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ON REGULAR SUBGROUPS OF THE AFFINE GROUP

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Abstract

Catino and Rizzo ['Regular subgroups of the affine group and radical circle algebras', *Bull. Aust. Math. Soc.* **79** (2009), 103–107] established a link between regular subgroups of the affine group and the radical brace over a field on the underlying vector space. We propose new constructions of radical braces that allow us to obtain systematic constructions of regular subgroups of the affine group. In particular, this approach allows to put in a more general context the regular subgroups constructed in Tamburini Bellani ['Some remarks on regular subgroups of the affine group' *Int. J. Group Theory*, **1** (2012), 17–23].

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1. Introduction

Let V be a vector space over a field F. Clearly, the group $T(V)$ of all translations of V and any of its conjugate subgroups by an element of $GL(V)$ are abelian regular subgroups of the affine group $AGL(V)$. Caranti, Dalla Volta and Sala [3] obtained a simple description of the abelian regular subgroups of the affine group $AGL(V)$ in terms of commutative associative radical algebras with the underlying vector space V. As an application, they provide an example of an abelian regular subgroup of the affine group over an infinite vector space that trivially intersects the group of translations. It is well known that such an example cannot be given when the vector space is allowed to be finite-dimensional. To answer a question which appeared in [10], Hegedűs [8] constructed regular subgroups of some affine groups over a finite vector space containing only the trivial translation. Of course, his construction leads to nonabelian examples. Other examples can be found, for instance, in [7, 17]. A description of the regular subgroups, not necessarily abelian, of the affine group is obtained by the first author and Rizzo [4] in terms of radical circle algebras, a generalization of radical algebras. A vector space V over a field F with a multiplication $\cdot$ is called a *(right) circle algebra* if, for all $\alpha \in F$ and for all $u, v, w \in V$, the following statements hold:

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(1) $\alpha(u \cdot v) = (\alpha u) \cdot v$;
(2) $(u + v) \cdot w = u \cdot w + v \cdot w$;
(3) $u \cdot (v + w + v \cdot w) = u \cdot v + u \cdot w + (u \cdot v) \cdot w$.

It is clear that any associative algebra is a circle algebra and that any commutative circle algebra is an associative algebra. These new structures are very closely related to (right) braces, introduced by Rump [14] to find nondegenerate involutive set-theoretic solutions of the Yang–Baxter equation. So, using Rump's terminology [16], (right) circle algebras hereafter are called *(right) braces over* F or F*-braces*. As in an ordinary algebra, let us introduce the *circle operation* in an F-brace V defined by $u \circ v := u + v + u \cdot v$, for all $u, v \in V$. Then $(V, \circ)$ is a semigroup. In particular, if $(V, \circ)$ is a group, then we say that the F-brace V is *radical*. The main result of [4] establishes the following link between regular subgroups of the affine group $AGL(V)$ and F-brace structures with the underlying vector space V.

THEOREM 1.1 [4]. *Let V be a vector space over a field F. Denote by $\mathcal{RB}$ the class of radical F-braces with the underlying vector space V and by $\mathcal{T}$ the set of all regular subgroups of the affine group $AGL(V)$.*

(1) *Let $V^\bullet$ be a radical F-brace with underlying vector space V. Then $T(V^\bullet) = \{\tau_x | x \in V\}$, where $\tau_x : V \to V$, $y \mapsto y \circ x$ is a regular subgroup of the affine group $AGL(V)$.*
(2) *The map*
$$f : \mathcal{RB} \longrightarrow \mathcal{T}, \quad V^\bullet \longmapsto T(V^\bullet)$$
is a bijection. In this correspondence, isomorphism classes of F-braces correspond to conjugacy classes under the action of $GL(V)$ of regular subgroups of $AGL(V)$.

We note that a generalization of the above result to the holomorph of an abelian group has recently been obtained in [12].

The open problem of determining all regular subgroups of an affine group, formulated in [11], may be replaced by that of determining all radical F-braces. The new constructions of radical F-braces presented in this paper allow us to obtain systematic constructions of regular subgroups of the affine group. In particular, this approach allows to put in a more general context the regular subgroups constructed in [17].

