R_1 \supseteq R_2 \supseteq \ldots \supseteq R_{2n}$, then $\bigcup_{k=1}^n (R_{2k-1} - R_{2k}) \leqq_1 \delta C$. Suppose $R_1 \supseteq R_2 \supseteq \ldots \supseteq R_{2n}$, and generate r.e. sets $W_{g(x)}$, $W_{h(x)}$ according to the following instructions: For $1 \leqq k \leqq n$,

(i) put all elements of D_{u_k} into $W_{g(x)} \Leftrightarrow x \in R_{2k-1}$;

(ii) put all elements of $D_{u_k} - \bigcup_{k<i\leqq n} D_{u_i}$ into $W_{h(x)} \Leftrightarrow x \in R_{2k}$.

Let $f(x) = \langle g(x), h(x) \rangle$; we claim that

$$x \in \bigcup_{k=1}^{n} (R_{2k-1} - R_{2k}) \Leftrightarrow f(x) \in \delta C.$$

Assume $x \in R_{2k-1} - R_{2k}$ for $1 \leqq k \leqq n$. Then $x \in R_j$ for all j, $1 \leqq j \leqq 2k - 1$, so that $\bigcup_{1\leqq i\leqq k} D_{u_i} \subseteq W_{g(x)}$. Moreover, $x \in R_{2j}$ if and only if $1 \leqq j < k$, so $z \in W_{h(x)} \Leftrightarrow (\exists j)_{1\leqq j<k}(z \in D_{u_j} - \bigcup_{j<i\leqq n} D_{u_i})$. But

$$D_{u_k} \cap \bigcup_{1\leqq j<k} \Big(D_{u_j} - \bigcup_{j<i\leqq n} D_{u_i}\Big) = \emptyset,$$

which implies $D_{u_k} \cap W_{h(x)} = \emptyset$; so $D_{u_k} \subseteq W_{g(x)} - W_{h(x)} = V_{f(x)}$ which implies $f(x) \in \delta C$. Conversely, suppose $x \notin \bigcup_{k=1}^n (R_{2k-1} - R_{2k})$. Since $R_1 \supseteq R_2 \supseteq \ldots \supseteq R_{2n}$, this implies $x \in \bar{R}_1 \cup R_{2n} \cup \bigcup_{k=1}^{n-1} (R_{2k} - R_{2k+1})$.

Case 1. $x \in \bar{R}_1$: Then $(\forall k)_{1\leqq k\leqq n} x \notin R_k$, which implies $W_{g(x)} = \emptyset$ and $V_{f(x)} = W_{g(x)} - W_{h(x)} = \emptyset$; hence $f(x) \notin \delta C$.

Case 2. $x \in R_{2n}$: Then $(\forall k)_{1\leqq k\leqq n} x \in R_k$, which implies $W_{g(x)} = W_{h(x)} = \bigcup_{1\leqq k\leqq n} D_{u_k}$. But then $V_{f(x)} = \emptyset$ and $f(x) \notin \delta C$.

Case 3. $x \in R_{2k} - R_{2k+1}$ for some k, $1 \leqq k < n$: Then $W_{g(x)} = \bigcup_{1\leqq j\leqq k} D_{u_j}$ and $W_{h(x)} = \bigcup_{1\leqq j\leqq k} (D_{u_j} - \bigcup_{j<i\leqq n} D_{u_i})$; hence

$$V_{f(x)} = W_{g(x)} - W_{h(x)} = \Big(\bigcup_{1\leqq j\leqq k} D_{u_j}\Big) \cap \Big(\bigcup_{k<j\leqq n} D_{u_j}\Big).$$

But then by the hypothesis that $\langle D_{u_1}, \ldots, D_{u_n}\rangle$ is regular, $(\forall u \in Z)$ $(D_u \not\subseteq V_{f(x)})$, which implies $f(x) \notin \delta C$. It follows that $f(x) \in \delta C \Rightarrow x \in \bigcup_{k=1}^{n} (R_{2k-1} - R_{2k})$, and $\bigcup_{k=1}^{n} (R_{2k-1} - R_{2k}) \leqq_1 \delta C$ via f.

