36
Four-quark correlators

We shall be concerned here with the two-point correlators associated with the four-quark operators. These operators can describe a four-quark state but play also a crucial rôle for describing the flavour changing $\Delta S = 1$ for the $\Delta I = 1/2$ rule processes of the weak hamiltonian and the $\Delta S = 2$ and $\Delta B = 2$ for the $K - \bar{K}$ and $B - \bar{B}$ oscillations.

36.1 Four-quark states

The two-point function associated to the colour singlet operator:

$$\mathcal{O}^{\pm} = \frac{1}{\sqrt{2}} \sum_{\Gamma=1,\gamma_5} \bar{s}\Gamma s(\bar{u}\Gamma u \pm \bar{d}\Gamma d) \tag{36.1}$$

has been evaluated in [465] to leading order in α_s and including non-perturbative corrections. It is shown in Fig. 36.1, and reads:

$$\begin{aligned}\Psi_4(q^2) = q^8 \ln \frac{-q^2}{\nu^2} \Bigg\{ &-\frac{1}{40960\pi^6} + \frac{1}{1020\pi^6}\left(\frac{m_s^2}{q^2}\right) + \frac{1}{q^4}\left[\frac{m_s\langle\bar{s}s\rangle}{128\pi^4} + \frac{\langle\alpha_s G^2\rangle}{64\pi^5}\right] \\ &- \frac{1}{q^6}\left[\frac{1}{64\pi^4}\left\langle\bar{s}\sigma_{\mu\nu}\frac{\lambda^a}{2}G_a^{\mu\nu}s\right\rangle + \frac{3}{8\pi^2}(\langle\bar{u}u\rangle^2 + \langle\bar{s}s\rangle^2)\right]\Bigg\} \\ &- \left(\frac{8}{3q^2}\right) m_s\langle\bar{s}s\rangle\langle\bar{u}u\rangle^2 ,\end{aligned} \tag{36.2}$$

which is free from non-local $\frac{1}{\epsilon} \ln -q^2/\nu^2$ pole absorbed by the addition of evanescent diagrams. The two-point correlator associated to the operator:

$$\mathcal{O}^{\pm} = \frac{1}{\sqrt{2}} \sum_{\Gamma=1,\gamma_5} \bar{s}\Gamma\lambda_a s(\bar{u}\Gamma\lambda_a u \pm \bar{d}\Gamma\lambda_a d) , \tag{36.3}$$

has been analysed in citeSN4Q and can be easily deduced from the former to leading order using the Fierz transform.

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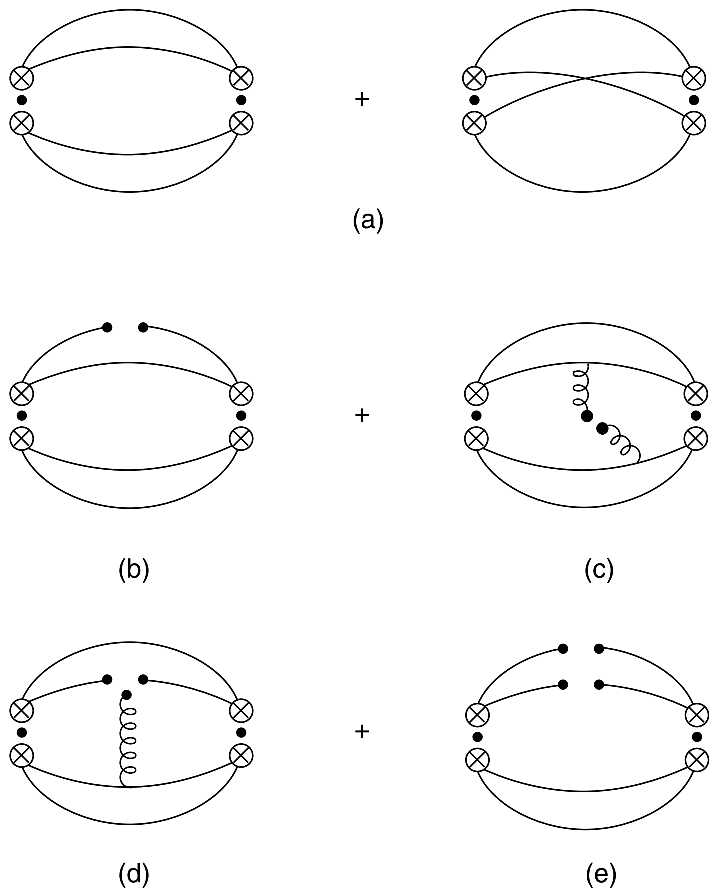

Fig. 36.1. Feynman diagrams corresponding to the OPE of the four-quark correlator: (a) perturbative; (b) quark condensate; (c) gluon condensate; (d) mixed condensate; (e) four-quark condensate.