2. Some preliminary results

Let F be a field and n a positive integer. We use the embedding of $AGL(n, F)$ into $GL(n + 1, F)$ as a parabolic subgroup corresponding to the partition of $n + 1$ as $(1, n)$. Thus, $AGL(n, F)$ acts on the right on the set of affine vectors $\Omega := \{(1, v) \mid v \in F^n\}$. Then, if T is a regular subgroup of $AGL(n, F)$, there exists a unique element γ_v of T such that the embedding in $GL(n + 1, F)$ has $(1, v)$ as first row, for every $v \in F^n$. Thus,

$$T = \left\{ \begin{pmatrix} 1 & v \\ 0 & \gamma_v \end{pmatrix} \;\middle|\; v \in F^n \right\}. \tag{2.1}$$

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By Theorem 1.1, we note that there is a unique F-brace $V^\bullet$ such that $T = T(V^\bullet)$. Moreover, the multiplication on the F-space V is,

$$\forall x, y \in V, \quad x \cdot y = x(\gamma_y) - x.$$

Conversely, if $V^\bullet$ is an F-brace and $\gamma_v : V \to V,\ x \mapsto x \cdot v + x$, then the regular subgroup $T(V^\bullet)$ has the description given for T in (2.1).

These remarks allow a different and more precise version of [17, Corollary 5.3]. Indeed, by the classification of nilpotent associative algebras of dimension 3 over an arbitrary field [6], we have the following proposition:

Proposition 2.1. *Every abelian regular subgroup of $AGL(3, F)$ is conjugate either to the group of translations or to exactly one of the following subgroups:*

$$\left\{ \left. \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & x_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| x_1, x_2, x_3 \in F \right\},$$

$$\left\{ \left. \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & x_1 & x_2 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| x_1, x_2, x_3 \in F \right\},$$

$$\left\{ \left. \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & \alpha x_1 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| x_1, x_2, x_3 \in F,\ \alpha \in R \right\}.$$

where $R \subseteq \dot{F}$ is a transversal of $\dot{F}/\dot{F}^2$.

Clearly, we may construct nonabelian regular subgroups of $AGL(3, F)$, for any field F. For example, choosing as an F-brace the (noncommutative) nilpotent algebra $A_{3,5}$ of [6], we have the subgroup

$$\left\{ \left. \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & -x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| x_1, x_2, x_3 \in F \right\}.$$

Observe that [4, Corollary 2] gives a description of the intersection with the group of translations of a regular subgroup obtained from an F-brace by means of considering left annihilators. We state this as our next result:

Proposition 2.2. *Let $V^\bullet$ be a radical F-brace with the underlying vector space V over a field F. Let $T(V^\bullet) = \{\tau_x | x \in V\}$, where $\tau_x : V \to V, y \mapsto y \circ x$, and let $T(V)$ be the group of translations. Then*

$$T(V) \cap T(V^\bullet) = \{\tau_x | x \in \mathrm{Ann}_L(V^\bullet)\}$$

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where $\mathrm{Ann}_L(V^\bullet)$ *is the set of the left annihilators of the radical* F*-brace* $V^\bullet$*, that is,* $\mathrm{Ann}_L(V^\bullet) := \{x \mid \ x \in V,\ \textit{for all } v \in V\ \ x \cdot v = 0\}$*. In particular, if* $V^\bullet$ *is a radical associative algebra of finite dimension, then* $T(V) \cap T(V^\bullet) \neq 1$.

Similarly to the situation in an ordinary algebra, we define the set of *right annihilators* of an F-brace V as follows:

$$\mathrm{Ann}_R(V) := \{x \mid \ x \in V,\ \forall v \in V,\ \forall \lambda \in F,\ v \cdot (\lambda x) = 0\}.$$

The set $\mathrm{Ann}(V) := \mathrm{Ann}_L(V) \cap \mathrm{Ann}_R(V)$ will be called the *annihilator* of the F-brace V. Of course, this is an ideal of V.

Definition 2.3. Let V be a brace over a field F. Then a subspace I of V is said to be an *ideal* of V if, for all $a, b \in V$ and $x \in I$, the following properties hold:

(1) $x \cdot a \in I$;
(2) $a \cdot (x + b) - a \cdot b \in I$.

Given a radical brace V over a field F, a subspace I of V is an ideal if and only if $x \cdot a \in I$ and $a \cdot x \in I$, for all $a \in V$ and $x \in I$ (see [14]).

Proposition 2.4. *Let* V *be a brace over a field* F *such that* $(V, \circ)$ *is a monoid. Then* $\mathrm{Ann}(V)$ *is an ideal of* V.

