EVOLUTION OF TURBULENT MAGNETIC FIELDS – APPROACH TO A STEADY STATE

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Abstract. The dynamical evolution of a weak, random, magnetic excitation in a turbulent electrically-conducting fluid is examined under varying kinematic conditions. It is found that the results of an earlier paper (Kraichnan and Nagarajan, 1967) can be reliably extended to a stage of evolution wherein the magnetic spectrum has reached local equipartition with the velocity. The transfer of the magnetic energy to smaller wavenumbers (larger scales) is considerable and significant. This result is highly pertinent to the *turbulent dynamo* question, which has been variously investigated recently. The relevance of the coupling of the rms magnetic field to the magnetic modes of all scales in deciding the efficiency of this transfer is discussed.

1. Introduction and Review

In a number of recent investigations, (Parker, 1970; Moffatt, 1970; Parker, 1969; Krause, 1968; Rädler, 1968; Steenbeck *et al.*, 1966; Steenbeck and Krause, 1966, 1967; Krause and Rädler, 1971; Fitremann and Frisch, 1969; Vainshtein, 1970), the question of regeneration of a magnetic field, by turbulent motions has been reconsidered, under a variety of kinematic assumptions about the turbulence. In an earlier paper (Kraichnan and Nagarajan, 1967), we have reviewed the previous work on this subject in great detail and found that simple intuitive statistical arguments like equipartition, or analogical and heuristic kinematic considerations like the vorticity analogy are highly inadequate in resolving this question. In a recent paper, Kraichnan (1970) has considered the analogous question of the growth and propagation of the deviations between the point-to-point velocity fields in two flow systems, which are statistically identical. Here again, one finds that the ultimate evolution depends on the quantitative competition between the local-enhancement and sweeping-away processes in the wave-number domain. One needs a considerable amount of knowledge of the internal dynamics and characteristic times, and assertions of kinematic nature based on universal equilibrium hypotheses are highly inadequate.

In our paper referred to earlier, we could not carry our calculations very much forward in time, because we had no reliable information about the internal time structure of the combined fields of velocity and magnetic field, at that time. In a more recent paper (Nagarajan, 1971), we have investigated the internal structure of the steady state spectra on the basis of a detailed *dynamical* theory. In this, we have also reviewed the relevance of the ideas of Kolmogorov to the hydromagnetic case, keeping in mind the Galilean non-invariance of the hydromagnetic equations to a random constant magnetic field transformation. The cascade of energy in the hydromagnetic case is not strictly local in the wave number domain. A large scale rms magnetic field presents the possibility of Alfvén wave propagation along it and thus provides a significant dynamical coupling between magnetic fields of large and small

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scales. Our steady state considerations provide us with the necessary information about the local internal relaxation features and their relative magnitudes, so much so we plan to extend our earlier study of evolution of weak magnetic fields – to a stage in which the spectrum of the magnetic field has evolved sufficiently to a point of dynamical feedback to the velocity field and consequently a statistical steady-state.

And since we are basing our calculations on a well-considered dynamical theory of turbulence, we will be able to throw some light on the nature of the transfer of energy in the magnetic spectrum: in particular, without using either oversimplifications or idealisations of the characteristic length and time scales of the magnetic field and turbulence as have been done by Moffatt (1970), Parker (1969), Fitremann and Frisch (1969) or Vainshtein (1970).

2. The Dynamical Model

We start with a steady turbulence with an extended inertial range. The choice of the kinematic parameters and the wave number range is made suitably, so that we can talk of an extended equilibrium range, without worrying about the sources of input of energy into the system from the geometric range. Further, there exists a sufficiently noticeable dissipative tail to the spectrum at the high wave number end. The form of the spectrum and parameters are chosen so as to be compatible with the asymptotic requirements of the direct interaction approximation of Kraichnan (1958, 1959, 1965, 1966), with suitable modifications to reproduce Kolmogorov scaling.

A disturbance which is localized in the wave number range of the magnetic spectrum is introduced at time $t=0$.

Following the notations of our earlier papers (Kraichnan, 1958; Kraichnan and Nagarajan, 1967; Nagarajan, 1971), we can write the equation for the secular evolution of the two spectra for times >0 as

$$\begin{aligned}\left(\frac{\partial}{\partial T} + 2\nu k^2\right) E^V(k;T) &= \iint \frac{k}{2pq}\, dp\, dq \left[\left\{k^2 a_{kpq} E^V(p;T)\, E^V(q;T)\, \theta^{VVV}_{kpq}\right.\right. \\ &\quad \left. - p^2 b_{kpq} E^V(k;T)\, E^V(q;T)\, \theta^{VVV}_{pqk}\right\} \\ &\quad + \left\{k^2 a_{kpq} E^M(p;T)\, E^M(q;T)\, \theta^{VMM}_{kpq}\right. \\ &\quad \left.\left. - p^2 C_{kpq} E^V(k;T)\, E^M(q;T)\, \theta^{MMV}_{pqk}\right\}\right] \end{aligned} \tag{1}$$

$$\begin{aligned}\left(\frac{\partial}{\partial T} + 2\lambda k^2\right) E^M(k;T) &= \iint_{\Delta} \frac{k}{2pq}\, dp\, dq \left[k^2 d_{kpq} E^M(p;T)\, E^V(q;T)\, \theta^{MMV}_{kpq}\right. \\ &\quad - p^2 h_{kpq} E^M(k;T)\, E^V(q;T)\, \theta^{MVM}_{pqk} \\ &\quad \left. - p^2 j_{kpq} E^M(k;T)\, E^M(q;T)\, \theta^{VMM}_{pqk}\right] \end{aligned} \tag{2}$$

The spectral functions $E^V(k;T)$ and $E^M(k;T)$ are connected to the velocity and magnetic fields as follows:

$$W^V(k;t,t') = (2\pi)^{-3}\int d^3(\mathbf{x}-\mathbf{y})\,\langle \mathbf{U}(\mathbf{x};t)\cdot\mathbf{U}(\mathbf{y};t')\rangle\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

$$W^M(k;t,t') = (2\pi)^{-3}\int d^3(\mathbf{x}-\mathbf{y})\,\langle \mathbf{W}(\mathbf{x};t)\cdot\mathbf{W}(\mathbf{y};t')\rangle\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

