Zeta Functions and 'Kontsevich Invariants' on Singular Varieties

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Abstract. Let *X* be a nonsingular algebraic variety in characteristic zero. To an effective divisor on *X* Kontsevich has associated a certain motivic integral, living in a completion of the Grothendieck ring of algebraic varieties. He used this invariant to show that birational (smooth, projective) Calabi-Yau varieties have the same Hodge numbers. Then Denef and Loeser introduced the invariant *motivic (Igusa) zeta function*, associated to a regular function on *X*, which specializes to both the classical *p*-adic Igusa zeta function and the topological zeta function, and also to Kontsevich's invariant.

This paper treats a generalization to singular varieties. Batyrev already considered such a 'Kontsevich invariant' for log terminal varieties (on the level of Hodge polynomials of varieties instead of in the Grothendieck ring), and previously we introduced a motivic zeta function on normal surface germs. Here on any $\mathbb{Q}$-Gorenstein variety *X* we associate a motivic zeta function and a 'Kontsevich invariant' to effective $\mathbb{Q}$-Cartier divisors on *X* whose support contains the singular locus of *X*.

Introduction

0.1

Let *k* be a field of characteristic zero. To a nonsingular (irreducible) variety *X* and a morphism $f\colon X \to \mathbb{A}^1$, both defined over *k*, was associated the invariant *motivic (Igusa) zeta function* by Denef and Loeser [DL2]. By definition it lives in a power series ring in one variable over the ring $\mathcal{M}_L$, where $\mathcal{M}$ is the Grothendieck ring of algebraic varieties over *k*, *L* is the class of $\mathbb{A}^1$ in $\mathcal{M}$, and $\mathcal{M}_L$ denotes localization. When $X = \mathbb{A}^d$ this invariant specializes to both the usual *p*-adic Igusa zeta function and the topological zeta function associated to a polynomial *f*. (In fact in [DL2] the authors treat an even more general invariant, involving motives instead of varieties, from which also the whole Hodge spectrum of *f* at any point of $f^{-1}\{0\}$ can be deduced.) This notion of motivic zeta function can easily be extended to an effective divisor *D* instead of just a morphism *f*.

The authors were inspired by Kontsevich's idea of *motivic integration*. In [Kon] Kontsevich associated to a nonsingular irreducible variety *X* and an effective divisor *D* on *X* an invariant $\mathcal{E}(D)$, living by definition in an appropriate completion $\hat{\mathcal{M}}$ of $\mathcal{M}_L$. He used this invariant to show that birationally equivalent (smooth, projective) Calabi-Yau varieties have the same Hodge numbers.

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0.2

There are important formulas for these invariants in terms of an embedded resolution (with strict normal crossings) $h\colon Y \to X$ of supp D. Let $\dim X = d$ and denote by E_i, $i \in T$, the irreducible components of $h^{-1}(\text{supp } D)$. To the E_i are associated natural multiplicities N_i and ν_i defined by $h^*D = \sum_{i \in T} N_i E_i$ and $\text{div}(h^*dx) = \sum_{i \in T}(\nu_i - 1)E_i$, where dx is a local generator of the sheaf of regular differential d-forms on X. Also we partition Y into the locally closed strata $E_I^\circ := (\bigcap_{i \in I} E_i) \setminus (\bigcup_{\ell \notin I} E_\ell)$ for $I \subset T$.

We denote the class of a variety V in $\mathcal{M}$ by $[V]$, and by analogy with the usual p-adic Igusa zeta function we denote the variable of the power series ring over $\mathcal{M}_L$ formally by L^{-s}. Then the motivic zeta function $\mathcal{Z}(D, s)$ of D is given by the formula

$$\mathcal{Z}(D, s) = L^{-d} \sum_{I \subset T} [E_I^\circ] \prod_{i \in I} \frac{(L-1)L^{-\nu_i}(L^{-s})^{N_i}}{1 - L^{-\nu_i}(L^{-s})^{N_i}}$$

and so it lives already in a localization of the polynomial ring $\mathcal{M}_L[L^{-s}]$. Kontsevich's invariant for D is given by

$$\mathcal{E}(D) = L^{-d} \sum_{I \subset T} [E_I^\circ] \prod_{i \in I} \frac{L-1}{L^{\nu_i + N_i} - 1}$$

and can thus in some sense be derived from $\mathcal{Z}(D, s)$ by 'substituting $s = 1$'.

0.3

One can specialize $\mathcal{Z}(D, s)$ and $\mathcal{E}(D)$ to more 'concrete' invariants, involving instead of the class $[V]$ of a variety V in $\mathcal{M}$ other additive invariants as the Hodge polynomial $H(V)$ or the Euler characteristic $\chi(V)$ of V. With a little work one obtains for instance from $\mathcal{Z}(D, s)$ the *topological zeta function*

$$z(D, s) = \sum_{I \subset T} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{\nu_i + sN_i} \in \mathbb{Q}(s),$$

which was introduced in [DL1] for $X = \mathbb{A}^d$ and $k = \mathbb{C}$, and the invariant

$$e(D) = \sum_{I \subset T} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{\nu_i + N_i} \in \mathbb{Q}.$$

0.4

Can the invariants above be generalized to singular (normal) varieties X such that analogous formulas in terms of an embedded resolution are valid? The main problem is whether these formulas are independent of the chosen resolution. Let D be an effective Weil divisor on X and $h\colon Y \to X$ an embedded resolution of $X_{\text{sing}} \cup \text{supp } D$ with irreducible components E_i, $i \in T$, of $h^{-1}(X_{\text{sing}} \cup \text{supp } D)$. Can we generalize

the multiplicities N_i and ν_i? When D is Cartier (or $\mathbb{Q}$-Cartier) the same expression $h^*D = \sum_{i \in T} N_i E_i$ makes sense. We think that the most natural generalization of the ν_i are the log discrepancies given by $K_Y = h^*K_X + \sum_{i \in T}(\nu_i - 1)E_i$, where $K.$ is the canonical divisor. To this end we need in general X to be Gorenstein (or $\mathbb{Q}$-Gorenstein). Up to now the following generalizations appeared (with $k = \mathbb{C}$):

(a) In dimension 2 these multiplicities are defined for arbitrary Weil divisors on normal surfaces. In [V3] we introduced a topological zeta function and a motivic zeta function for effective divisors on normal surface germs. We could have done this as well globally, associating to an effective Weil divisor D on a normal surface X for which $X_{\text{sing}} \subset \text{supp}\, D$ the zeta functions $\mathcal{Z}(D, s)$ and $z(D, s)$, given by the same formulas as above.