36.2 $\Delta S = 1$ correlator and $\Delta I = 1/2$ rule

In these weak processes, the short-distance Hamiltonian can be described by the four-quark operators $Q_i(x)$ obtained from the operator product expansion:

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_i C_i(\mu^2) Q_i \ , \tag{36.4}$$

where V_{uq} are elements of the CKM mixing matrix, while C_i is the Wilson coefficient obtained from pQCD calculation. The relevant two-point function for these processes is:

$$\Psi(q^2) \equiv i \int dx\, e^{iqx} \, \langle 0| \, T\{ \mathcal{H}_{\text{eff}}(x) \, \mathcal{H}_{\text{eff}}(0)^\dagger \} |0\rangle$$
$$= \left(\frac{G_F}{\sqrt{2}} \right)^2 |V_{ud} V_{us}^*|^2 \sum_{i,j} C_i(\mu^2) \, C_j^*(\mu^2) \, \Psi_{ij}(q^2) \, . \tag{36.5}$$

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This vacuum-to-vacuum correlator can be studied with perturbative QCD methods, allowing for a consistent combination of Wilson-coefficients $C_i(\mu^2)$ and two-point functions of the four-quark operators, Ψ_{ij}, in such a way that the renormalization scheme and scale dependences exactly cancel (to the computed order). The associated spectral function $\frac{1}{\pi}\mathrm{Im}\Psi^{\Delta S=1}(q^2)$ is a quantity with definite physical information. It describes in an inclusive way how the weak Hamiltonian couples the vacuum to physical states of a given invariant mass. In the following we shall analyse the four-quark correlators but build a RS combination that is useful for the physical processes. Here, we shall consider the correlators associated to the $\Delta S = 1$ operators:

$$Q_1 = 4\,(\bar{s}_L\gamma^\mu d_L)\,(\bar{u}_L\gamma_\mu u_L)\,, \qquad Q_2 = 4\big(\bar{s}^\alpha_L\gamma^\mu d^\beta_L\big)\big(\bar{u}^\beta_L\gamma_\mu u^\alpha_L\big)\,. \tag{36.6}$$

It is usual to work in the diagonal basis:

$$Q_\pm = \frac{1}{2}\,(Q_1 \pm Q_2)\ , \tag{36.7}$$

and to define the RS-invariant operators [476]:

$$\bar{Q}_\pm \equiv \left[1 + \left(\frac{\alpha_s}{\pi}\right) B_\pm\right] Q_\pm\ , \tag{36.8}$$

where in the t'Hooft–Veltman (HV) and naïve dimensional regularization (NDR) schemes (see Chapter 8):

$$B^{HV}_\pm = \frac{7}{8}\left(\pm 1 - \frac{1}{N}\right)\ , \qquad B^{NDR}_\pm = \frac{11}{8}\left(\pm 1 - \frac{1}{N}\right)\ . \tag{36.9}$$

In this basis, the corresponding correlator is:

$$\bar{\Psi}_{\pm\pm} = \frac{1}{2}\left[1 + 2\left(\frac{\alpha_s}{\pi}\right) B_\pm\right][\Psi_{11} \pm \Psi_{12}]\ , \tag{36.10}$$

and is RS-invariant.