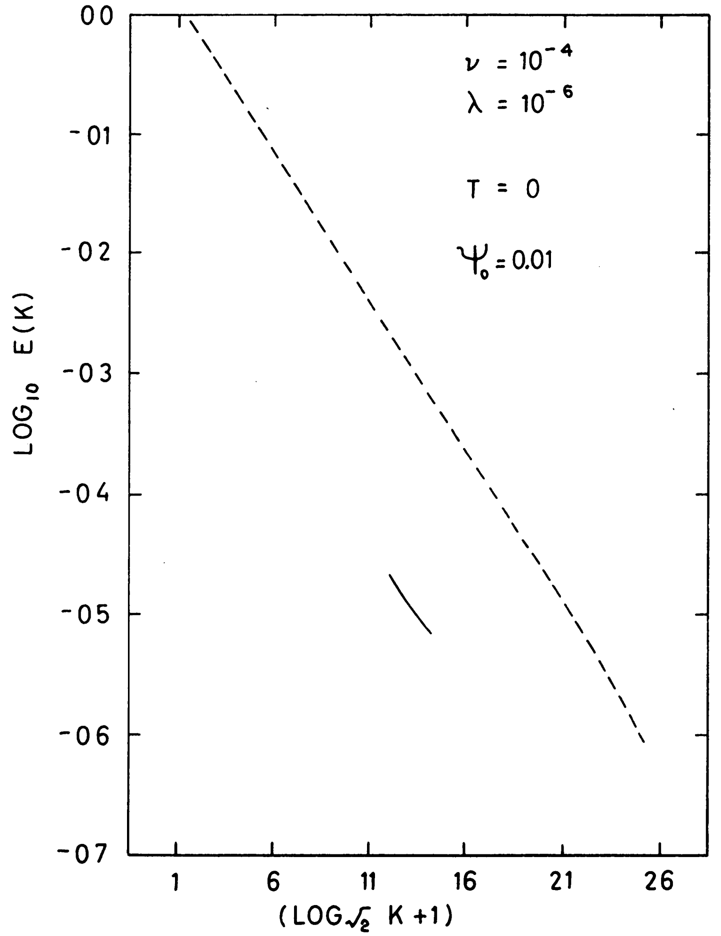

Fig. 1.

where $\mathbf{U}(\mathbf{x}, t)$ is the fluid velocity and $(4\pi\mu\varrho)^{1/2}\,\mathbf{W}(\mathbf{x}, t)$ is the magnetic induction field, ϱ is the fluid density, μ the magnetic susceptibility of the fluid, ν and λ are the kinematic viscosity and magnetic diffusivity respectively.

We assume the turbulence to be homogeneous and isotropic

$$\tfrac{1}{2}W^V(k;t,t') = (4\pi k^2)^{-1}\, E^V\left(k;\frac{t+t'}{2}\right) R^V(k;t-t').$$

$$\tfrac{1}{2}W^M(k;t,t') = (4\pi k^2)^{-1}\, E^M\left(k;\frac{t+t'}{2}\right) R^M(k;t-t')$$

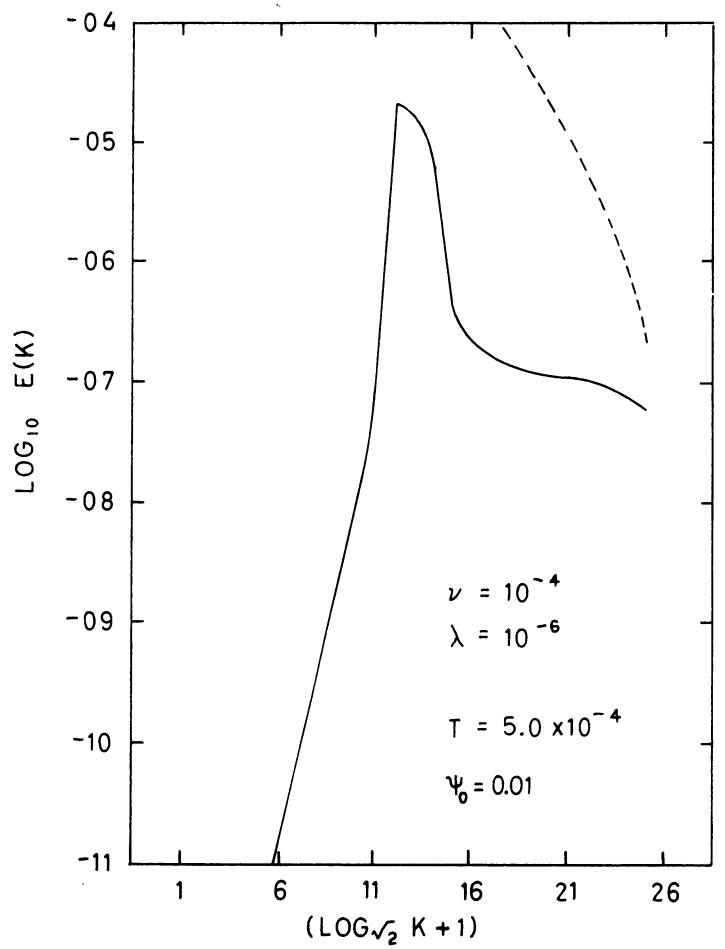

Fig. 2.

where $W^V\{\ \}$ and $W^M\{\ \}$ are energy functions and $R^V\{\ \}$ and $R^M\{\ \}$ are modal correlation functions.

The θ'-s which appear in Equations (1) and (2) are the effective memory times of the interaction between the three respective wave numbers. They are given by

$$\theta_{lmn}^{abc}(T) = \int_{-\infty}^{\infty} G_l^a(T-s)\, R_m^b(T+s)\, R_n^c(T+s)\, ds$$

(where a, b, $c=V$ or M) and $G^V(k;T)$ and $G^M(k;T)$ are the averaged response functions of the velocity and magnetic fields for the given wave number respectively.