Proof. Note that $0 \in \mathrm{Ann}(V)$ since $(V, \circ)$ is a monoid. Let $c, d \in \mathrm{Ann}(V)$. Then

$$\begin{aligned} x \cdot (\lambda(c + d)) &= x \cdot (\lambda c + \lambda d) = x \cdot (\lambda c + \lambda d + (\lambda c) \cdot (\lambda d)) \\ &= x \cdot (\lambda c) + x \cdot (\lambda d) + (x \cdot (\lambda c)) \cdot (\lambda d) = 0, \\ (c + d) \cdot x &= c \cdot x + d \cdot x = 0, \\ x \cdot (\lambda(-c)) &= x \cdot (\lambda((-1)c)) = x \cdot ((-\lambda)c) = 0, \end{aligned}$$

for all $x \in V$ and $\lambda \in F$. Now let $c \in \mathrm{Ann}(V)$ and $\mu \in F$. Then

$$(\mu c) \cdot x = \mu (c \cdot x) = 0$$

and

$$x \cdot (\lambda (\mu c)) = x \cdot ((\lambda\mu) c) = 0,$$

for all $x \in V, \lambda \in F$, that is, $\mu c \in \mathrm{Ann}(V)$. Therefore $\mathrm{Ann}(V)$ is a subspace of V. Moreover, let $a, b \in V$ and $c \in \mathrm{Ann}(V)$. Then

$$\begin{aligned} x \cdot (\lambda(c \cdot a)) &= 0, \\ (c \cdot a) \cdot x &= 0, \\ x \cdot (\lambda(a \cdot (c + b) - a \cdot b)) &= x \cdot (\lambda(a \cdot (c + b + c \cdot b) - a \cdot b)) \\ &= x \cdot (\lambda(a \cdot c + a \cdot b + (a \cdot c) \cdot b - a \cdot b)) \\ &= x \cdot (\lambda(a \cdot c + (a \cdot c) \cdot b)) = x \cdot (\lambda 0) = 0, \\ (a \cdot (c + b) - a \cdot b) \cdot x &= (a \cdot (c + b + c \cdot b) - a \cdot b) \cdot x \\ &= (a \cdot c + a \cdot b + (a \cdot c) \cdot b - a \cdot b) \cdot x \\ &= (a \cdot c + (a \cdot c) \cdot b) \cdot x = 0 \cdot x = 0, \end{aligned}$$

for all $x \in V$ and $\lambda \in F$. □

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3. Hochschild product

In this section we exhibit a description of all finite-dimensional radical (right) F-braces with nontrivial annihilator. In this way, we obtain examples of regular subgroups of the affine group with nontrivial intersection with the translation group. For this purpose, we introduce some cohomological tools in analogy with the method employed by de Graaf in [6] for the classification of nilpotent associative algebras of dimensions 2 and 3 over any field. We translate the concepts of 2-*cocycles*, *coboundaries* and the *Hochschild product* from the context of associative algebras into that of F-braces (see [2, 13]).

Definition 3.1. Let A be an F-brace and V a vector space over a field F. A map $\theta : A \times A \to V$ with the properties:

(1) $(\lambda a + \mu b, c)\theta = \lambda((a, c)\theta) + \mu((b, c)\theta)$,

(2) $(a, b + c + b \cdot c)\theta = (a, b)\theta + (a, c)\theta + (a \cdot b, c)\theta$,

for all $a, b, c \in A$ and $\lambda, \mu \in F$, is called a 2-*cocycle* of A with values in V.

Thus 2-cocycles of F-algebras (see [13]) are particular cases of 2-cocycles of F-braces. But an F-algebra, viewed as F-brace, may have more 2-cocycles, in the sense of Definition 3.1.

Example 3.2. Let N be the zero algebra of dimension n over a field F and τ an endomorphism of the additive group of F. Then the map

$$\theta : N \times N \longrightarrow F, \qquad \left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} y_i e_i\right) \longmapsto \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)\tau$$

is a 2-cocycle of the F-brace N but, in general, not of the F-algebra N. In particular, θ is a 2-cocycle of the n-dimensional zero algebra if and only if τ is linear.

Definition 3.3. Let A be an F-brace, V an F-vector space and $\theta : A \oplus A \to V$ a 2-cocycle. Put $A_\theta := A \oplus V$. For all $a, b \in A$ and $v, w \in V$ we define

$$(a + v) \cdot (b + w) := a \cdot b + (a, b)\theta.$$

The F-brace A_θ is called a *Hochschild product* of A by V, a terminology which refers to a special case considered by Hochschild [9, proof of Theorem 6.2] (see [2]).