LEMMA 6.5. *Let C be an open class of d.r.e. sets with finite core, and assume $n \geqq 1$. If C has no regular sequence of length $n + 1$, then $\delta C \leqq_1 (\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot n$.*

Proof. Suppose C is an open class with finite core Z. If C has no regular sequence of length $n + 1$, then for all $u_1, u_2, \ldots, u_{n+1} \in Z$,

$$(\exists k)_{1\leqq k\leqq n}(\exists u \in Z)\left[D_u \subseteq \Big(\bigcup_{1\leqq i\leqq k} D_{ui}\Big) \cap \Big(\bigcup_{k< i\leqq n+1} D_{ui}\Big)\right].$$

By Proposition 6.2(a), it suffices to show that $\delta C = \bigcup_{k=1}^{n} (R_{2k-1} - R_{2k})$ for some r.e. sets $R_1, R_2, \ldots, R_{2n}$. This is done as follows: For notational convenience, we will use $(\exists u_1, \ldots, u_j \in Z)$ to abbreviate $(\exists u_1 \in Z)(\exists u_2 \in Z) \ldots (\exists u_j \in Z)$, and for $1 \leqq j < n$, we let $\Phi_j(v_1, \ldots, v_j, u_1, \ldots, u_{j+1})$ denote the following formula:

$$D_{u_{j+1}} \subseteq W_{\pi_1(x)} \,\&\, D_{v_j} \subseteq W_{\pi_2(x)} \,\&\, D_{u_{j+1}} \cap \Big(\bigcup_{1\leqq i\leqq j} D_{vi}\Big) = \emptyset$$
$$\& \ (\forall u \in Z)\Big(D_u \subseteq \bigcup_{1\leqq i\leqq j} D_{ui} \to D_u \cap D_{vj} \neq \emptyset\Big).$$

Define the sets $R_1, R_2, \ldots, R_{2n}$ as follows:

$$R_1 = \{x | (\exists u_1 \in Z)(D_{u_1} \subseteq W_{\pi_1(x)})\};$$
$$R_2 = \{x | (\exists u_1 \in Z)[D_{u_1} \subseteq W_{\pi_1(x)} \,\&\, D_{u_1} \cap W_{\pi_2(x)} \neq \emptyset]\};$$

and for $1 < k \leqq n$,

$$\begin{aligned} R_{2k-1} = \{x | (\exists u_1, u_2, \ldots, u_k \in Z)(\exists v_1, \ldots, v_{k-1})[&D_{u_1} \subseteq W_{\pi_1(x)} \\ &\& \ \Phi_1(v_1, u_1, u_2) \ \& \ \Phi_2(v_1, v_2, u_1, u_2, u_3) \\ &\& \ldots \& \ \Phi_{k-1}(v_1, \ldots, v_{k-1}, u_1, \ldots, u_k)]\}; \\ R_{2k} = \{x | (\exists u_1, \ldots, u_k \in Z)(\exists v_1, \ldots, v_{k-1})[&D_{u_1} \subseteq W_{\pi_1(x)} \\ &\& \ \Phi_1(v_1, u_1, u_2) \ \& \ \Phi_2(v_1, v_2, u_1, u_2, u_3) \\ &\& \ldots \& \ \Phi_{k-1}(v_1, \ldots, v_{k-1}, u_1, \ldots, u_k) \\ &\& \ (\forall u \in Z)(D_u \subseteq \bigcup_{1\leqq i\leqq k} D_{u_i} \to D_u \cap W_{\pi_2(x)} \neq \emptyset)]\}. \end{aligned}$$

It is clear that since Z is a fixed finite set, each $\Phi_j(v_1, \ldots, v_j, u_1, \ldots, u_{j+1})$ is an r.e. condition and hence each set R_i is r.e., $1 \leqq i \leqq 2n$; in addition it is easily verified that $R_1 \supseteq R_2 \supseteq \ldots \supseteq R_{2n}$. It remains to show that $\bigcup_{k=1}^{n} (R_{2k-1} - R_{2k}) = \delta C$.