In a general turbulent system in which a weak macroscopic (i.e. geometric range) disturbance in the magnetic spectrum is introduced at time $t=0$, the θ'-s will be very complicated functions of the correlation and response features of the turbulence

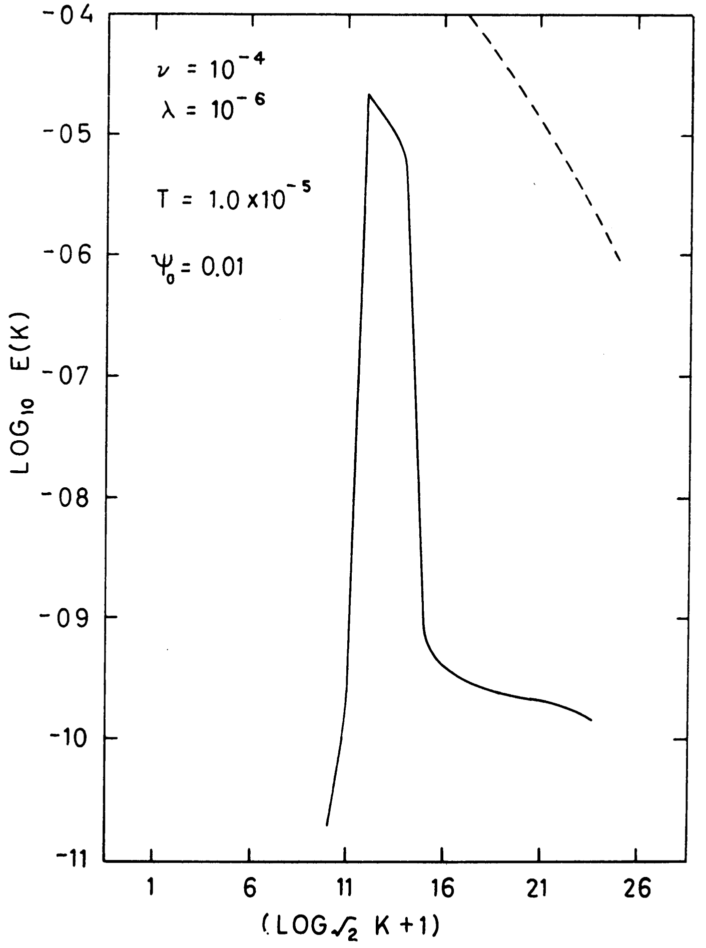

Fig. 3.

and initial magnetic field. But if we assume that the weak magnetic excitation is sufficiently localized in the inertial range, the secular time dependence of the θ'-s can be ignored. This point has been discussed in detail by Kraichnan (1959) in the hydrodynamic context. In the magnetic situation also much of the argument goes through unaltered.

We choose a form for the correlation and relaxation functions and the θ'-s from Nagarajan, 1971.

$$R^a(k;t) = \exp\{-\tfrac{1}{4}\pi(\zeta_a(k)\,t)^2\}$$
$$G^a(k;T) = \exp\{-\tfrac{1}{4}\pi(\eta_a(k)\,t)^2\}$$

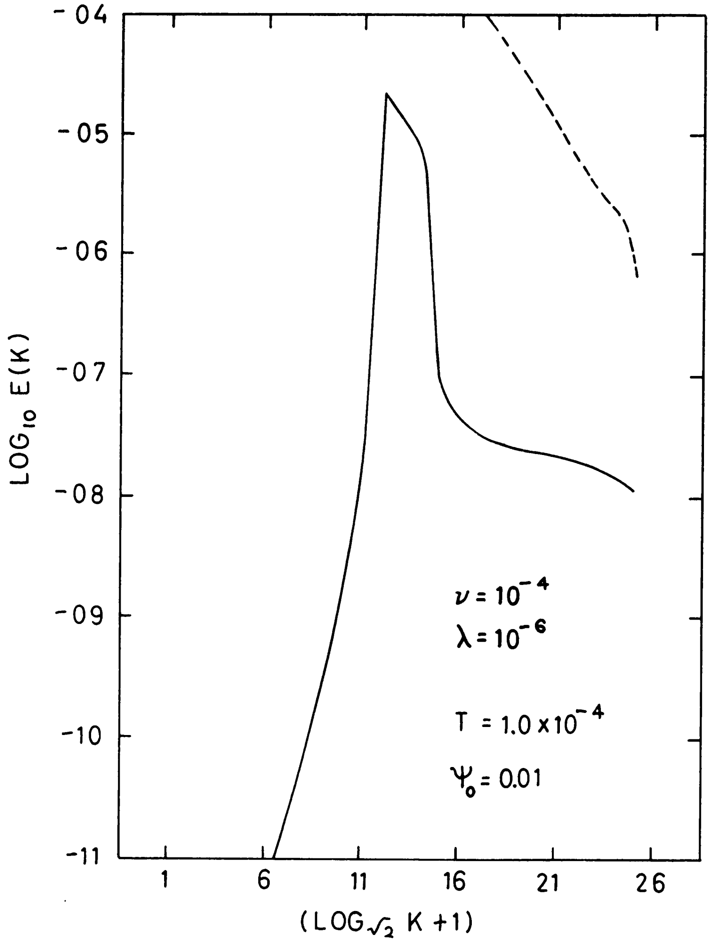

Fig. 4.

which gives for θ

$$\theta^{abc}_{kpq} = [\{\eta_a(k)\}^2 + \{\zeta_b(p)\}^2 + \{\zeta_c(q)\}^2]^{-1/2}.$$

Our elaborate study of the various extreme considerations of Galilean-invariance and Kolmogorov's arguments on the one hand and Galilean non-invariant Eulerian solutions on the other in the steady-state case (Nagarajan, 1971) convinces us that in so far as energy transfer information is concerned, the details of the internal correlation times are not very important. Using the results of this study, we evolve a quasi-Lagrangian scheme. We take the velocity correlations and relaxations to be Kolmogorovian i.e. decided by the local parameters of the position in the wave number spectrum. The magnetic terms are modulated by energy range parameters as in the unmodified direct interaction approximation of Kraichnan (1959, 1965). With