(b) In arbitrary dimension Batyrev [B2] considered the case $D = 0$ and associated 'Kontsevich-like' invariants to a log terminal X on the level of Hodge polynomials and Euler characteristics. The last one, which he called *stringy Euler number*, is given by the formula for $e(D)$ in (0.3) with all $N_i = 0$. The invariant on Hodge polynomial level was used in [B2] to define *stringy Hodge numbers* for projective canonical Gorenstein varieties, and to formulate a topological mirror duality test for canonical Calabi-Yau varieties.

(c) Batyrev [B3] also extended his construction to Kawamata log terminal pairs (X, D), *i.e.*, pairs such that $K_X + D$ is $\mathbb{Q}$-Cartier and all $a_i > 0$ in the expression $K_Y = h^*(K_X + D) + \sum_{i \in T}(a_i - 1)E_i$. On the Euler characteristic level this invariant is given by the formula

$$e\big((X, D)\big) = \sum_{I \subset T} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{a_i}.$$

In [B3] these invariants are used to prove a version of Reid's McKay correspondence conjecture.

We should mention that Batyrev is naturally restricted to the log terminality conditions above (all $\nu_i > 0$ and all $a_i > 0$, respectively) by applying motivic integration techniques to show that the formulas above are independent of the chosen resolution; see [B2, Theorem 6.28].

We also want to remark that $\mathcal{E}(D)$ is generalized in [DL3] in a different way (see 3.5).

0.5

In this paper we extend the invariants above beyond the log terminal case to the following general situation. Now let X be any normal $\mathbb{Q}$-Gorenstein variety and D an effective $\mathbb{Q}$-Cartier divisor with $X_{\text{sing}} \subset \text{supp}\, D$. We associate first to these data zeta functions $\mathcal{Z}(D, s)$, $Z(D, s)$ and $z(D, s)$ on 'motivic' level, Hodge polynomial level and Euler characteristic level, respectively, such that the same formulas as in (0.2) and (0.3) are valid. Then we *define* 'Kontsevich' invariants $\mathcal{E}(D)$, $E(D)$ and $e(D)$ on the analogous levels by taking the limit for $s \rightarrow 1$ in the associated zeta functions

(admitting the value ∞). In particular when all $\nu_i + N_i \neq 0$ the formulas in (0.2) and (0.3) are again valid.

Furthermore taking the limit for $s \to -1$ in the zeta functions we obtain invariants $\mathcal{E}\big((X,D)\big)$, $E\big((X,D)\big)$ and $e\big((X,D)\big)$ of the pair (X,D) on the same levels, the last one given by the same formula as in (0.4).

In fact we can relax our condition $X_{\text{sing}} \subset \text{supp}\,D$ to $\text{LCS}(X) \subset \text{supp}\,D$, where $\text{LCS}(X)$ is the locus of log canonical singularities of X. In particular this locus is empty when X is log terminal; so we really generalize the invariants of [B2].

0.6

In Section 1 we recall the motivic zeta function of Denef and Loeser and the invariant of Kontsevich on smooth varieties X, generalizing the first one to effective divisors instead of regular functions. As an introduction to singular varieties we treat the easy case of a canonical X in Section 2; there we also consider an application to minimal models. For $\mathbb{Q}$-Gorenstein varieties X the zeta functions $Z(D,s)$ and $z(D,s)$ on the level of Hodge polynomials and Euler characteristics, respectively, are constructed in an elementary way in Section 3. We provide some examples in Section 4. The 'motivic' version requires more work. In Section 5 we first introduce a motivic zeta function $\mathcal{Z}(D,J,s)$ on a smooth X, associated to both an effective divisor D and an invertible subsheaf J of the sheaf of regular differential forms on X. (This can be compared with associating a p-adic Igusa zeta function to both a polynomial and a differential form.) Then we use this object to define the motivic zeta function $\mathcal{Z}(D,s)$ for a $\mathbb{Q}$-Gorenstein X in Section 6. We include an appendix indicating how to extend the original Kontsevich invariant on smooth X to $\mathbb{Q}$-divisors instead of (ordinary) divisors, needing a finite extension of $\hat{\mathcal{M}}$.

Remark 0.7 After this work was finished we learned about the proofs of Włodarczyk [Wł] and of Abramovich *et al.* [AKMW] of the weak factorization conjecture for birational maps. Using weak factorization we can give another proof that the zeta functions in this paper are well defined.

1 Smooth Varieties

1.1

Let k be a field of characteristic zero; the varieties and morphisms we will consider are assumed to be defined over k. (A variety is a reduced separated scheme of finite type over k, not necessarily irreducible.)

We fix some terminology concerning resolution. A *resolution* of an irreducible variety X is a proper birational morphism $h\colon Y \to X$ from a smooth variety Y, which is an isomorphism outside the set X_{sing} of singular points of X. A *log resolution* or *embedded resolution* of an irreducible variety X is a resolution $h\colon Y \to X$ of X for which $h^{-1}(X_{\text{sing}})$ is a divisor with strict normal crossings, *i.e.,* with smooth irreducible components intersecting transversely. A *log resolution* or *embedded resolution* of a reduced Weil divisor D on a normal variety X is a proper birational morphism $h\colon Y \to X$ from

a smooth Y, which is an isomorphism outside $X_{\text{sing}} \cup D$, and such that $h^{-1}(X_{\text{sing}} \cup D)$ is a divisor with strict normal crossings.

We denote by $\mathcal{M}$ the Grothendieck ring of (algebraic) varieties over k. This is the free abelian group generated by the symbols $[V]$, where $[V]$ is a variety, subject to the relations $[V] = [V']$ if $V \cong V'$ and $[V] = [V \setminus V'] + [V']$ if V' is closed in V. Its ring structure is given by $[V] \cdot [V'] := [V \times V']$. We abbreviate $L := [\mathbb{A}^1]$ and denote by $\mathcal{M}_L = \mathcal{M}[L^{-1}]$ the localization of $\mathcal{M}$ w.r.t. the multiplicative set $\{L^n, n \in \mathbb{N}\}$.

1.2

For $[V] \in \mathcal{M}$ we denote by $H(V) \in \mathbb{Z}[u, v]$ its Hodge polynomial and by $\chi(V)$ its Euler characteristic. We briefly explain these notions.

