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AREA AND LENGTH MAXIMA FOR UNIVALENT FUNCTIONS

SHINJI YAMASHITA

Let S be the family of functions $f(z) = z + a_2 z^2 + \dots$ which are analytic and univalent in $|z| < 1$. We find the value

$$\max_{f \in S} \iint_{|z|<r} \left|(z/f(z))'\right|^2 dx\, dy$$

as a function of r, $0 < r < 1$. The known lower estimate of

$$\sup_{f \in S} \int_{|z|=r} \left|f'(z)\right| |dz|$$

is improved. Relations with the growth theorem are considered and the radius of univalence of $f(z)/z$ is discussed.

For g analytic in $D = \{|z| < 1\}$, we set

$$\Delta(r, g) = \iint_{|z|<r} |g'(z)|^2 \, dxdy, \qquad 0 < r \leqslant 1, \quad z = x + iy.$$

We call g Dirichlet–finite if $\Delta(1, g) < \infty$. Let S be the family of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and univalent in D and set

$$F_f(z) = f(z)/z, \quad z \in D, \quad f \in S.$$

As a consequence of the celebrated de Branges theorem: $|a_n| \leqslant n$ $(n \geqslant 2)$ for $f \in S$, (see [**1**]) we have immediately

$$\pi^{-1}\Delta(r, F_f) = \sum_{n=1}^{\infty} n\, |a_{n+1}|^2 r^{2n} \leqslant \sum_{n=1}^{\infty} n(n+1)^2 r^{2n} = \pi^{-1}\Delta(r, F_K),$$

where $K(z) = z/(1-z)^2$ is the Koebe function. Therefore

$$\max_{f \in S} \Delta(r, F_f) = 2\pi r^2 \left(r^2 + 2\right)\left(1 - r^2\right)^{-4}$$

for $0 < r < 1$. For each r, $0 < r < 1$, the maximum is attained only by the rotations of the Koebe function: $K_\theta(z) = e^{-i\theta} K\left(e^{i\theta} z\right)$, where θ is real. We first prove:

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Theorem 1. *We have*

$$\max_{f\in S}\Delta(r,1/F_f)=2\pi r^2\left(r^2+2\right)\quad\text{for}\quad 0<r\leqslant 1.$$

For each r, $0<r\leqslant 1$, the maximum is attained only by K_θ's.

Proof: Given $f\in S$, we can apply the area theorem [**3**, p.29] to

$$f(1/z)^{-1}=z-a_2+\sum_{n=1}^{\infty}b_nz^{-n}\quad(|z|>1)$$

to obtain

$$\sum_{n=1}^{\infty}n\left|b_n\right|^2\leqslant 1. \tag{2}$$

Since
$$1/F_f(z)=1-a_2z+\sum_{n=1}^{\infty}b_nz^{n+1},\quad z\in D,$$

it follows from (2), together with $|a_2|\leqslant 2$, that

$$\pi^{-1}\Delta(r,1/F_f)=\left|a_2\right|^2r^2+2r^4\sum_{n=1}^{\infty}2^{-1}(n+1)\left|b_n\right|^2r^{2n-2}$$
$$\leqslant 4r^2+2r^4\sum_{n=1}^{\infty}n\left|b_n\right|^2\leqslant 2r^2\left(r^2+2\right).$$

Since $\Delta\left(r,1/F_{K_\theta}\right)=2\pi r^2\left(r^2+2\right)$, we now have the identity. If the maximum is attained by f, then $|a_2|=2$, so that $f=K_\theta$ for some θ. □

It follows that $\Delta(1,1/F_f)\leqslant 6\pi$. This shows that each function $f\in S$ is the quotient of two functions, z and $1/F_f(z)$, both of which are bounded and Dirichlet-finite in D; see estimate (6) for the bound $|1/F_f|\leqslant 4$.

Each $f\in S$ maps $\{|z|=r\}$ onto a curve of length

$$L(r,f)=r\int_0^{2\pi}\left|f'\left(re^{it}\right)\right|dt\qquad(0<r<1).$$

It is known that, for $0<r<1$,

$$2^{-1}\pi r(1+r)(1-r)^{-2}<L(r,K)\leqslant\sup_{f\in S}L(r,f); \tag{3}$$

see [**2**, Theorem 2] and [**3**, p.39]. Now, as another application of the de Branges theorem we have

$$\max_{f\in S}\Delta(r,f)=\Delta(r,K)=\pi r^2\left(r^4+4r^2+1\right)\left(1-r^2\right)^{-4}, \tag{4}$$

for $0<r<1$. The maximum is attained only by K_θ's.

We improve (3) in

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Theorem 2. *For $0 < r < 1$ we have*

$$2\pi r\left(r^4 + 4r^2 + 1\right)^{1/2}\left(1 - r^2\right)^{-2} \leqslant L(r, K) \leqslant \sup_{f\in S} L(r, f). \tag{5}$$

Proof: This is a consequence of the expression of $\Delta(r, K)$ in (4), without appealing to the expression of $L(r, K)$ in terms of elliptic integrals (see [2]). We only apply to K the isoperimetric inequality:

$$\Delta(r, f) \leqslant \pi\{L(r, f)/(2\pi)\}^2 \text{ for } f \in S,$$

which says that, of all rectifiable Jordan curves with the given perimeter $L(r, f)$, $(0 < r < 1)$, the circle has interior of maximum area. □

Since

$$\inf_{0<r<1} \left(r^4 + 4r^2 + 1\right)^{1/2}(1 + r)^{-3} = \sqrt{6}/8 > 1/4,$$

estimate (5) is better than (3).