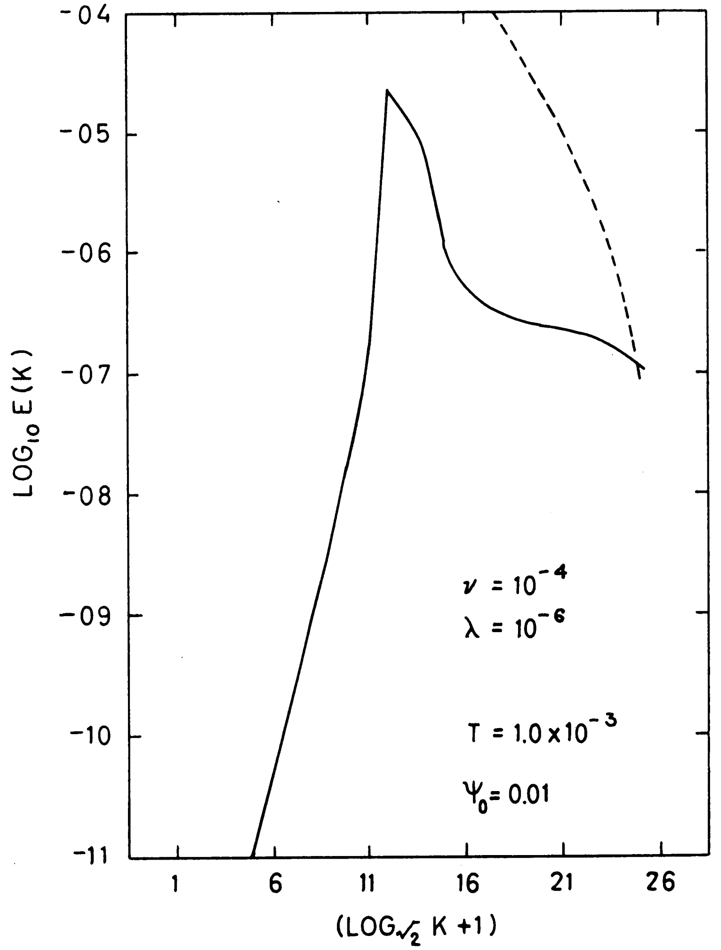

Fig. 5.

these preliminaries one can write

$$\zeta_V(k) = [E^V(k;T)\,k^3]^{1/2}$$
$$\eta_V(k) = [\{\zeta_V(k)\}^2 + (\nu k^2)^2]^{1/2}$$
$$\zeta_m(k) = (v_0 k)$$
$$\eta_m(k) = [\{\zeta_m(k)\}^2 + (\lambda' k^2)^2]^{1/2}.$$

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Here v_0 is the rms velocity in the energy range. (It will be apparent that this energy-range mixing was the reason why we chose the initial magnetic excitation to be localized in the inertial range. But for that the results of the hydrodynamic case or even the steady-state study will be inapplicable.) We choose a convenient unit of wave numbers and time scales such that $v_0 = 1$.

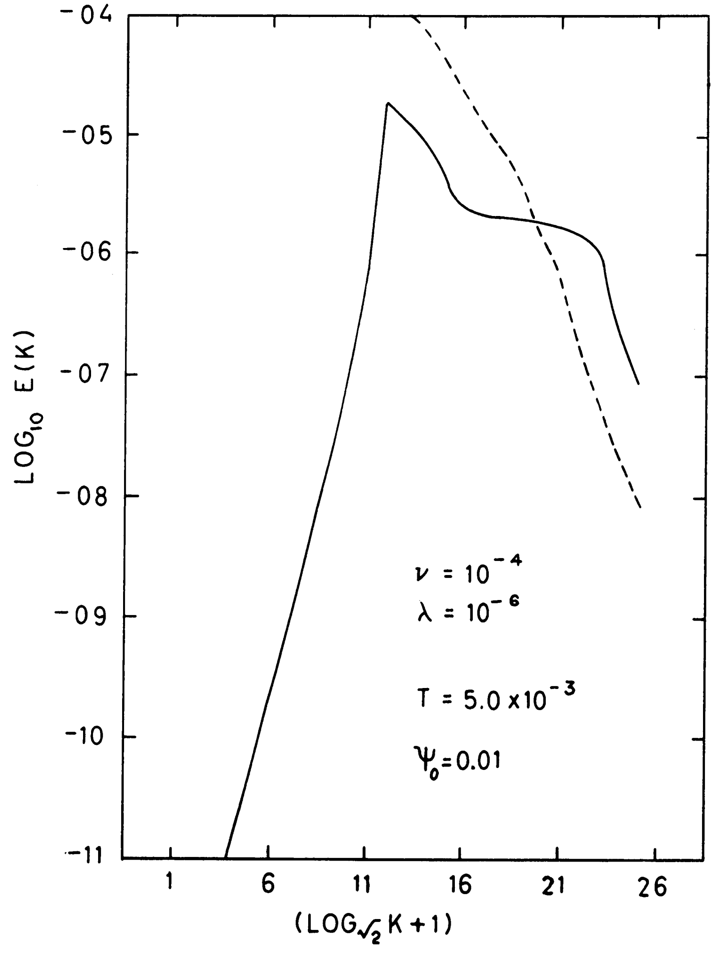

Fig. 6.

3. Evolution Study

Now that all the quantities in Equations (1) and (2) are completely defined, we integrate them forward in time. In time, they have the character of a set of non-linear coupled differential equations. But for each time value there is an integral to be per-

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formed over the contributions from various regions of wave number space. We discretise the wave number region into twenty-five logarithmic half-octave intervals.