Let first $k = \mathbb{C}$. Then for a variety V we denote by $h^{p,q}\big(H_c^i(V, \mathbb{C})\big)$ the rank of the (p, q)-Hodge component of its i-th cohomology group with compact support and we set $e^{p,q}(V) := \sum_{i \geq 0}(-1)^i h^{p,q}\big(H_c^i(V, \mathbb{C})\big)$. The *Hodge polynomial* of V is $H(V) = H(V; u, v) := \sum_{p,q} e^{p,q}(V)u^p v^q \in \mathbb{Z}[u, v]$.

Precisely by the defining relations of $\mathcal{M}$ there is a well defined ring morphism $H \colon \mathcal{M} \to \mathbb{Z}[u, v]$ determined by $[V] \mapsto H(V)$.

We denote by $\chi(V)$ the topological Euler characteristic of V, *i.e.*, the alternating sum of the ranks of its Betti or de Rham cohomology groups. Clearly $\chi(V) = H(V; 1, 1)$ and we also obtain a ring morphism $\chi \colon \mathcal{M} \to \mathbb{Z}$ determined by $[V] \to \chi(V)$.

For arbitrary k (of characteristic zero) we choose an embedding of the field of definition of the variety V into $\mathbb{C}$. Then we can define the same morphisms H and χ on $\mathcal{M}$ starting from the $e^{p,q}(V)$; they are independent of the chosen embedding since for a smooth projective V we have that $e^{p,q}(V) = (-1)^{p+q} \dim_k H^q(V, \Omega_V^p)$.

1.3

Till the end of this section we let X be a smooth irreducible variety of dimension d and W a subvariety of X.

In [DL2] Denef and Loeser associate to $W \subset X$ and a morphism $f \colon X \to \mathbb{A}^1$ an invariant named *motivic Igusa zeta function*. We recall here briefly its definition but generalize immediately to effective divisors D (instead of functions f). We refer to [DL2] for more details and motivation, and for the relation with the usual p-adic Igusa zeta function.

We denote by $\mathcal{L}(X)$ the *scheme of germs of arcs* on X. It is a scheme over k whose k-rational points are the morphisms $\operatorname{Spec} k[[t]] \to X$ (called the germs of arcs on X). In fact $\mathcal{L}(X)$ is defined as the projective limit $\varprojlim \mathcal{L}_n(X)$ of the schemes of truncated arcs $\mathcal{L}_n(X)$, whose k-rational points are the morphisms $\operatorname{Spec}(k[t]/t^{n+1}k[t]) \to X$ (see [DL2] and [BLR, p. 276]). There are canonical morphisms $\pi_n \colon \mathcal{L}(X) \to \mathcal{L}_n(X)$, induced by truncation. Remark also that $\mathcal{L}_0(X) = X$.

Now let D be an effective divisor on X. For $n \in \mathbb{N}$ we define $Y_{n,D,W}$ as the subscheme of $\mathcal{L}(X)$ whose K-rational points, for any field $K \supset k$, are the morphisms $\varphi \colon \operatorname{Spec} K[[t]] \to X$ satisfying the following conditions:

(i) φ sends the closed point of Spec $K[[t]]$ to a point P in W;
(ii) if f is a local equation of D at P, then the power series in t given by $f \circ \varphi$ must be exactly of order n. (This is clearly independent of the choice of f.)

We then denote by $X_{n,D,W}$ the image of $Y_{n,D,W}$ in $\mathcal{L}_n(X)$, viewed as a reduced subscheme of $\mathcal{L}_n(X)$. The *motivic zeta function* of D (and $W \subset X$) is

$$\mathcal{Z}_W(D,s) = \mathcal{Z}_W(X,D,s) := \sum_{n\in\mathbb{N}} [X_{n,D,W}] L^{-(n+1)d-ns} \in \mathcal{M}_L[[L^{-s}]].$$

Here L^{-s} is just a variable and in the power series ring $\mathcal{M}_L[[L^{-s}]]$ we abbreviate $L^a \cdot (L^{-s})^b$ by L^{a-sb} for $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. (When D is given by a global function f on X Denef and Loeser denoted this invariant by $\int_W^\sim f^s$ in [DL2].)

One can think here mainly about W as being X itself, the divisor $\{f = 0\}$, or a point of $\{f = 0\}$. This W-formalism enables us to treat these cases together, and the greater generality is also useful.

1.3.1

We briefly compare this with the classical p-adic situation. Let $f \in \mathbb{Q}_p[x] = \mathbb{Q}_p[x_1,\ldots,x_d]$ and denote by $|z| = p^{-\operatorname{ord}_p z}$ the p-adic absolute value of $z \in \mathbb{Q}_p$. *Igusa's local zeta function* of f is

$$Z_p(f,s) := \int_{\mathbb{Z}_p^d} |f(x)|^s \, |dx|$$

for $s \in \mathbb{C}$ with $\Re(s) > 0$, where $|dx|$ denotes the Haar measure on $\mathbb{Q}_p^d$ such that $\mathbb{Z}_p^d$ has measure 1. When $f \in \mathbb{Z}_p[x]$ it is not difficult to verify that

$$Z_p(f,s) = \sum_{n\in\mathbb{N}} \operatorname{card}(X_{n,f}) p^{-(n+1)d-ns},$$

where $X_{n,f}$ is the image in $(\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)^d$ of $Y_{n,f} = \{x \in \mathbb{Z}_p^d \mid \operatorname{ord}_p f(x) = n\}$. See [D2] for an introduction and an overview on Igusa's local zeta function.

1.4

There is an important formula for $\mathcal{Z}_W(D,s)$ in terms of a log resolution of supp D. In particular it implies the rationality result that $\mathcal{Z}_W(D,s)$ belongs in fact already to a certain localization of the polynomial ring $\mathcal{M}_L[L^{-s}]$.

Let $h\colon Y \to X$ be a log resolution of supp D. We denote by E_i, $i \in T$, the irreducible components of $h^{-1}(\text{supp } D)$ and by N_i and $\nu_i - 1$ the multiplicities of E_i in h^*D and the divisor of h^*dx, respectively, where dx is a local generator of the sheaf Ω_X^d of regular differential d-forms. We partition Y into the locally closed strata $E_I^\circ := (\bigcap_{i\in I} E_i) \setminus (\bigcup_{\ell\notin I} E_\ell)$ for $I \subset T$. (Here $E_\phi^\circ = Y \setminus \bigcup_{\ell\in T} E_\ell$.)