Obviously, different 2-cocycles can lead to Hochschild products that are isomorphic to each other.

We observe that the passage from A to A_θ preserves the property of being radical.

Proposition 3.4. *Let A be a radical F-brace and θ a 2-cocycle of A with values in an F-vector space V. Then A_θ is radical.*

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PROOF. Let $x \in A_\theta$ and $a \in A, v \in V$ such that $x = a + v$. Let a^- be the inverse of a with respect to the adjoint operation. Then

$$y := a^- + (-v - \theta(a, a^-))$$

is the inverse of x with respect to the adjoint operation in A_θ. □

The following elementary assertion will be useful for the proof of Theorem 3.6.

LEMMA 3.5. *Let A be an F-brace such that $(A, \circ)$ is a monoid. Then*

$$x \cdot (y + c) = x \cdot y$$

for all $x, y \in V$ and $c \in \mathrm{Ann}(V)$.

PROOF. Let $x, y \in V$ and $c \in \mathrm{Ann}(V)$. Then

$$x \cdot (y + c) = x \cdot (y + c + y \cdot c) = x \cdot y + x \cdot c + (x \cdot y) \cdot c = x \cdot y. \qquad \square$$

THEOREM 3.6. *Let B be a radical F-brace such that $\mathrm{Ann}(B) \neq \{0\}$. Then there exist an F-brace A, an F-vector space V and a 2-cocycle $\theta : A \times A \to V$ such that B is isomorphic to A_θ.*

PROOF. Let $V := \mathrm{Ann}(B)$, $A := B/\mathrm{Ann}(B)$ and $\pi : B \to A$ be the projection map. Choose a linear map $\sigma : A \to B$ such that $(x\sigma)\pi = x$, for all $x \in A$. Clearly, σ is injective. Moreover, we have

$$(x\sigma \cdot y\sigma - (x \cdot y)\sigma)\pi = x \cdot y - x \cdot y = 0,$$

for all $x, y \in A$, that is, $x\sigma \cdot y\sigma - (x \cdot y)\sigma \in V$. Thus we obtain a function θ from $A \times A$ into V by defining

$$(x, y)\theta := x\sigma \cdot y\sigma - (x \cdot y)\sigma.$$

We prove that θ is a 2-cocycle. It is easy to check that θ is linear in the first variable. Now let $x, y, z \in A$. Since $(y, z)\theta \in V$, by Lemma 3.5, we have

$$\begin{aligned}
x\sigma \cdot (y\sigma + z\sigma + (y \cdot z)\sigma) &= x\sigma \cdot (y\sigma + z\sigma + (y \cdot z)\sigma + y\sigma \cdot z\sigma - y\sigma \cdot z\sigma) \\
&= x\sigma \cdot (y\sigma + z\sigma + y\sigma \cdot z\sigma - (y\sigma \cdot z\sigma - (y \cdot z)\sigma)) \\
&= x\sigma \cdot (y\sigma + z\sigma + y\sigma \cdot z\sigma - (y, z)\theta) \\
&= x\sigma \cdot (y\sigma + z\sigma + y\sigma \cdot z\sigma) \\
&= x\sigma \cdot y\sigma + x\sigma \cdot z\sigma + (x\sigma \cdot y\sigma) \cdot z\sigma.
\end{aligned}$$

Since $(x, y)\theta \in V$, we also obtain

$$\begin{aligned}
(x\sigma \cdot y\sigma) \cdot z\sigma &= -(x, y)\theta \cdot z\sigma + (x\sigma \cdot y\sigma) \cdot z\sigma \\
&= -(x\sigma \cdot y\sigma - (x \cdot y)\sigma) \cdot z\sigma + (x\sigma \cdot y\sigma) \cdot z\sigma \\
&= (x \cdot y)\sigma \cdot z\sigma + (x\sigma \cdot y\sigma) \cdot z\sigma - (x\sigma \cdot y\sigma) \cdot z\sigma \\
&= (x \cdot y)\sigma \cdot z\sigma.
\end{aligned}$$