The details of this procedure are much the same as in an earlier paper (Nagarajan, 1971). We perform the time integration using a fourth-order variable-step Runge-Kutta Scheme. The details of the numerical scheme are given elsewhere (Nagarajan,

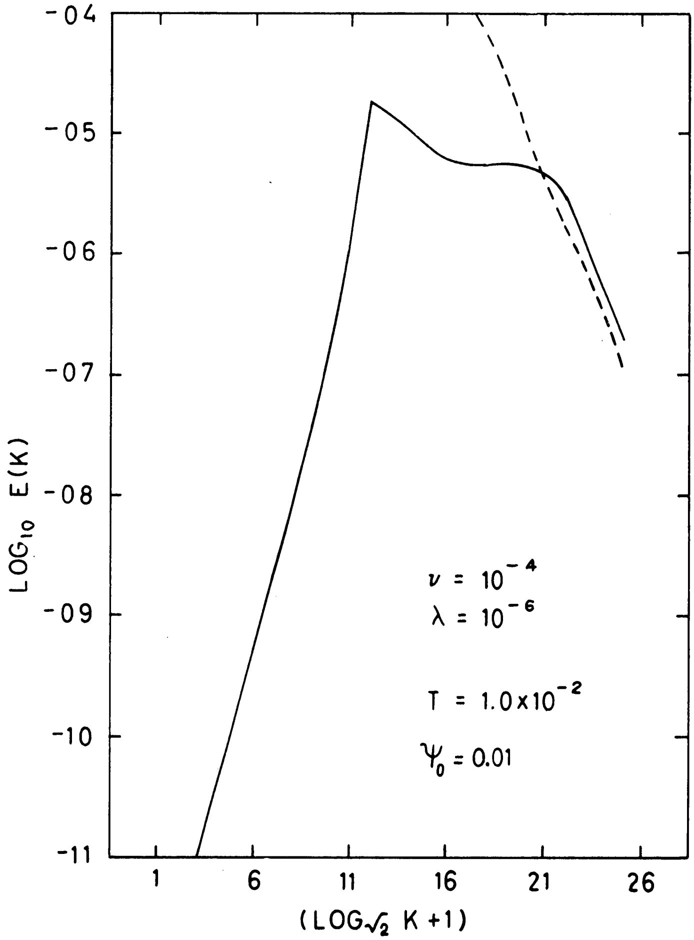

Fig. 7.

1970). We shall here consider only the results and their astrophysical implications.

Figure 1 shows the initial spectral disposition in one of the runs. The dotted line gives the velocity spectrum, and the continuous line, the magnetic disturbance. ψ_0 is the value of the initial ratio of the magnetic spectrum to the velocity spectrum at nonzero points, which is a parameter of the run. Though we are going to display

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here only initial disturbances which have the same spectral shape as the velocity and are localised in wave number space in a delta-function way, we had performed a number of runs with a variety of initial shapes $ak^n \exp(-bk^m)$ and initial ratio ψ_0. There was no pathological feature arising from the initial choice either numerically or otherwise.

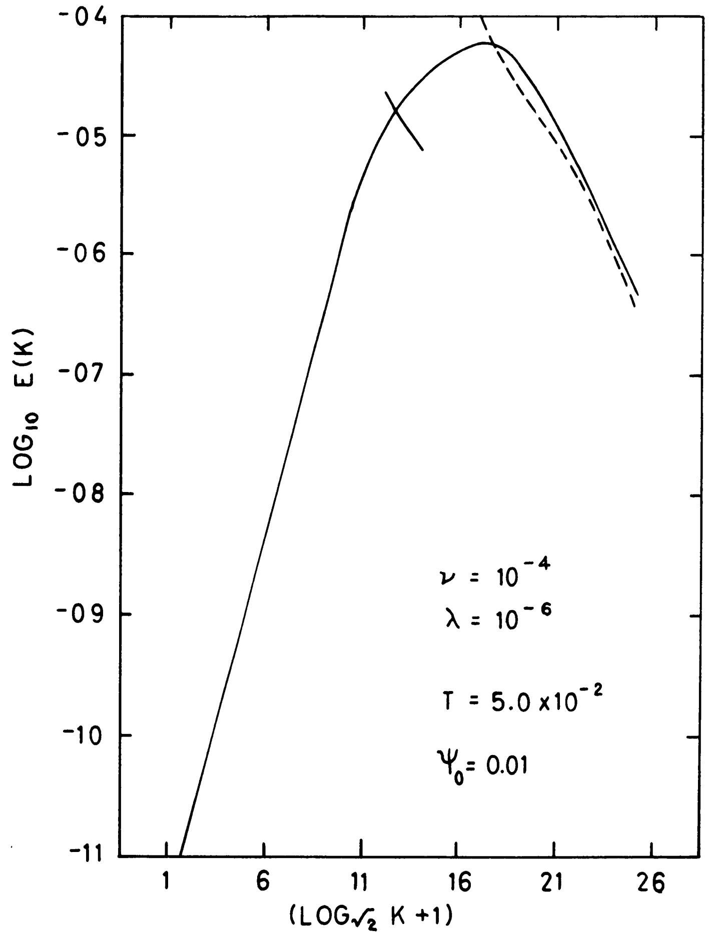

Fig. 8.

Figures 2 and 3 give the spectra at characteristic times $t=1.0\times10^{-5}$ and $t=1.0\times10^{-4}$. These time scales are so normalised that they are unity for the largest wave numbers in our system. The noteworthy feature of the curves is that the energy has now moved both to higher and lower wave numbers. The rate of transfer to lower wave numbers is essentially smaller than the rate of transfer to higher wave numbers, because the characteristic times of transfer are of the order of the internal times of the given scale.

Figures 4 and 5 give the spectra at $t = 5.0 \times 10^{-4}$ and 1.0×10^{-3}. Already, within a time of the order of the local eddy-circulation time in the largest wave numbers, the magnetic spectrum has wrapped up sufficiently to almost equality with the velocity