Theorem   *We have the formula*

$$\mathcal{Z}_W(D,s) = L^{-d} \sum_{I\subset T} [E_I^\circ \cap h^{-1}W] \prod_{i\in I} \frac{L-1}{L^{\nu_i+sN_i} - 1}$$

(where $\frac{1}{L^{\nu_i+sN_i}-1} := \frac{L^{-\nu_i-sN_i}}{1-L^{-\nu_i-sN_i}}$*).* *So* $\mathcal{Z}_W(D,s)$ *belongs already to the localization* $\mathcal{M}_L[L^{-s}]_{(1-L^{-n-Ns})_{n,N\in\mathbb{N}\setminus\{0\}}}$ *of the polynomial ring* $\mathcal{M}_L[L^{-s}]$.

1.4.1

One should compare this formula with the classical formula of Denef [D1, Theorems 2.4 and 3.1] for Igusa's local zeta function $Z_p(f,s)$ of $f \in \mathbb{Q}[x_1,\ldots,x_d]$ in terms of a resolution $h\colon Y \to \mathbb{A}^d$ of $\{f = 0\}$. Using the notation above we have for all but finitely many p that

$$Z_p(f,s) = p^{-d} \sum_{I\subset T} \#(E_I^\circ)_{\mathbb{F}_p} \prod_{i\in I} \frac{p-1}{p^{\nu_i+sN_i}-1},$$

where $\#(\cdot)_{\mathbb{F}_p}$ denotes the number of $\mathbb{F}_p$-rational points of the reduction mod p. See [D1], [D2] for more details.

1.5

Here we generalize $\mathcal{Z}_W(D,s)$ to effective $\mathbb{Q}$-divisors on X. Now let D be an effective $\mathbb{Q}$-divisor on X and say that rD is a divisor for $r \in \mathbb{N}\setminus\{0\}$. We define $\mathcal{Z}_W(D,s) := \mathcal{Z}_W(rD,s/r)$, meaning by this the motivic zeta function of 1.3 for the divisor rD, where the variable L^{-s} is replaced by a variable $(L^{-s})^{1/r}$. This definition is easily checked to be independent of the chosen r, using Theorem 1.4.

Moreover Theorem 1.4 is still valid in this context. The only difference is that the N_i, $i \in T$, are now rational numbers (of the form a/r with $a \in \mathbb{N}\setminus\{0\}$), and one should consider L^{-sN_i} as an abbreviation of $\left((L^{-s})^{1/r}\right)^{rN_i}$.

1.6

One can specialize the motivic zeta functions $\mathcal{Z}_W(D,s)$ to more 'concrete' invariants on the level of Hodge polynomials and on the level of Euler characteristics.

(i) Let D be an effective divisor on X. Since the Hodge polynomial $H(\mathbb{A}^1) = uv$ the morphism $H\colon \mathcal{M} \to \mathbb{Z}[u,v]$ extends naturally to a ring morphism $H\colon \mathcal{M}_L \to \mathbb{Z}[u,v]_{uv} = \mathbb{Z}[u,v][(uv)^{-1}]$ (and further to a morphism on power series rings over these rings). We define

$$Z_W(D,s) = Z_W(X,D,s) := H\big(\mathcal{Z}_W(D,s)\big) = \sum_{n\in\mathbb{N}} H(X_{n,D,W})(uv)^{-(n+1)d-ns},$$

where now we denote the variable of the power series ring over $\mathbb{Z}[u,v]_{uv}$ by $(uv)^{-s}$. Using the notation of 1.4 we have the formula

$$Z_W(D,s) = (uv)^{-d} \sum_{I\subset T} H(E_I^\circ \cap h^{-1}W) \prod_{i\in I} \frac{uv-1}{(uv)^{\nu_i+sN_i}-1}$$

$$\in \mathbb{Z}[u,v]_{uv}[(uv)^{-s}]_{(1-(uv)^{-n-Ns})_{n,N\in\mathbb{N}\setminus\{0\}}} \subset \mathbb{Q}(u,v)\big((uv)^{-s}\big).$$

(ii) To specialize further to the level of Euler characteristics one takes heuristically the limit of the expression above for $u, v \to 1$. We briefly explain the exact argument; see [DL2, (2.3)] for the argument starting from $\mathcal{Z}_W(D, s)$. Let R denote the subring of $\mathbb{Z}[u, v]_{uv}[[(uv)^{-s}]]$ generated by $\mathbb{Z}[u, v]_{uv}[(uv)^{-s}]$ and the elements $\frac{uv-1}{1-(uv)^{-n-Ns}}$, where $n, N \in \mathbb{N} \setminus \{0\}$. ($Z_W(D, s)$ lives in R.) By expanding $(uv)^{-s}$ and $\frac{uv-1}{1-(uv)^{-n-Ns}}$ formally into series in $uv - 1$, one constructs a canonical algebra morphism

$$R \to \mathbb{Z}[u, v]_{uv}[s][(n + sN)^{-1}]_{n, N \in \mathbb{N} \setminus \{0\}}[[uv - 1]],$$

where $[[uv - 1]]$ denotes completion with respect to the ideal $(uv - 1)$. Composing this morphism with the quotient map given by dividing out $(uv - 1)$ in this last algebra yields a morphism

$$\varphi \colon R \to \frac{\mathbb{Z}[u, v]_{uv}}{(uv - 1)}[s][(n + sN)^{-1}]_{n, N \in \mathbb{N} \setminus \{0\}}.$$

In this last ring the evaluation $u = v = 1$ is well defined; we put

$$z_W(D, s) = z_W(X, D, s) := \lim_{u, v \to 1} \varphi\big(Z_W(D, s)\big)$$

$$= \sum_{I \subset T} \chi(E_I^\circ \cap h^{-1}W) \prod_{i \in I} \frac{1}{\nu_i + sN_i} \in \mathbb{Q}(s).$$

When $X = \mathbb{A}^n$ and D is given by a polynomial f these invariants are just the *topological zeta functions* $Z_{\text{top}}(f, s)$ and $Z_{\text{top},0}(f, s)$ of [DL1] if we take $W = X$ and $W = \{0\}$, respectively.

(iii) As in 1.5 we can consider $Z_W(D, s)$ and $z_W(D, s)$ also for $\mathbb{Q}$-divisors D.

1.7

Now we recall the original motivic integral, introduced by Kontsevich in [Kon], using the notation of 1.3. We refer to [DL3] for a detailed exposition in a much more general setting; see also the appendix. A nice introduction is [C].