($\subseteq$) Assume $x \in R_1 - R_2$. Then for some $u_1 \in Z$, $D_{u_1} \subseteq W_{\pi_1(x)}$. If for this u_1, $D_{u_1} \cap W_{\pi_2(x)} \neq \emptyset$ then $x \in R_2$, contrary to assumption. So $x \in R_1 - R_2 \Rightarrow (\exists u_1 \in Z)(D_{u_1} \subseteq W_{\pi_1(x)} - W_{\pi_2(x)})$ which implies $x \in \delta C$. Next assume $x \in R_{2k-1} - R_{2k}$ for some k, $1 < k \leqq n$. Then for some $u_1, \ldots, u_k \in Z$,

$$(\exists v_1, \ldots, v_{k-1})[D_{u_1} \subseteq W_{\pi_1(x)} \,\&\, \Phi_1(v_1, u_1, u_2)$$
$$\& \ldots \& \ \Phi_{k-1}(v_1, \ldots, v_{k-1}, u_1, \ldots, u_k)].$$

Note that for each j,

$$\Phi_j(v_1, \ldots, v_j, u_1, \ldots, u_{j+1}) \Rightarrow D_{u_{j+1}} \supseteq W_{\pi_1(x)},$$

so that $\bigcup_{1 \leqq i \leqq k} D_{u_i} \subseteq W_{\pi_1(x)}$. If for all $u \in Z$, $D_u \subseteq \bigcup_{1 \leqq i \leqq k} D_{u_i}$ implies $D_u \cap W_{\pi_2(x)} \neq \emptyset$, then $x \in R_{2k}$, contrary to assumption. So

$$x \in R_{2k-1} - R_{2k} \Rightarrow (\exists u \in Z)\left[D_u \subseteq \bigcup_{1 \leqq i \leqq k} D_{u_i} \subseteq W_{\pi_1(x)} \,\&\, D_u \cap W_{\pi_2(x)} = \emptyset\right]$$

$$\Rightarrow (\exists u \in Z)\,[D_u \subseteq W_{\pi_1(x)} - W_{\pi_2(x)} = V_x] \Rightarrow x \in \delta C.$$

Hence $\bigcup_{k=1}^{n} (R_{2k-1} - R_{2k}) \subseteq \delta C$.

($\supseteq$) Assume $x \notin \bigcup_{k=1}^{n}(R_{2k-1} - R_{2k})$. Then again since $R_1 \supseteq R_2 \supseteq \ldots \supseteq R_{2n}$, $x \in \bar{R}_1 \cup R_{2n} \cup \bigcup_{k=1}^{n-1} (R_{2k} - R_{2k+1})$.

Case 1. $x \in \bar{R}_1$: Then $(\forall u_1 \in Z)(D_{u_1} \not\subseteq W_{\pi_1(x)})$, which implies $(\forall u_1 \in Z)$ $(D_{u_1} \not\subseteq V_x = W_{\pi_1(x)} - W_{\pi_2(x)})$. Hence $x \in \bar{R}_1 \Rightarrow x \notin \delta C$.