$$\begin{aligned}\frac{1}{\pi}\mathrm{Im}\bar{\Psi}_{\pm\pm}(s,\mu^2)\ &= \theta(s)\,\frac{s^4}{(4\pi)^6}\,A_\pm\left\{1 + \left(\frac{\alpha_s}{\pi}\right)\left[\frac{3}{2}\left(\pm 1 - \frac{1}{N}\right)\ln\left|\frac{s}{\mu^2}\right|\right.\right.\\ &\qquad \left.\left. + \frac{3}{4}N \mp \frac{101}{20} + \frac{43}{10}\frac{1}{N}\right]\right\}\ ,\end{aligned} \tag{36.11}$$

with:

$$A_\pm = \frac{2}{45}N(N \pm 1)\,. \tag{36.12}$$

The coefficient of the logarithm is just equal to the leading-order anomalous dimensions $\gamma^{(1)}_\pm$ of $Q_\pm$. Introducing the μ^2-dependent Wilson coefficient:

$$C_\pm(\mu^2) = \alpha_s(\mu^2)^{\gamma^{(1)}_\pm/\beta_1}\left[1 - \frac{\alpha_s(\mu^2)}{4\pi}\,R_\pm\right]\ , \tag{36.13}$$

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where the NLO correction $R_\pm$ can be found in [476], it is possible to form the RGI spectral functions:

$$\frac{1}{\pi}\mathrm{Im}\hat{\Psi}_{\pm\pm}(s) = \frac{1}{\pi}\mathrm{Im}\bar{\Psi}_{\pm\pm}(s)C_\pm^2(s) \tag{36.14}$$

For $N = 3$ the two spectral functions read:

$$\begin{aligned}\frac{1}{\pi}\mathrm{Im}\hat{\Psi}_{++}(s) &= \theta(s)\,\frac{8}{15}\,\frac{s^4}{(4\pi)^6}\,\alpha_s(s)^{-4/9}\left[1-\frac{3649}{1620}\left(\frac{\alpha_s}{\pi}\right)\right],\\ \frac{1}{\pi}\mathrm{Im}\hat{\Psi}_{--}(s) &= \theta(s)\,\frac{4}{15}\,\frac{s^4}{(4\pi)^6}\,\alpha_s(s)^{8/9}\ \left[1+\frac{9139}{810}\left(\frac{\alpha_s}{\pi}\right)\right].\end{aligned} \tag{36.15}$$

Taking $\alpha_s(s)/\pi \approx 0.1$, at the NLO we find a moderate suppression of $\mathrm{Im}\hat{\Psi}_{++}$ by roughly 20%, whereas $\mathrm{Im}\hat{\Psi}_{--}$ acquires a huge enhancement on the order of 100%. Because $\mathrm{Im}\hat{\Psi}_{++}$ solely receives contributions from $\Delta I = 3/2$, and $\mathrm{Im}\hat{\Psi}_{--}$ is a mixture of both $\Delta I = 1/2$ and $\Delta I = 3/2$, this pattern of the radiative corrections entails a strong enhancement of the $\Delta I = 1/2$ amplitude, which can provide a promising picture for the emergence of the $\Delta I = 1/2$–rule.

36.3 The $\Delta S = 2$ correlator

Here, we shall consider the correlator associated to the $\Delta S = 2$ operator:

$$\mathcal{O}_{\Delta S=2} = (\bar{s}_L\gamma^\mu d_L)(\bar{s}_L\gamma_\mu d_L) \tag{36.16}$$

where:

$$\psi_L \equiv \frac{1}{2}(1-\gamma_5)\psi\ . \tag{36.17}$$

We shall analyse its phenomenological application in the next chapter. The QCD expression of the spectral function reads [468]:

$$\begin{aligned}\frac{1}{\pi}\mathrm{Im}\Psi_{\Delta S=2}(t) = \frac{1}{(16\pi^2)^3}\frac{1}{10}\left(1+\frac{1}{N}\right)t^4\alpha_s(t)^{-4/9}\Bigg\{1 - A\left(\frac{\alpha_s}{\pi}\right)\\ -\frac{40\bar{m}_s^2}{t} - \frac{20\pi^2}{t^2}(16\pi m_s\langle\bar{s}s\rangle - \langle\alpha_s G^2\rangle)\Bigg\}\ .\end{aligned} \tag{36.18}$$