We say that $\dim M \le n$ for $M \in \mathcal{M}$ if M can be expressed as a $\mathbb{Z}$-linear combination of classes of algebraic varieties of dimension at most n. We consider the decreasing filtration $(F^m)_{m \in \mathbb{Z}}$ on $\mathcal{M}_L$, where F^m is the subgroup of $\mathcal{M}_L$ generated by $\{[V]L^{-i} \mid \dim V - i \le -m\}$, and we denote by $\hat{\mathcal{M}}$ the completion of $\mathcal{M}_L$ with respect to this filtration.

Let again D be an effective divisor on X. We set

$$\mathcal{E}_W(D) = \mathcal{E}_W(X, D) := \sum_{n \in \mathbb{N}} \frac{[X_{n,D,W}]}{L^{(n+1)d}} L^{-n} \in \hat{\mathcal{M}};$$

this expression converges in $\hat{\mathcal{M}}$ since $\dim[X_{n,D,W}] \le (n + 1)d$. This invariant was denoted as $[\int_X e^D]$ by Kontsevich (for $W = X$) and as $\int_{\pi_0^{-1}W} L^{-\operatorname{ord}_t \mathcal{O}(-D)} d\mu$ in [DL3]. In this last paper Denef and Loeser develop an integration theory for semi-algebraic subsets of $\mathcal{L}(X)$ with values in $\hat{\mathcal{M}}$ such that $[X_{n,D,W}]/L^{(n+1)d}$ is just the volume of $Y_{n,D,W}$. See also Section 5 and the appendix.

Remark 1.8 As far as we know it is not clear whether or not the natural morphism $\mathcal{M}_L \to \hat{\mathcal{M}}$ is injective; its kernel is $\bigcap_{m\in\mathbb{Z}} F_m$. However for an algebraic variety V we have that $H(V)$ and $\chi(V)$ only depend on the image of $[V]$ in $\hat{\mathcal{M}}$, see 1.12.

Theorem 1.9 (**[Kon], [DL3, (6.5)]**) *Using the notation of 1.4 we have the following formula for $\mathcal{E}_W(D)$ in terms of a log resolution $h\colon Y \to X$ of* supp *D:*

$$\mathcal{E}_W(D) = L^{-d} \sum_{I\subset T} [E_I^\circ \cap h^{-1}W] \prod_{i\in I} \frac{L-1}{L^{\nu_i+N_i}-1} \quad in\ \hat{\mathcal{M}}.$$

In particular $\mathcal{E}_W(D)$ belongs to the image of $\mathcal{M}_L[(L^n-1)^{-1}]_{n\in\mathbb{N}\setminus\{0\}}$ in $\hat{\mathcal{M}}$.

So by Theorem 1.4 we obtain that $\mathcal{E}_W(D) = \mathcal{Z}_W(D)|_{s=1}$ in $\hat{\mathcal{M}}$, where the evaluation '$s=1$' means substituting L^{-1} for the variable L^{-s}.

1.10

The following important change of variables formula is a special case of [DL3, Lemma 3.3], and was also mentioned in [Kon].

Theorem *Let also X' be a smooth irreducible variety and $\rho\colon X' \to X$ a proper birational morphism. Let D be an effective divisor on X. Then*

$$\mathcal{E}_W(X,D) = \mathcal{E}_{\rho^{-1}W}(X', \rho^*D + K_{X'|X})$$

where $K_{X'|X} = K_{X'} - \rho^*K_X$ is the relative canonical divisor or discrepancy divisor.

1.11

It is possible to generalize the set-up in 1.7–1.10 to effective $\mathbb{Q}$-divisors. We treat this in the appendix. In particular we obtain for an effective $\mathbb{Q}$-divisor D on X, such that rD is a divisor for an $r \in \mathbb{N} \setminus \{0\}$, an analogous invariant $\mathcal{E}_W(D) \in \hat{\mathcal{M}}[L^{1/r}]$. It is given in terms of a log resolution $h\colon Y \to X$ (as in 1.4) by the same formula as in 1.9, where now the N_i belong to $\frac{1}{r}(\mathbb{N} \setminus \{0\})$. So $\mathcal{E}_W(D)$ belongs to the image of $\mathcal{M}[L^{-1/r}][(L^{n/r}-1)^{-1}]_{n\in\mathbb{N}\setminus\{0\}}$ in $\hat{\mathcal{M}}[L^{1/r}]$.

When supp D has strict normal crossings we extend in the appendix the notion of $\mathcal{E}_W(D)$ further to the case that all coefficients of D are > -1. Remark that then in Theorem 1.9 (with $h = \mathrm{Id}_X$) all $\nu_i = 1$, and our condition on the coefficients of D is thus precisely that all $\nu_i + N_i > 0$.

1.12

One can also specialize the invariant $\mathcal{E}_W(D)$ to the level of Hodge polynomials and Euler characteristics. We only consider expressions in terms of log resolutions (using the notation of 1.4).

The morphism $H\colon \mathcal{M} \to \mathbb{Z}[u,v]$ extends canonically to a morphism $H\colon$ $\mathcal{M}_L[(L^n-1)^{-1}]_{n\in\mathbb{N}\setminus\{0\}} \to \mathbb{Z}[uv]_{uv}\big[\big((uv)^n-1\big)^{-1}\big]_{n\in\mathbb{N}\setminus\{0\}} \subset \mathbb{Q}(u,v)$. Since the kernel of the natural map $\mathcal{M}_L \to \hat{\mathcal{M}}$ is killed by H we can in fact consider H as a morphism from the image of $\mathcal{M}_L[(L^n-1)^{-1}]_{n\in\mathbb{N}\setminus\{0\}}$ in $\hat{\mathcal{M}}$ into $\mathbb{Q}(u,v)$.

We define for an effective divisor D on X the invariants

$$E_W(D) = E_W(X,D) := H\big(\mathcal{E}_W(D)\big)$$

$$= (uv)^{-d} \sum_{I\subset T} H(E_I^\circ \cap h^{-1}W) \prod_{i\in I} \frac{uv-1}{(uv)^{\nu_i+N_i}-1} \in \mathbb{Q}(u,v)$$

and

$$e_W(D) = e_W(X,D) := \lim_{u,v\to 1} E_W(X,D) = \sum_{I\subset T} \chi(E_I^\circ \cap h^{-1}W) \prod_{i\in I} \frac{1}{\nu_i+N_i} \in \mathbb{Q}.$$

The extended notions of $\mathcal{E}_W(D)$ for $\mathbb{Q}$-divisors of 1.11 can analogously be specialized. We obtain the same expressions where now the N_i are rational; then $E_W(D)$ is a rational function in u, v with 'fractional powers'. For $W = X$ this was already considered by Batyrev [B3].