Case 2. $x \in R_{2k} - R_{2k+1}$ for some k, $1 \leqq k < n$: Suppose $x \in \delta C$; then for some $u_0 \in Z$, $D_{u_0} \subseteq W_{\pi_1(x)} - W_{\pi_2(x)}$. Now $x \in R_{2k}$ implies that for some $u_1, \ldots, u_k \in Z$ and some $v_1, \ldots, v_k$,

$$D_{u_1} \subseteq W_{\pi_1(x)} \wedge \Phi_1(v_1, u_1, u_2) \,\&\, \ldots \,\&\, \Phi_{k-1}(v_1, \ldots, v_{k-1}, u_1, \ldots, u_k)$$
$$\&\, (\forall u \in Z)\left(D_u \subseteq \bigcup_{1 \leqq i \leqq k} D_{u_i} \Rightarrow D_u \cap W_{\pi_2(x)} \neq \emptyset\right).$$

Then since for each j, $\Phi_j(v_1, \ldots, v_j, u_1, \ldots, u_{j+1}) \Rightarrow D_{v_j} \subseteq W_{\pi_2(x)}$, $\bigcup_{1 \leqq i < k} D_{v_i} \subseteq W_{\pi_2(x)}$ which implies $D_{u_0} \cap \bigcup_{1 \leqq i < k} D_{v_i} = \emptyset$. Let

$$Z_k = \{u \in Z | D_u \subseteq \bigcup\nolimits_{1 \leqq i \leqq k} D_{u_i}\};$$

then $D_u \cap W_{\pi_2(x)} \neq \emptyset$ for each $u \in Z_k$. For each $u \in Z_k$, choose $z_u \in D_u \cap W_{\pi_2(x)}$, and let v_k be such that $D_{v_k} = \{z_u | u \in Z_k\}$. Then $D_{v_k} \subseteq W_{\pi_2(x)}$, and, since $D_u \cap D_{v_k} \neq \emptyset$ for each $u \in Z_k$, it follows that

$$(\forall u \in Z)\left(D_u \subseteq \bigcup_{1 \leqq i < k} D_{u_i} \Rightarrow D_u \cap D_{v_k} \neq \emptyset\right).$$

Hence $\Phi_k(v_1, \ldots, v_k, u_1, \ldots, u_k, u_0)$ holds, which implies that

$$(\exists u_1, \ldots, u_{k+1} \in Z)(\exists v_1, \ldots, v_k)[D_{u_1} \subseteq W_{\pi_1(x)}$$
$$\&\, \Phi_1(v_1, u_1, u_2) \,\&\, \ldots \,\&\, \Phi_{k-1}(v_1, \ldots, v_{k-1}, u_1, \ldots, u_k)$$
$$\&\, \Phi_k(v_1, \ldots, v_k, u_1, \ldots, u_{k+1})].$$

But then $x \in R_{2k+1}$, contrary to assumption. Hence $x \in R_{2k} - R_{2k+1} \Rightarrow x \notin \delta C$.

Case 3. $x \in R_{2n}$: Then for some $u_1, \ldots, u_n \in Z$ and some $v_1, \ldots, v_{n-1}$,

$$D_{u_1} \subseteq W_{\pi_1(x)} \,\&\, \Phi_1(v_1, u_1, u_2) \,\&\, \ldots \,\&\, \Phi_{n-1}(v_1, \ldots, v_{n-1}, u_1, \ldots, u_n) \,\&$$
$$(\forall u \in Z)\left(D_u \subseteq \bigcup_{1 \leqq i \leqq n} D_{u_i} \Rightarrow D_u \cap W_{\pi_2(x)} \neq \emptyset\right).$$

Suppose $x \in \delta C$. Then for some $u_{n+1} \in Z$, $D_{u_{n+1}} \subseteq W_{\pi_1(x)} - W_{\pi_2(x)}$. By the hypothesis that C has no regular sequence of length $n + 1$, there is a $u' \in Z$

and a k', $1 \leqq k' \leqq n$, satisfying

$$D_{u'} \subseteq \Big(\bigcup_{1\leqq i\leqq k'} D_{ui}\Big) \cap \Big(\bigcup_{k'<i\leqq n+1} D_{ui}\Big).$$