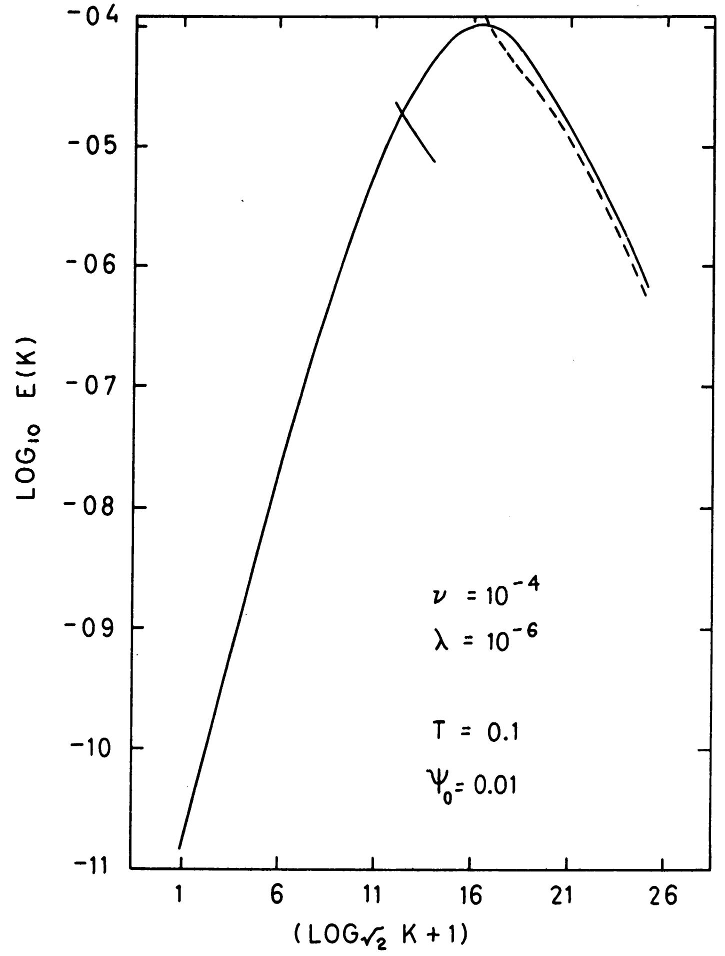

Fig. 9.

spectrum at the highest wave number. Figure 5 to some extent and Figure 6, in a more profound way show that the magnetic spectrum has overshot significantly above the velocity at lowest scales. This arises because of two reasons: (1) The choice of kinematic parameters ν and λ. In this run λ is very much smaller, so much so the magnetic spectrum has a *longer* dissipative tail. (2) The second reason for the overshooting is the fact that the form of the spectrum is still non-equilibrium so much so the approach to local equipartition is in an overstable way.

Figure 7 and more prominently Figure 8 show how the feature of equipartition is transferred to smaller wave numbers, much in the same way as argued by Biermann and Schlüter (1951). By now the evolution has reached a stage in which any peculiar

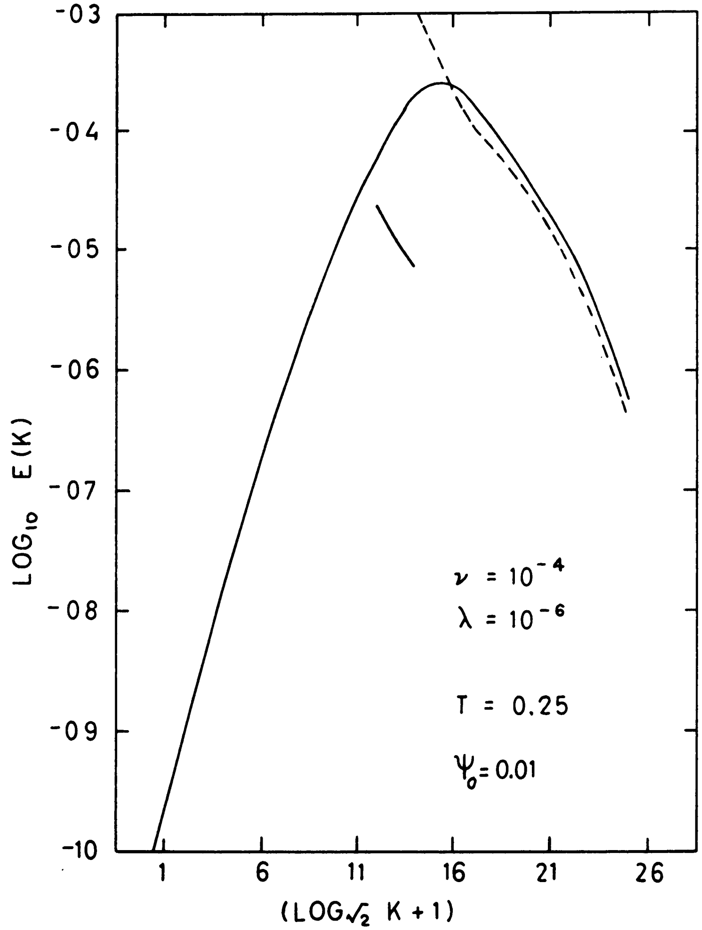

Fig. 10.

dependence on choice of initial form has been completely lost. Figures 9 and 10, which are for the same run for times $t=0.1$ and 0.25 show that by now the evolution has reached a stage when one can safely conclude about ultimate features. The numerical integration times involved at this stage are so large that one stops the calculations because no new features are likely to evolve from further evolution study.

Figures 11, 12 and 13 feature the final and initial spectra for a few other runs which start different initial ratios and kinematic parameters. These are meant for the purist to show that pathological features are not included in the choice of initial assumptions.

In all these runs, at a fairly advanced evolution, the spectral shape reaches an approximate form $A(t)\,k^4 \exp\{-B(t)\,k^2\}$. Thereafter the integral features of the spectrum evolve more or less without change of form.

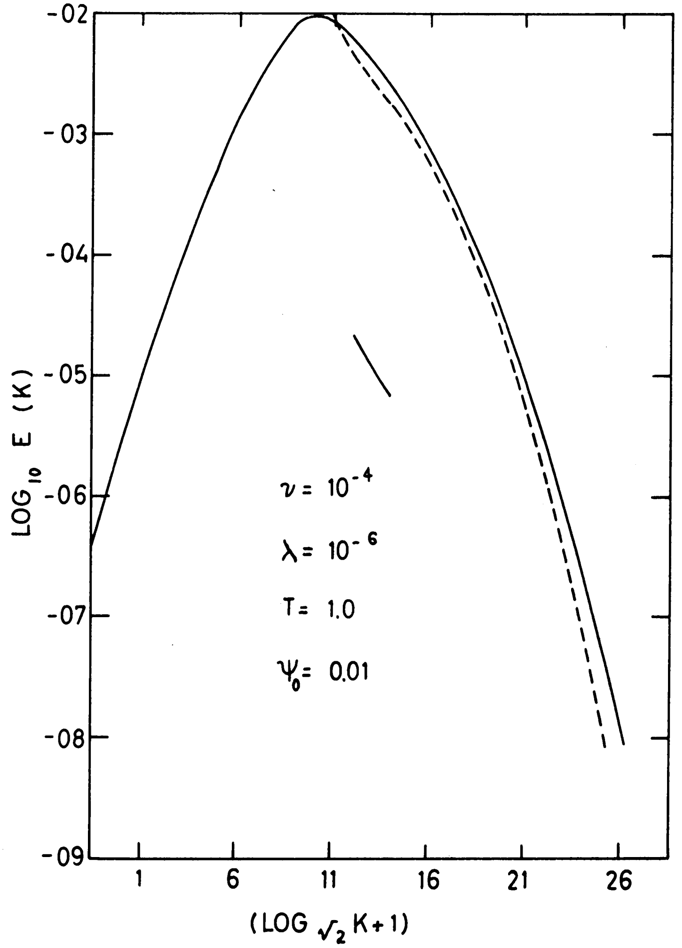

Fig. 11.