2 Immediate Generalizations and Applications

2.1

We recall some terminology with origins in the Minimal Model Program. See for example [KM], [KMM], [Kol].

On any normal variety V there is a well-defined linear equivalence class of canonical Weil divisors, denoted by K_V. An arbitrary Weil divisor D on V is called $\mathbb{Q}$-Cartier if rD is Cartier for some $r \in \mathbb{N} \setminus \{0\}$. A normal variety V is called ($\mathbb{Q}$-)Gorenstein if K_V is ($\mathbb{Q}$-)Cartier.

Let X be a normal variety and D a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. (In particular we can have $D = 0$ and then X is $\mathbb{Q}$-Gorenstein.) Let $\rho\colon Y \to X$ be a log resolution of supp D and denote by E_i, $i \in T$, the irreducible components of $h^{-1}(X_{\text{sing}} \cup \text{supp } D)$. Then we can write

$$K_Y = \rho^*(K_X + D) + \sum_{i\in T}(a_i - 1)E_i$$

in Pic $Y \otimes \mathbb{Q}$ and $a_i = a_i(X,D;E_i)$ is called the *log discrepancy* (with respect to the pair (X,D)) of E_i for $i \in T$. This number a_i does not depend on the chosen resolution (it is determined by the valuation on $k(X)$ associated to E_i). Remark that when X is smooth and $D = 0$ the numbers ν_i defined in 1.4 are just log discrepancies.

(i)   First let $D = 0$. The variety X is called *terminal*, *canonical*, *log terminal* and *log canonical* if for some (or, equivalently, any) log resolution of X we have that $a_i > 1$, $a_i \geq 1$, $a_i > 0$ and $a_i \geq 0$, respectively, for all $i \in T$.

(ii)   When $D \neq 0$ the pair (X, D) is said to be *Kawamata log terminal* (shortly *klt*) if
        for some (or any) log resolution of supp D we have that $a_i > 0$ for all $i \in T$. In
        particular this implies that, if $D = \sum_i d_i D_i$ with the D_i irreducible, all $d_i < 1$.
        (See [Kol], [S] for a discussion of other log terminality notions for pairs.)

(iii) A closed subvariety $C \subset X$ is called a *log canonical centre* of X if for some log
        resolution $\rho\colon Y \to X$ there exists $i \in T$ such that $\rho(E_i) = C$ and $a_i \leq 0$. The
        locus of log canonical singularities of X, denoted by $\mathrm{LCS}(X)$, is the union of all
        log canonical centres of X. In particular $\mathrm{LCS}(X) = \varnothing \Leftrightarrow X$ is log terminal.
        (Hence a more appropriate notation for this locus, proposed by Kollár, would
        be $\mathrm{Nlt}(X)$, indicating the locus where X is not log terminal.)

2.2

A natural idea, inspired by Theorem 1.10, to generalize the invariant $\mathcal{E}_W(X, D)$ to a
($\mathbb{Q}$-)divisor D on a singular variety X is as follows. Take a resolution $h\colon Y \to X$ of
X and define $\mathcal{E}_W(X, D)$ as $\mathcal{E}_{h^{-1}W}(Y, h^*D + K_{Y|X})$, whenever this makes sense, and
verify independency of the chosen resolution. So we want X to be $\mathbb{Q}$-Gorenstein and
$h^*D + K_{Y|X}$ to be effective, or at least that its coefficients are > -1 if its support has
normal crossings.

Below we treat the 'instructional' case that X is ($\mathbb{Q}$-)Gorenstein and canonical and
D is an effective ($\mathbb{Q}$-)Cartier divisor.

2.3   Definition-Proposition

(i)   Let X be a Gorenstein and canonical variety and W a subvariety of X; let D be an
       effective Cartier divisor on X. Take a resolution $h\colon Y \to X$ of X. Then we define

$$\mathcal{E}_W(X, D) := \mathcal{E}_{h^{-1}W}(Y, h^*D + K_{Y|X}) \in \hat{\mathcal{M}}.$$

(ii) More generally let X be $\mathbb{Q}$-Gorenstein and canonical and W a subvariety of X;
       let D be an effective $\mathbb{Q}$-Cartier divisor on X. Say rK_X and rD are Cartier for an
       $r \in \mathbb{N} \setminus \{0\}$. Take a resolution $h\colon Y \to X$ of X. Then we define $\mathcal{E}_W(X, D) \in$
       $\hat{\mathcal{M}}[L^{1/r}]$ as above.

Proof (i) The divisor $h^*D + K_{Y|X}$ is effective since $K_{Y|X}$ is effective, which is equiva-
lent to X being canonical. Let now $h'\colon Y' \to X$ be another log resolution of X. Since
two such resolutions are always dominated by a third it is sufficient to consider the
case that h' factors through h as $h'\colon Y' \xrightarrow{\pi} Y \xrightarrow{h} X$. Then by Theorem 1.10 we
have

$$\mathcal{E}_{h^{-1}W}(Y, h^*D + K_{Y|X}) = \mathcal{E}_{\pi^{-1}h^{-1}W}\left(Y', \pi^*(h^*D + K_{Y|X}) + K_{Y'|Y}\right)$$
$$= \mathcal{E}_{h'^{-1}W}(Y', h'^*D + K_{Y'|X}).$$

(ii) Completely analogous, using the extended theory for $\mathbb{Q}$-divisors mentioned
in 1.11.   ∎

When $h\colon Y \to X$ is a log resolution of supp D we have the same formula as in Theorem 1.9, where the ν_i must be generalized according to their meaning as log discrepancies. More precisely, denoting the irreducible components of $h^{-1}(X_{\mathrm{sing}} \cup \mathrm{supp}\, D)$ by E_i, $i \in T$, we set $h^*D = \sum_{i \in T} N_i E_i$ and $K_Y = h^*K_X + \sum_{i \in T}(\nu_i - 1)E_i$. Then $h^*D + K_{Y|X} = \sum_{i \in T}(\nu_i + N_i - 1)E_i$ and so

$$\mathcal{E}_W(X, D) = L^{-d} \sum_{I \subset T} [E_I^\circ \cap h^{-1}W] \prod_{i \in I} \frac{L-1}{L^{\nu_i + N_i} - 1},$$

where d is the dimension of X.

2.4

With essentially the same arguments, but needing more material from the appendix, we could introduce $\mathcal{E}_W(X, D)$ for a $\mathbb{Q}$-Gorenstein variety X and a $\mathbb{Q}$-Cartier divisor D on X such that the pair $(X, -D)$ is klt. (Check that this is more general than the case in 2.3!) On the level of Hodge polynomials this would be possible using [B2, Theorems 6.27 and 6.28]. We do not pursue this here; our invariants $E_W(X, D)$ in Section 3 and $\mathcal{E}_W(X, D)$ in Section 6 cover this case anyhow.