If $k' = n$, then $D_{u'} \subseteq D_{u_{n+1}}$ which since u', $u_{n+1} \in Z$ implies $D_{u'} = D_{u_{n+1}}$ by minimality. But then $D_{u_{n+1}} \subseteq \bigcup_{1\leqq i\leqq n} D_{ui}$ which implies $D_{u_{n+1}} \cap W_{\pi_2(x)} \neq \emptyset$, which contradicts $D_{u_{n+1}} \subseteq W_{\pi_1(x)} - W_{\pi_2(x)}$. So $k' < n$, which implies $\Phi_{k'}(v_1, \ldots, v_{k'}, u_1, \ldots, u_{k'+1})$ holds. Then $D_{u'} \subseteq \bigcup_{1\leqq i\leqq k'} D_{ui} \Rightarrow D_{u'} \cap D_{vk'} \neq \phi$. Let $z \in D_{u'} \cap D_{vk'}$; then $z \in D_{u'} \cap D_{vk'} \cap W_{\pi_2(x)}$ since

$$\Phi_{k'}(v_1, \ldots, v_{k'}, u_1, \ldots, u_{k'+1}) \Rightarrow D_{vk'} \subseteq W_{\pi_2(x)}. \text{ Now}$$

$$D_{u'} \subseteq \bigcup_{k'<i\leqq n+1} D_{ui} \Rightarrow z \in \bigcup_{k'<i\leqq n+1} D_{ui}.$$

But for each $j, j' \leqq j < n$, $\Phi_j(v_1, \ldots, v_j, u_1, \ldots, u_{j+1}) \Rightarrow D_{u_{j+1}} \cap \bigcup_{1\leqq i\leqq j} D_{vi} = \emptyset$, which implies $z \notin D_{u_{j+1}}$; so that $z \notin \bigcup_{k'<i\leqq n} D_{ui}$. Thus $z \in D_{u_{n+1}}$, which implies $D_{u_{n+1}} \cap W_{\pi_2(z)} \neq \emptyset$ again contradicting $D_{u_{n+1}} \subseteq W_{\pi_1(x)} - W_{\pi_2(x)}$. Hence $x \in R_{2n} \Rightarrow x \notin \delta C$, which completes the proof that $\delta C \subseteq \bigcup_{k=1}^{n} (R_{2k-1} - R_{2k})$.

We can now classify the 1-degree of index sets of open classes of d.r.e. sets with finite core, by means of the following theorem:

THEOREM 6.6. *Let C be a non-trivial open class of d.r.e. sets with finite core $Z = \{u_1, \ldots, u_m\}$, $m \geqq 1$. Then*

(a) $\delta C \equiv (\mathbf{e}_1 \wedge \mathbf{a}_1) \Leftrightarrow m = 1$;

(b) *if $m \geqq 2$, let n_C be the largest n such that C has a regular sequence of length n. Then $2 \leqq n_C \leqq m$, and $\delta C \equiv (\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot n_C$.*

Proof. Assume C is an open class with core containing m elements. If $m = 1$, then C is a basic open class; hence δC is d.r.e. by Theorem 5.1 and, since δC is non-trivial by hypothesis, $\delta C \equiv K \times \bar{K}$ by Corollary 4.3. But $\bar{K} \times K \equiv \mathbf{e}_1 \wedge \mathbf{a}_1$ by Propositions 6.2(a) and 3.2(a). Conversely, $\delta C \equiv \mathbf{e}_1 \wedge \mathbf{a}_1$ implies δC is d.r.e. by Proposition 6.2(a); that $m = 1$ then follows from Theorem 5.1. Now assume $m \geqq 2$, and let $R_C = \{n | C$ has a regular sequence of length $n\}$. By definition of "regular sequence," $n \in R_C \Rightarrow n \leqq m$, while by Lemma 6.3, $2 \in R_C$. Hence $n_C = \max R_C$ exists and satisfies $2 \leqq n_C \leqq m$. By Lemma 6.4, $(\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot n_C \leqq_1 \delta C$, and since $n_C + 1 \notin R_C$, it follows that $\delta C \leqq_1 (\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot n_C$. Hence $\delta C \equiv (\mathbf{e}_1 \wedge \mathbf{a}_1)\cdot n_C$.