The coefficient of the perturbative correction is RS dependent. In [471], it has been shown that one can define a RS invariant combination $\hat{Q}_{\Delta S=2}$:

$$\hat{Q}_{\Delta S=2} \equiv \alpha_s(\nu)^{\gamma_{\Delta S=2}/\beta_1}\left[1-\left(\frac{\alpha_s}{\pi}\right)Z\right]Q_{\Delta S=2}\ , \tag{36.19}$$

where Z depends on the regularization scheme used [475]; $\gamma_{\Delta S=2}$ is the anomalous dimension of the operator $Q_{\Delta S=2}$ defined as:

$$Q_{\Delta S=2} \equiv \frac{1}{2}\left[\mathcal{O}_{\Delta S=2} + \left(\bar{s}_L^\alpha\gamma^\mu d_L^\beta\right)\left(\bar{s}_L^\beta\gamma_\mu d_L^\alpha\right)\right]. \tag{36.20}$$

It coincides with $\mathcal{O}_{\Delta S=2}$ in the HV scheme since in HV Fierz symmetry is respected for current–current operators while $\mathcal{O}_{\Delta S=2}$ renormalizes into itself. This is not the case for the NDR scheme where the γ_5 matrix is naïvely anti-commuting while the rest of the calculation is done in n-dimensions. Within this RS invariant combination one obtains [471]:

$$A = -\frac{3649}{1620} , \tag{36.21}$$

where the global effect reduces by about 20% the lowest-order result.

36.4 The $\Delta B = 2$ correlator

We shall consider the two-point correlator:

$$\psi_{\Delta B=2}(q^2) \equiv i \int d^4x \, e^{iqx} \, \langle 0|\mathcal{T}\mathcal{O}_q(x)(\mathcal{O}_q(0))^\dagger|0\rangle , \tag{36.22}$$

built from the $\Delta B = 2$ weak operator $\mathcal{O}_q$ defined as:

$$\mathcal{O}_q(x) \equiv (\bar{b}\gamma_\mu Lq)(\bar{b}\gamma_\mu Lq) , \tag{36.23}$$

with: $L \equiv (1-\gamma_5)/2$ and $q \equiv d, s$. This correlator has been firstly evaluated to lowest order in [472] in the case of massless light quark mass and including non-perturbative corrections. The perturbative radiative corrections including *non-factorizable* corrections have been obtained in [473]. The $SU(3)$ breaking correction has been evaluated in [474]. The lowest-order perturbative contribution for $m_s \neq 0$ to the two-point correlator is [474]:

$$\begin{aligned}
\frac{1}{\pi}\text{Im}\psi^{\text{pert}}_{\Delta B=2}(t) = {} & \theta(t-4(m_b+m_s)^2) \times \frac{t^4}{1536\pi^6} \times \int_{(\sqrt{\delta}+\sqrt{\delta'})^2}^{(1-\sqrt{\delta}-\sqrt{\delta'})^2} dz \int_{(\sqrt{\delta}+\sqrt{\delta'})^2}^{(1-\sqrt{z})^2} du \, zu \\
& \times \lambda^{1/2}(1,z,u)\lambda^{1/2}\left(1,\frac{\delta}{z},\frac{\delta'}{z}\right)\lambda^{1/2}\left(1,\frac{\delta}{u},\frac{\delta'}{u}\right) \\
& \times \left[4f\left(\frac{\delta}{z},\frac{\delta'}{z}\right)f\left(\frac{\delta}{u},\frac{\delta'}{u}\right) - 2f\left(\frac{\delta}{z},\frac{\delta'}{z}\right)g\left(\frac{\delta}{u},\frac{\delta'}{u}\right)\right. \\
& \left. -2g\left(\frac{\delta}{z},\frac{\delta'}{z}\right)f\left(\frac{\delta}{u},\frac{\delta'}{u}\right) + \frac{(1-z-u)^2}{zu}g\left(\frac{\delta}{z},\frac{\delta'}{z}\right)g\left(\frac{\delta}{u},\frac{\delta'}{u}\right)\right] .
\end{aligned} \tag{36.24}$$

Here $\delta \equiv m_b^2/t$ and $\delta' \equiv m_s^2/t$, respectively. The functions $f(x, y)$ and $g(x, y)$ are defined by

$$\begin{aligned}
f(x,y) &\equiv 2 - x - y - (x-y)^2 , \\
g(x,y) &\equiv 1 + x + y - 2(x-y)^2 .
\end{aligned} \tag{36.25}$$

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The function $\lambda(x, y, z)$ is a phase space factor,