2.5

In the rest of this section we present an application on minimal models, taking $k = \mathbb{C}$.

Recall that an irreducible projective variety V is called a *minimal model* if V is terminal and K_V is *numerically effective* (shortly *nef*), i.e., the intersection number $K_V \cdot C \geq 0$ for any irreducible curve C on V. The Minimal Model Program predicts the existence of a minimal model in every birational equivalence class $\mathcal{C}$ of nonnegative Kodaira dimension; furthermore one should be able to transform every smooth irreducible projective variety in $\mathcal{C}$ by a finite number of divisorial contractions and flips to a minimal model.

In dimension 2 it is well known that each such class has a unique minimal model, which is moreover smooth (then divisorial contractions are just blowing-downs and flips do not occur). In dimension 3 the existence and desired property of minimal models were proved by Mori; here it is crucial to allow terminal singularities, and minimal models are *not* unique in a given birational equivalence class of nonnegative Kodaira dimension. In dimension ≥ 4 the Minimal Model Program is still a major conjecture in algebraic geometry and is becoming a working hypothesis.

It is natural and important in this context to look for invariants which are shared by birationally equivalent minimal models. In [Wa] Wang proved that birationally equivalent *smooth* minimal models have the same Betti numbers, using the following result [Wa, Corollary 1.10].

Proposition 2.6 *Let $f\colon V \dashrightarrow V'$ be a birational map between two minimal models. Then there exist a smooth projective variety Y and birational morphisms $\varphi\colon Y \to V$, $\varphi'\colon Y \to V'$ such that $\varphi^*K_V = \varphi'^*K_{V'}$.*

(In fact Wang only needs V and V' to be terminal varieties for which K_V and $K_{V'}$ are nef along the exceptional loci of f in V and V', respectively, to conclude.) This result has more interesting consequences.

Theorem 2.7 *Let V and V' be birationally equivalent minimal models. Then*

(i) $\mathcal{E}_V(V, 0) = \mathcal{E}_{V'}(V', 0)$, *and*
(ii) *if V and V' are smooth, then $[V] = [V']$ in $\hat{\mathcal{M}}$.*

Proof (i) Take $V \xleftarrow{\varphi} Y \xrightarrow{\varphi'} V'$ as in Proposition 2.6. Then by Theorem 1.10 (and its generalization in 1.11) we have

$$\mathcal{E}_V(V, 0) = \mathcal{E}_Y(Y, K_Y - \varphi^* K_V) = \mathcal{E}_Y(Y, K_Y - \varphi'^* K_{V'}) = \mathcal{E}_{V'}(V', 0).$$

(ii) For any smooth variety X we have that $\mathcal{E}_X(X, 0) = [X]$.   ∎

As a corollary birationally equivalent *smooth* minimal models have the same Hodge numbers and *a fortiori* the same Betti numbers. In particular this is true for smooth Calabi-Yau varieties. See also [B1, Theorems 1.1 and 4.2].

2.8

Assuming the Minimal Model Program in some dimension d we can use Theorem 2.7 to define a birational invariant. For any birational equivalence class $\mathcal{C}$ of nonnegative Kodaira dimension the expression $\mathcal{E} := \mathcal{E}_X(X, 0)$ is independent of a chosen minimal model X. Looking at 2.3 it is given by the following formula in terms of any log resolution $h\colon Y \to X$ of any minimal model X of $\mathcal{C}$. Denote by E_i, $i \in T$, the irreducible components of $h^{-1}(X_{\text{sing}})$ and set $K_Y = h^* K_X + \sum_{i \in T}(\nu_i - 1)E_i$. Then

$$\mathcal{E} = L^{-d} \sum_{I \subset T} [E_I^\circ] \prod_{i \in I} \frac{L - 1}{L^{\nu_i} - 1}$$

in some integral extension of $\hat{\mathcal{M}}$. Using weak factorization (see Remark 0.7), we can consider $\mathcal{E}$ more precisely in a suitable localization of an integral extension of $\mathcal{M}$.

One could extract 'minimal stringy Hodge numbers' from (the Hodge polynomial version of) this invariant, see [B2]; and maybe it is related to a 'minimal cohomology theory' as explained in [Wa].

3 Singular Varieties; On the Level of Hodge Polynomials and Euler Characteristics

3.1

Our aim in this paper is to associate zeta functions and 'Kontsevich' invariants to effective $\mathbb{Q}$-Cartier divisors D on arbitrary $\mathbb{Q}$-Gorenstein varieties X for which $X_{\text{sing}} \subset \text{supp}\, D$, generalizing the notions in Section 1. In this section we realize this on the level of Hodge polynomials and Euler characteristics in a fairly elementary way. The more general case on the level of the Grothendieck ring will be treated in Section 5.

3.2

We fix notation for this section. Let X be a $\mathbb{Q}$-Gorenstein variety and D an effective $\mathbb{Q}$-Cartier divisor on X. (When $\dim X = 2$ we only need that X is normal and D can be any effective Weil divisor with rational coefficients, see [V3].) For a log resolution $h\colon Y \to X$ of supp D we denote by E_i, $i \in T$, the irreducible components of $h^{-1}(X_{\text{sing}} \cup \text{supp } D)$ and we put $E_I^\circ := (\bigcap_{i \in I} E_i) \setminus (\bigcup_{\ell \notin I} E_\ell)$ for $I \subset T$. We also set $h^*D = \sum_{i \in T} N_i E_i$ and $K_Y = h^*K_X + \sum_{i \in T}(\nu_i - 1)E_i$. Remember that now the $\nu_i \in \mathbb{Q}$ and they can be negative or zero.

In the sequel we will again consider arbitrary subvarieties W of X. One can think mainly about W being for example X, supp D, X_{sing} or a point of X_{sing}.

3.3 Definition-Proposition

Let X be a $\mathbb{Q}$-Gorenstein variety of dimension d and W a subvariety of X. Let D be an effective $\mathbb{Q}$-Cartier divisor on X such that $X_{\text{sing}} \subset \text{supp } D$. Take $r \in \mathbb{N} \setminus \{0\}$ with rK_X and rD Cartier.