To complete the classification it remains to show that in Theorem 6.6(b), n_C can take on all possible values. If $a, b \in N$ we use $[a, b]$ to denote $\{k \in N | a \leqq k \leqq b\}$ if $a \leqq b$ and the empty set otherwise. We require the following lemma:

LEMMA 6.7. *Assume $n < m$. For each j, $1 \leqq j \leqq m - n + 2$, let $F_j = [1, j-1] \cup [j+1, m-n+2]$.*

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(a) *Suppose* $1 \leqq j_1, j_2, j_3 \leqq m - n + 2$ *and* $j_2 \neq j_3$; *then* $F_{j_1} \subseteq F_{j_2} \cup F_{j_3}$.

(b) *Let* $\langle H_1, \ldots, H_p \rangle$ *be a sequence of finite sets with* $p \geqq 3$ *and* $H_i \neq H_j$ *for* $i \neq j$. *Suppose that for some* $i_1, i_2, i_3, j_1, j_2, j_3$ *satisfying* $1 \leqq i_1 < i_2 < i_3 \leqq p$ *and* $1 \leqq j_1, j_2, j_3 \leqq m - n + 2$, $H_{i_1} = F_{j_1}$, $H_{i_2} = F_{j_2}$ *and* $H_{i_3} = F_{j_3}$. *Then*

$$H_{i_1} \subseteq \Big(\bigcup_{1 \leqq i \leqq i_1} H_i \Big) \cap \Big(\bigcup_{i_1 < i \leqq p} H_i \Big).$$

Proof. (a) Suppose $1 \leqq j_1, j_2, j_3 \leqq m - n + 2$ and $j_2 \neq j_3$; then $j_2 \in F_{j_2} = [1, m - n + 2] - \{j_3\}$. Hence $F_{j_1} \subseteq [1, m - n + 2] = ([1, m - n + 2] - \{j_2\}) \cup \{j_2\} \subseteq F_{j_2} \cup F_{j_3}$.

(b) Suppose $1 \leqq i_1 < i_2 < i_3 \leqq p$ and $H_{i_1} = F_{j_1}, H_{i_2} = F_{j_2}, H_{i_3} = F_{j_3}$ for $1 \leqq j_1, j_2, j_3 \leqq m - n + 2$. Now $i_2 \neq i_3 \Rightarrow H_{i_2} \neq H_{i_3} \Rightarrow F_{j_2} \neq F_{j_3} \Rightarrow j_2 \neq j_3$. Then by (a), $H_{i_1} = F_{j_1} \subseteq F_{j_2} \cup F_{j_3} = H_{i_2} \cup H_{i_3} \subseteq \bigcup_{i_1 < i \leqq p} H_i$ since $i_1 < i_2, i_3$. Hence

$$H_{i_1} \subseteq \Big(\bigcup_{1 \leqq i \leqq i_1} H_i \Big) \cap \Big(\bigcup_{i_1 < i \leqq p} H_i \Big).$$

THEOREM 6.8. *For each* $m \geqq 2$ *and* n *satisfying* $2 \leqq n \leqq m$ *there is an open class of d.r.e. sets whose core contains* m *elements and whose index set has* 1-*degree* $(\mathbf{e}_1 \wedge \mathbf{a}_1) \cdot n$.

Proof. By Theorem 6.7(b), it suffices to show that for all $m \geqq 2$ and any n satisfying $2 \leqq n \leqq m$ there is an open class X of d.r.e. sets whose core contains m sets and whose longest regular sequence has length n. We define the core of C as the set of canonical indices of the finite sets $F_1, \ldots, F_m$ defined by

$$F_j = [1, j - 1] \cup [j + 1, m - n + 2], \text{ for } 1 \leqq j \leqq m - n + 2,$$
$$F_j = [1, m - n) \cup \{j\}, \quad \text{for } m - n + 2 < j \leqq m.$$

Case 1. $m = n$: Then $F_1 = \{2\}$, $F_2 = \{1\}$ and $F_j = \{j\}$ for $3 \leqq j \leqq m$, and it is easily verified that the sequence $\langle F_1, \ldots, F_m \rangle$ is regular for C; hence the longest regular sequence for C has length $n = m$.