4. Conclusions

Apart from the fact that this evolution study fills many a gap in our earlier study, this proves more or less conclusively that there is no reason to expect, in evolving non-equilibrium hydromagnetic turbulence, that the transfer will take place only to larger wave numbers. In fact, the transfer to smaller wave numbers is significant and this

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can provide just the missing link in the *turbulent dynamo* problem. The regeneration of larger magnetic loops through a co-operative interaction of the velocity fluctuations of all scales and magnetic fluctuations of smaller scales is not only feasible but very significant. In our study, we find that this is facilitated by two dynamical requirements. Firstly, the non-equilibrium feature of the magnetic spectrum: the ultimate steady-

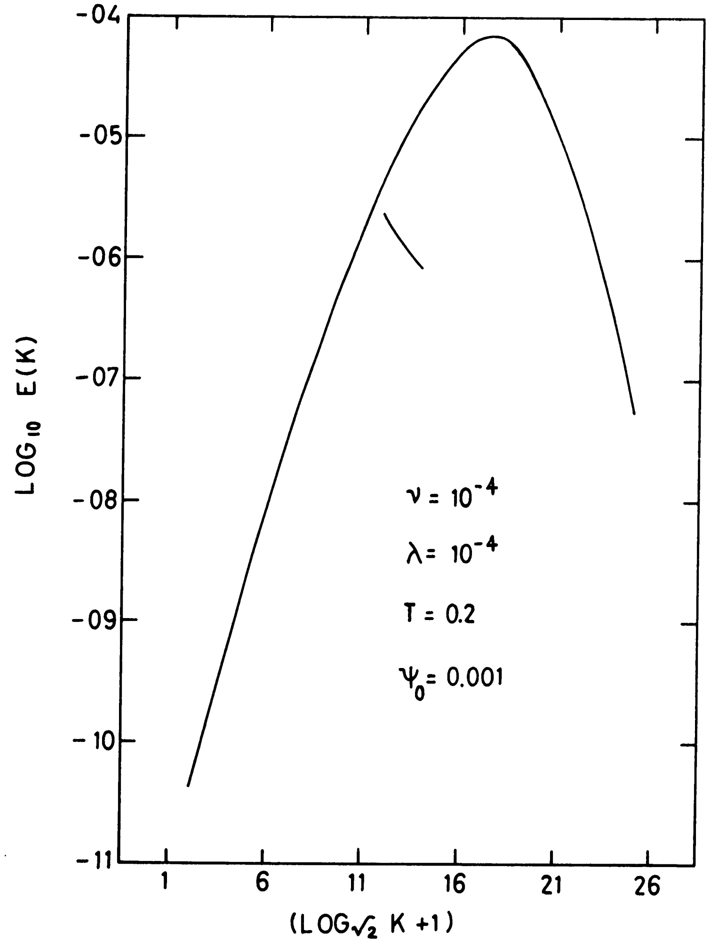

Fig. 12.

state magnetic spectrum will be in equipartition with the velocity in all scales other than the ones where either the inputs of energy from external sources of the train of energy through molecular dissipation depresses or raises either of them. Any other form of the spectral ratio is not an invariant form which will be left invariant by the non-linear interaction. The non-linear interaction will change the ratio to get into the

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equilibrium form. Secondly, the Galilean non-invariance: The fact that a magnetic field cannot be gauged out makes a profound modification in the internal dynamics. Here probably one can stretch our comparison a bit with other recent studies. Krause (1968), Rädler (1968), Steenbeck *et al.* (1966), Steenbeck and Krause (1966, 1967), Krause and Rädler (1971) and Moffatt (1970) have considered the α-effect of regenera-

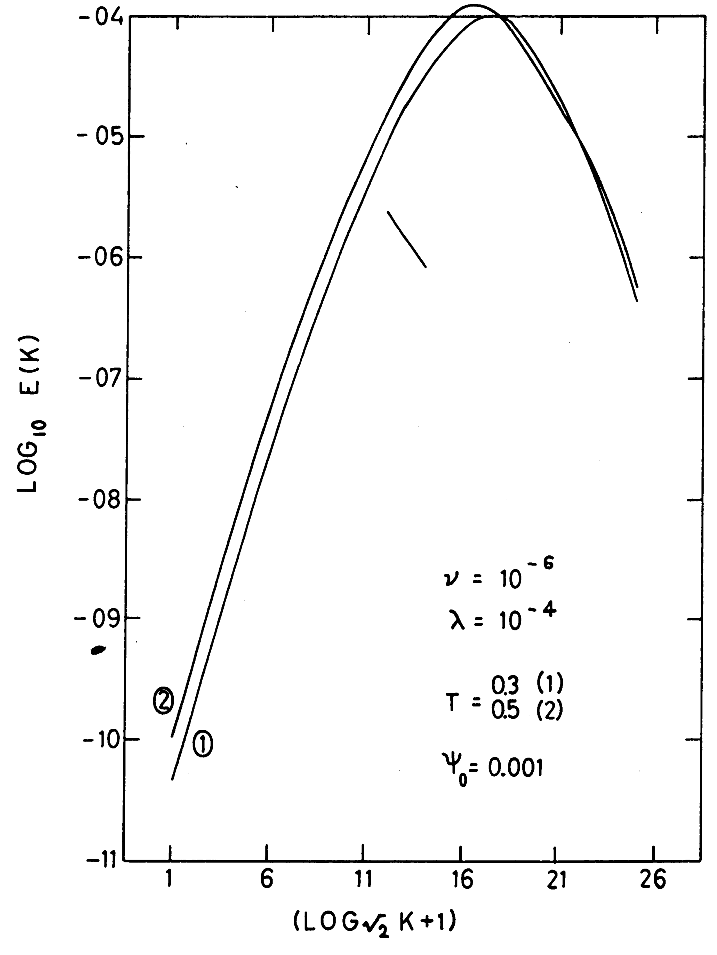

Fig. 13.