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \ . \tag{36.26}$$

We include the α_s correction from factorizable diagrams to the m_s contribution by using the results for the two-point correlators of currents [477]. This can be done using the convolution formula:

$$\frac{1}{\pi}\text{Im}\psi^{\alpha_s}_{\Delta B=2}(t) = \theta(t - 4(m_b + m_s)^2) \times \frac{t^2}{6\pi^4}\int_{(\sqrt{\delta}+\sqrt{\delta'})^2}^{(1-\sqrt{\delta}-\sqrt{\delta'})^2} dz \int_{(\sqrt{\delta}+\sqrt{\delta'})^2}^{(1-\sqrt{z})^2} du\lambda^{1/2}(1, z, u)$$
$$\times \left\{\text{Im}\Pi^0_{\mu\nu}(zt)\text{Im}\Pi^{\alpha_s\mu\nu}(ut) + \text{Im}\Pi^{\alpha_s}_{\mu\nu}(zt)\text{Im}\Pi^{0\mu\nu}(ut)\right\} \tag{36.27}$$

Here $\Pi^0_{\mu\nu}(q^2)$ and $\Pi^{\alpha_s}_{\mu\nu}(q^2)$ are respectively the lowest and the next-to-leading order QCD contribution to the two-point correlator $\Pi_{\mu\nu}(q^2)$ which is defined by

$$\Pi_{\mu\nu}(q^2) \equiv i\int d^4x e^{iqx} \times \langle 0|T(\bar{b}_L(x)\gamma_\mu s_L(x))(\bar{s}_L(0)\gamma_\nu b_L(0))|0\rangle \ . \tag{36.28}$$

The quark condensate contribution reads:

$$\frac{1}{\pi}\text{Im}\psi^{\bar{s}s}_{\Delta B=2}(t) = \theta(t - 4(m_b + m_s)^2)\frac{1}{384\pi^3}m_s\langle\bar{s}s\rangle$$
$$\times \int_{(m_b+m_s)^2}^{(\sqrt{t}-m_b)^2} dq_1^2\sqrt{\lambda_1}\left(4 + 2q^2\frac{\partial}{\partial q^2}\right)$$
$$\times \left[\sqrt{\lambda_0}\left\{\lambda_1\left(1 + \frac{m_b^2}{q^2} - \frac{q_1^2}{q^2}\right)q_1^2\right.\right.$$
$$\left.\left. + f_1\left(1 - \frac{m_b^2}{q^2} + \frac{q_1^2}{q^2}\right)\left(q^2 - m_b^2 - q_1^2\right)\right\}\right]. \tag{36.29}$$

Here λ_0, λ_1, and f_1 are defined by

$$\lambda_0 \equiv \lambda\left(1, \frac{q_1^2}{q^2}, \frac{m_b^2}{q^2}\right),$$
$$\lambda_1 \equiv \lambda\left(1, \frac{m_b^2}{q_1^2}, \frac{m_s^2}{q_1^2}\right),$$
$$f_1 \equiv 1 + \frac{m_b^2}{q_1^2} + \frac{m_s^2}{q_1^2} - 2\frac{\left(m_b^2 - m_s^2\right)^2}{q_1^4}\ . \tag{36.30}$$

The gluon condensate contribution reads in the case $m_s = 0$ [472]:

$$\frac{1}{\pi}\text{Im}\psi^{G^2}_{\Delta B=2}(t) = \frac{t^2}{(16\pi^2)^2}\frac{1}{\pi}\langle\alpha_s G^2\rangle\int_{x_0}^1 dx\int_{y_-}^{y_+} dy$$
$$\times \{-(\Delta/2y^2)[\Delta - y(1-y)][2xy + (1-x)^2(1-y)]$$

$$+(\delta x/3y^3)(1-x)^2(1-y)^3[2\Delta - y(1-y)]\}$$
$$-\int_{\delta}^{(1-\sqrt{\delta})^2} dz\, z(1-\delta/z)^2\lambda^{1/2}(1,z,\delta)\,, \tag{36.31}$$

where:

$$\Delta \equiv \delta(y/x+1-y) - y(1-y) \tag{36.32}$$

and the parametric integration limits are given by:

$$x_0 = \delta/(1-\sqrt{\delta})^2\,,$$
$$y_\pm = \frac{1}{2}[1+\delta(1-1/x) \pm \lambda^{1/2}(1,\delta,\delta/x)]\,. \tag{36.33}$$

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