Case 2. $n < m$: We first show that C has a regular sequence of length n: Consider the sequence $\langle G_1, \ldots, G_n \rangle$ where $G_i = F_{m-n+i}$, $1 \leqq i \leqq n$. Then $G_1 = [1, m - n] \cup \{m - n + 2\}$, $G_2 = [1, m - n + 1]$ and $G_i = [1, m - n] \cup \{m - n + i\}$ for $3 \leqq i \leqq n$. For each k, $1 \leqq k < n$, let $P_k = \bigcup_{1 \leqq i \leqq k} G_i$, $S_k = \bigcup_{k < i \leqq n} G_i$. We claim that for each j, k, $1 \leqq j \leqq m$, $1 \leqq k < n$, $F_j \not\subseteq P_k \cap S_k$. For suppose $1 \leqq j \leqq m - n + 1$. Then $m - n + 2 \in F_j$, but it is easily checked that $m - n + 2 \in G_1$ while $m - n + 2 \notin G_i$ for $1 < i \leqq n$. Hence for each k, $1 \leqq k < n$, $m - n + 2 \in P_k - S_k$, and $F_j \not\subseteq P_k \cap S_k$ if $1 \leqq j \leqq m - n + 1$. Now suppose $j = m - n + 2$; then $F_j = [1, m - n + 1]$. But $m - n + 1 \notin G_1 = P_1$, so $F_j \not\subseteq P_1 \cap S_1$, while $m - n + 1 \notin G_i$ for $i > 2$, so $F_j \not\subseteq S_k$ for $k \geqq 2$. Hence $F_j \not\subseteq P_k \cap S_k$ if $j = m - n + 2$. Finally suppose $m - n + 3 \leqq j \leqq m$. Then $j \in F_j$, while $j \in G_i$ only if $i = j - m + n$; but then for each k, $1 \leqq k \leqq n$, $j \in P_k - S_k$ if $k \geqq j - m + n$ and $j \in S_k - P_k$ if $k < j - m + n$. This completes the proof that the sequence $\langle G_1, \ldots, G_n \rangle$ is regular for C. It remains to show that C has no regular sequence of length $> n$. Suppose $\langle H_1, \ldots, H_p \rangle$ is such a sequence; then $p > n \geqq 2$ and $H_i \neq H_j$ for $i \neq j$. Since at most $n - 2$ of the sets H_i can be from among $\{F_j | m - n + 3 \leqq j \leqq m\}$, at least 3 of the sets H_i must be from among $\{F_j | 1 \leqq j \leqq m + n - 2\}$, say $H_{i_1} = F_{j_1}$, $H_{i_2} = F_{j_2}$, $H_{i_3} = F_{j_3}$ for $1 \leqq i_1 < i_2 < i_3 \leqq p$, $1 \leqq j_1, j_2, j_3 \leqq m + n - 2$. But then by Lemma 6.7(b),

$$H_{i_1} \subseteq \Big(\bigcup_{1 \leqq i \leqq i_1} H_i \Big) \cap \Big(\bigcup_{i_1 < i \leqq p} H_i \Big),$$

which contradicts the assumption that $\langle H_1, \ldots, H_p \rangle$ is regular. Hence the longest regular sequence for C has length n, and the 1-degree of δC is $(\mathbf{e}_1 \wedge \mathbf{a}_1) \cdot n$.

COROLLARY 6.9. *If $m \geqq 2$, there are $m - 1$ possible 1-degrees for index sets δC where C is an open class of d.r.e. sets whose core contains m elements.*

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