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Making use of these two equalities, we obtain

$$\begin{aligned}
&(x, y+z+y\cdot z)\theta \\
&\quad = x\sigma\cdot(y+z+y\cdot z)\sigma - (x\cdot(y+z+y\cdot z))\sigma \\
&\quad = x\sigma\cdot(y\sigma+z\sigma+(y\cdot z)\sigma) - ((x\cdot y)\sigma+(x\cdot z)\sigma+((x\cdot y)\cdot z)\sigma) \\
&\quad = x\sigma\cdot y\sigma + x\sigma\cdot z\sigma + (x\sigma\cdot y\sigma)\cdot z\sigma - (x\cdot y)\sigma - (x\cdot z)\sigma - ((x\cdot y)\cdot z)\sigma \\
&\quad = x\sigma\cdot y\sigma + x\sigma\cdot z\sigma + (x\cdot y)\sigma\cdot z\sigma - (x\cdot y)\sigma - (x\cdot z)\sigma - ((x\cdot y)\cdot z)\sigma \\
&\quad = (x,y)\theta + (x,z)\theta + (x\cdot y, z)\theta.
\end{aligned}$$

Furthermore, B is isomorphic to A_θ, because

$$\phi : B \longrightarrow A_\theta, \quad x \longmapsto (x+\mathrm{Ann}(B), x-(x+\mathrm{Ann}(B))\sigma)$$

is an isomorphism of F-braces. Obviously θ is bijective and linear.

Let $x_1, x_2 \in B$ and $y_1 := x_1+\mathrm{Ann}(B)$, $y_2 := x_2+\mathrm{Ann}(B)$, $z_1 := x_1 - y_1\sigma$ and $z_2 := x_2 - y_2\sigma$. Then

$$(x_1\phi)\cdot(x_2\phi) = (y_1+z_1)\cdot(y_2+z_2) = y_1\cdot y_2 + (y_1,y_2)\,\theta.$$

This equality and Lemma 3.5 imply that

$$\begin{aligned}
(x_1\phi)\cdot(x_2\phi) &= y_1\cdot y_2 + (y_1,y_2)\theta \\
&= ((y_1\cdot y_2)\sigma + (y_1,y_2)\theta)\phi \\
&= (y_1\sigma\cdot y_2\sigma)\phi \\
&= (y_1\sigma\cdot y_2\sigma + z_1\cdot y_2\sigma)\phi \\
&= ((y_1\sigma+z_1)\cdot y_2\sigma)\phi \\
&= ((y_1\sigma+z_1)\cdot(y_2\sigma+z_2))\phi \\
&= (x_1\cdot x_2)\phi.
\end{aligned}$$

This concludes the proof. □

The following result shows that the maps in (3.2) are the unique 2-cocycles of a one-dimensional zero algebra. Hence, by Theorem 3.6, we obtain all two-dimensional F-braces with nontrivial annihilator.

Proposition 3.7. *Let N be the zero algebra of dimension one over a field F with a basis (e_1) and let θ be a 2-cocycle on N. Then there exists $\tau \in \mathrm{Aut}(F^+)$ such that*

$$\forall x_1,\ y_1 \in F, \quad (x_1e_1, y_1e_1)\theta = x_1(y_1\tau).$$

Proof. Let

$$\tau : F \longrightarrow F, \quad x \longmapsto (e_1, xe_1)\theta.$$

It follows that

$$\begin{aligned}
(x+y)\tau = (e_1,(x+y)e_1)\theta &= (e_1, xe_1+ye_1)\theta \\
&= (e_1, xe_1+ye_1+(xe_1)\cdot(ye_1))\theta \\
&= (e_1,xe_1)\theta + (e_1,ye_1)\theta + (e_1\cdot xe_1, ye_1)\theta \\
&= (e_1,xe_1)\theta + (e_1,ye_1)\theta + (0,ye_1)\theta \\
&= (e_1,xe_1)\theta + (e_1,ye_1)\theta = x\tau + y\tau,
\end{aligned}$$

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for all $x, y \in F$, and so $\tau \in \mathrm{Aut}(F^+)$. Finally,

$$(x_1 e_1, y_1 e_1)\theta = x_1((e_1, y_1 e_1)\theta) = x_1(y_1\tau),$$

for all $x_1, y_1 \in F$. □

Clearly, the previous proposition can also be extended to the zero algebra of dimension $n \geq 2$.