We recall that

$$L(r, f) \leqslant 2\pi r(1 - r)^{-2} \equiv \gamma(r)$$

for $f \in S$ and $0 < r < 1$ [3, p.40]. Estimate (5) now yields

$$\left(\sqrt{6}/4\right)\gamma(r) \leqslant \sup_{f\in S} L(r, f) \leqslant \gamma(r).$$

Note that $\gamma(r)$ is the length of the boundary circle of $\delta_r = \{|z| < r(1 - r)^{-2}\}$.

We recall the growth theorem for $f \in S$:

$$(1 + |z|)^{-2} \leqslant |F_f(z)| \leqslant (1 - |z|)^{-2}, \quad z \in D; \tag{6}$$

see [3, p.33]. The image $f(\{|z| < r\})(f \in S)$ is contained in the disc δ_r with area $\pi r^2(1 - r)^{-4}$ and

$$\Delta(r, f)/\{\pi r^2(1 - r)^{-4}\}$$

is at most:

$$\left(r^4 + 4r^2 + 1\right)(1 + r)^{-4}, \quad 0 < r < 1,$$

which decreases from 1 to 3/8 as r increases from 0 to 1. Therefore, one may say that the upper estimate of (6) becomes "worse" as r increases because $f(\{|z| < r\})$ occupies only a small part of δ_r in area. We next assume that F_f is nonconstant. The Riemann surface Φ_r (Φ_r^*, respectively) which is the image of $\{|z| < r\}$ by F_f ($1/F_f$, respectively), by (6), has projection contained in the disc with centre 0 and radius $(1-r)^{-2}$ ($(1+r)^2$, respectively). The "sheet–number" of the covering surface Φ_r (Φ_r^*, respectively) over this disc:

$$\Delta(r, F_f)/\{\pi(1-r)^{-4}\}\Big(\Delta(r, 1/F_f)/\{\pi(1+r)^4\}, \text{ respectively}\Big)$$

is at most $2r^2(r^2+2)(1+r)^{-4}$ which increases from 0 to $3/8$ as r increases from 0 to 1. In this sense (6) yields little information on the distribution of the values of $F_f(z)$ ($1/F_f(z)$, respectively), for $|z| < r$.

Let C be the family of all $f \in S$ such that $f(D)$ is convex. With the aid of the coefficient estimate [**3**, p.45, Corollary] we have

$$\max_{f\in C} \Delta(r, F_f) = \pi r^2(1-r)^{-2}, \qquad 0 < r < 1.$$

For each r, $0 < r < 1$, the maximum is attained only by $J_\theta(z) \equiv z/(1-e^{i\theta}z)$, $z \in D$, θ real. A natural conjecture is that

$$\max_{f\in C} \Delta(r, 1/F_f) = \pi r^2, \qquad 0 < r \leqslant 1,$$

where the maximum is attained only by J_θ's.

Remark. If $a_2 = 0$ in (1) for $f \in S$, then $F_f'(0) = a_2 = 0$, so that F_f is not univalent in any disc with centre 0. To consider the case $a_2 \neq 0$, we first note that the function

$$\varphi(x) = -\log\left(1-x^2\right) + \left(3x^2 - 2x^4\right)\left(1-x^2\right)^{-2}$$

increases from 0 to $+\infty$ as x increases from 0 to 1. Therefore there exists a number $R \equiv R(a_2)$, $0 < R < 1$, such that $\varphi(R) = |a_2|^2$. We shall show that F_f is univalent in $\{|z| < R(a_2)\}$. The expression for $1/F_f$ in the proof of Theorem 1 shows that

$$g(z) \equiv \{1 - 1/F_f(z)\}/a_2 = z - a_2^{-1}\sum_{n=2}^{\infty} b_{n-1}z^n \text{ in } D.$$

The Schwarz inequality, together with (2), yields that

$$\begin{aligned} &|a_2|^{-1}\sum_{n=2}^{\infty} n\,|b_{n-1}|\,R^{n-1} \\ &\leqslant |a_2|^{-1}\Big\{\sum_{n=2}^{\infty}(n-1)\,|b_{n-1}|^2\Big\}^{1/2}\Big\{\sum_{n=2}^{\infty} n^2(n-1)^{-1}\left(R^2\right)^{n-1}\Big\}^{1/2} \\ &\leqslant |a_2|^{-1}\varphi(R)^{1/2} = 1. \end{aligned}$$

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By [**3**, p.73, Problem 24 (b)] we have that $R^{-1}g(Rz)$ is univalent and starlike in D. Thus g, and hence F_f, are univalent in $\{|z| < R\}$ as we wished. We note that the image of $\{|z| < R\}$ under $1/F_f$ is starlike with respect to 1 also. Unfortunately we cannot claim that $R(a_2)$ is sharp. In fact, for the Koebe function K with $a_2 = 2$ we have

$$R(2) = 0.6823\ldots\ldots,$$

while F_K is univalent in D. Finally, since $|a_2| \leqslant 2$ for $f \in S$, we have $R(a_2) \leqslant R(2)$.

References

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Department of Mathematics
Tokyo Metropolitan University
Fukasawa, Setagaya, Tokyo 158
Japan