tion in great detail. A certain aspect of the α-effect is included in our Galilean non-invariance picture, because a larger magnetic loop, when it is impressed on a system of smaller magnetic and velocity fluctuations, introduces a condition of reflectional non-invariance. Beyond this point one cannot carry the analogies because their inferences about the values of the α-effect are based on equilibrium transfer theory, which as our study has clearly shown, are inapplicable.

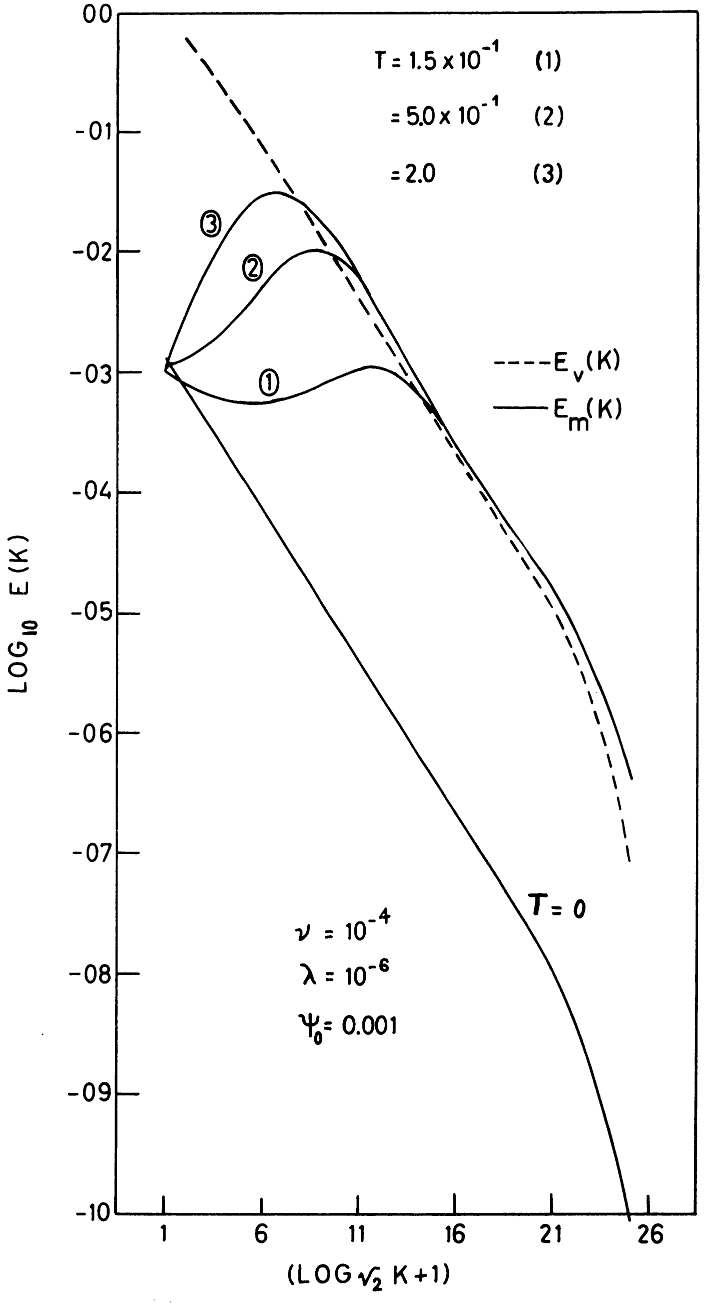

Fig. 14.

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Parker (1969) and Vainshtein (1970) have asked much the same question, as we have, but since they had to invoke some extreme idealisations to get their results, the physical validity of their conclusions is in doubt. Qualitatively, our results corroborate theirs.

Robinson and Rusbridge (1971), in a study of Plasma turbulence in the Zeta plasmas, have found that plasma turbulence seems to resemble fluid turbulence except that the turbulent elements are enlarged along the mean magnetic field to form rolls and suggest that an appropriate comparison would have to explain the existence of significant transfer to large scales from small-scales, as against isotopic hydrodynamic theory, which will not permit this. One hopes that it will not be too presumptuous to believe that the effect, they find is contained in our procedure. Further the importance of this to heat transfer in the presence of magnetic turbulence is also very tempting.

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Discussion

Nakagawa: What is your assumption concerning the initial velocity and magnetic field spectra?

Nagarajan: The initial velocity is in quasi-equilibrium with an extended inertial range. The magnetic spectrum is localized in the middle of the inertial range in all but one of the runs, with a level of excitation very much lower than the velocity.

Weiss: After equipartition has been achieved for intermediate wave numbers, is your steady energy spectrum maintained over periods comparable with the resistive decay time for the smallest wave numbers?

Nagarajan: Yes. We follow the time evolution until the initial form dependence is washed out. Essentially this turns out to be larger than the resistive time scale of the initial specimen. But after that time, the further buildup of the spectrum – even towards smaller wave numbers – takes energy from the velocity spectrum. This time-invariant self-preserving form with the tail in steady-state with the velocity, keeps growing in over-all energy and extent. This may look like a violation of simple physical and statistical requirements. But it is not.

Cowling: In many ways the assumptions made (nature of background fields, motions, statistical assumptions) appear to be as important in the theory of magnetohydrodynamic turbulence as the detailed theory.

Nagarajan: True: statistical description does not in any sense minimize the number of necessary assumptions. But the statistical theory has an advantage in that one requires only on-the-average features. So many of the phasing requirements are weakened. But the main feature of this investigation has been to show that the back-transfer in wave-number spectrum is significant, which can have truly deep conceptual consequences.