Unfortunately, Theorem 3.6 is not exhaustive because there are examples of F-braces with trivial annihilator, as shown by the following construction. Let $F := \mathbb{R}$ be the field of real numbers and let ϵ be a homomorphism from the additive group of $\mathbb{R}$ to its multiplicative group. We define on V the operation

$$(x_1 e_1 + x_2 e_2) \cdot (y_1 e_1 + y_2 e_2) := x_1(y_2\epsilon - 1)e_1$$

for all $x_1, x_2, y_1, y_2 \in F$. Then V is a radical F-brace. We note that the homomorphism ϵ is either the null function or

$$\epsilon : \mathbb{R} \longrightarrow \mathbb{R}, \quad y \longmapsto e^{(yf)},$$

where f is an additive endomorphism of $\mathbb{R}$ (see [1]). It is easy to check that if ϵ is not null, then $\mathrm{Ann}(V)$ is trivial.

4. Semidirect products of F-braces

The aim of this section is to exhibit other examples of F-braces by means of a construction that is a generalization of the *semidirect product of braces* introduced in [15]. (See also the version in [5].)

Let A and V be radical F-braces of dimension m and n, respectively. Let $\alpha : A^\circ \to GL(V) \cap \mathrm{Aut}(V^\circ)$ be a group homomorphism. For all $v \in V$, $b \in A$, we set $v^b := v(b\alpha)$. On the vector space $A \oplus V$, define the product

$$(a, v) \cdot (b, w) := (a \cdot b, v^b + v^b \cdot w - v),$$

for all $a, b \in A$ and $v, w \in V$, giving a radical F-brace. The properties for a radical brace follow from [5]. Moreover,

$$\begin{aligned}
\lambda((a, v) \cdot (b, w)) &= \lambda(a \cdot b, v^b + v^b \cdot w - v) = (\lambda(a \cdot b), \lambda(v^b + v^b \cdot w - v)) \\
&= ((\lambda a) \cdot b, \lambda(v^b) + \lambda(v^b \cdot w) - \lambda v) \\
&= ((\lambda a) \cdot b, \lambda(v^b) + (\lambda v^b) \cdot w - \lambda v) \\
&= ((\lambda a) \cdot b, (\lambda v)^b + (\lambda v)^b \cdot w - \lambda v) \\
&= (\lambda a, \lambda v) \cdot (b, w) = (\lambda(a, v)) \cdot (b, w),
\end{aligned}$$

for all $\lambda \in F$, $a, b \in A$ and $v, w \in V$. We call this radical F-brace a *semidirect product of the F-braces A and V* and denote it by $A \ltimes_\alpha V$.

Among others, the examples in [17, after (1.2)] may be obtained by means of our construction of semidirect products of F-braces. Thus, let F be a field and N_1, N_n the zero algebras on F of dimension 1 and n, respectively. Let $i, j \in \{1, 2, \dots, n\}$, $i \neq j$, and $\alpha_x : N_n \to N_n$ such that

$$\forall v \in N_n, \quad v\alpha_x = v(I_n + xE_{ij}).$$

Clearly, $\alpha_x \in GL(N_n) \cap \mathrm{Aut}(N_n^\circ)$, for all $x \in N_1$. Let

$$\alpha : N_1^\circ \to GL(N_n) \cap \mathrm{Aut}(N_n^\circ), \quad x \mapsto \alpha_x.$$

We have that

$$\begin{aligned} v((x \circ y)\alpha) = v\alpha_{x\circ y} = v\alpha_{x+y} &= v(I_n + (x + y)E_{ij}) \\ &= v(I_n + xE_{ij} + yE_{ij}) = (v(I_n + xE_{ij}))\alpha_y \\ &= (v\alpha_x)\alpha_y = v(x\alpha)(y\alpha), \end{aligned}$$

that is, α is a homomorphism of right braces. Consequently, $V := N_1 \ltimes_\alpha N_n$ is a radical F-brace of dimension $n + 1$ on F. Note that

$$\begin{aligned} (y, w)\gamma_{(x,v)} &= (y, w) \cdot (x, v) + (y, w) = (y \cdot x, w^x \cdot v + w^x - w) + (y, w) \\ &= (y \cdot x + y, w^x \cdot v + w^x) = (y, w^x) = (y, w(I_n + xE_{hk})). \end{aligned}$$

Therefore the regular subgroup of $AGL_{n+1}(V)$ associated to the radical F-brace $N_1 \ltimes_\alpha N_n$ is

$$\left\{ \left(\begin{matrix} 1 & x & v \\ 0 & 1 & 0_n \\ 0_n^t & 0_n^t & I_n + xE_{ij} \end{matrix} \right) \middle| \; x \in F, v \in N_n \right\}.$$

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