(i) The zeta function $Z_W(D, s) = Z_W(X, D, s)$ is the unique rational function in the variable $(uv)^{-s/r}$ and with coefficients in (the fraction field of) $\mathbb{Z}[u, v][(uv)^{1/r}]$ such that for $n \gg 0$
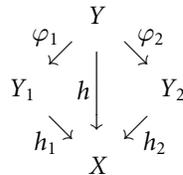$$Z_W(D, n) = E_{h^{-1}W}(Y, nh^*D + K_{Y|X}),$$
where $h\colon Y \to X$ is a resolution of X.

(ii) Let $h\colon Y \to X$ be a log resolution of supp D. With the notation of 3.2 we have that
$$Z_W(D, s) = \frac{1}{(uv)^d} \sum_{I \subset T} H(E_I^\circ \cap h^{-1}W) \prod_{i \in I} \frac{uv - 1}{(uv)^{\nu_i + sN_i} - 1}.$$

Proof Let $h_i\colon Y_i \to X$ be resolutions of X for $i = 1, 2$. We first show that the defining expressions for $Z_W(D, n)$ using Y_1 and Y_2 are equal when $n \gg 0$. Take n such that $nh_i^*D + K_{Y_i|X}$ is effective for $i = 1, 2$ (here we need that $X_{\text{sing}} \subset \text{supp } D$), and take a resolution $h\colon Y \to X$ of X dominating both Y_1 and Y_2.

$$
\begin{array}{ccc}
 & Y & \\
{}^{\varphi_1}\swarrow & \big\downarrow & \searrow{}^{\varphi_2} \\
Y_1 \quad h & & \quad Y_2 \\
{}_{h_1}\searrow & \big\downarrow & \swarrow{}_{h_2} \\
 & X &
\end{array}
$$

Then by Theorem 1.10 (for $\mathbb{Q}$-divisors and on the level of Hodge polynomials) we have for $i = 1, 2$ that
$$E_{h_i^{-1}W}(Y_i, nh_i^*D + K_{Y_i|X}) = E_{\varphi_i^{-1}h_i^{-1}W}\big(Y, \varphi_i^*(nh_i^*D + K_{Y_i|X}) + K_{Y|Y_i}\big)$$
$$= E_{h^{-1}W}(Y, nh^*D + K_{Y|X}).$$

Choosing now $h\colon Y \to X$ as a log resolution for supp D we have that

$$E_{h^{-1}W}(Y, nh^*D + K_{Y|X}) = \frac{1}{(uv)^d} \sum_{I \subset T} H(E_I^\circ \cap h^{-1}W) \prod_{i \in I} \frac{uv - 1}{(uv)^{\nu_i + nN_i} - 1}.$$

Hence for $n \gg 0$ the stated rational function in (ii) indeed yields $E_{h^{-1}W}(Y, nh^*D + K_{Y|X})$ when evaluating in $s = n$ (i.e., in $(uv)^{-s/r} = (uv)^{-n/r}$).

Finally this rational function must be unique since a polynomial over the domain $\mathbb{Z}[u, v][(uv)^{1/r}]$ can have at most finitely many zeroes.   ∎

Definition 3.4   With the same notation as in 3.3 we define the *topological zeta function* of D as

$$z_W(D, s) = z_W(X, D, s) := \sum_{I \subset T} \chi(E_I^\circ \cap h^{-1}W) \prod_{i \in I} \frac{1}{\nu_i + sN_i} \in \mathbb{Q}(s).$$

We can justify this definition either by an analogous proof or by obtaining $z_W(D, s)$ from $Z_W(D, s)$ by a limit argument as in 1.6.

3.5

In the following we extend Kontsevich's construction $E_W(X, D)$ to $\mathbb{Q}$-Gorenstein varieties X. We should remark here that in [DL3] Denef and Loeser also generalized in a different way $\mathcal{E}_W(X, D)$ to (arbitrary) singular varieties X. We consider their point of view as more 'integrational' and ours as more 'geometrical'. Our idea is simply to substitute $s = 1$ in $Z_W(D, s)$ when this makes sense or, more generally, to take the limit for $s \to 1$.

Definition 3.6   Let X be a $\mathbb{Q}$-Gorenstein variety and W a subvariety of X. Let D be an effective $\mathbb{Q}$-Cartier divisor on X such that $X_{\text{sing}} \subset$ supp D. Take $r \in \mathbb{N} \setminus \{0\}$ with rK_X and rD Cartier. Then we put

$$E_W(D) = E_W(X, D) := \lim_{s \to 1} Z_W(X, D, s) \in \mathbb{Q}(u^{1/r}, v^{1/r}) \cup \{\infty\}.$$

Remarks   (1) By $\lim_{s \to 1}$ we mean taking the limit $(uv)^{-s/r} \to (uv)^{-1/r}$. This is well defined since $Z_W(D, s)$ is a rational function in the variable $(uv)^{-s/r}$ over a field.

(2) If there exists a log resolution $h\colon Y \to X$ of supp D for which $\nu_i + N_i \neq 0$ for all $i \in T$, then, because of the formula in 3.3 (ii), we obtain $E_W(D)$ from $Z_W(D, s)$ simply by *substituting* $(uv)^{-1/r}$ for $(uv)^{-s/r}$. (We formulate this below as Proposition 3.7.) If on the other hand there does not exist such a log resolution, then in general we will have $E_W(D) = \infty$. However there are cases where our definition then yields an element in $\mathbb{Q}(u^{1/r}, v^{1/r})$, see Example 4.1.

Proposition 3.7 *Let $W \subset X$ and D be as in 3.6. Let $h\colon Y \to X$ be a log resolution of* supp D *for which $\nu_i + N_i \neq 0$ for all $i \in T$ (using the notation of 3.2). Then*

$$E_W(D) = \frac{1}{(uv)^d} \sum_{I \subset T} H(E_I^\circ \cap h^{-1}W) \prod_{i \in I} \frac{uv - 1}{(uv)^{\nu_i + N_i} - 1}.$$

So indeed we extended Kontsevich's invariant for smooth X on the level of Hodge polynomials (1.12).

3.8 Definition-Proposition

Let $W \subset X$ and D be as in 3.6. We define

$$e_W(D) = e_W(X, D) := \lim_{s \to 1} z_W(X, D, s) \in \mathbb{Q} \cup \{\infty\}.$$

Let $h\colon Y \to X$ be a log resolution of supp D for which $\nu_i + N_i \neq 0$ for all $i \in T$. Then

$$e_W(D) = \sum_{I \subset T} \chi(E_I^\circ \cap h^{-1}W) \prod_{i \in I} \frac{1}{\nu_i + N_i}.$$

3.9

Next we introduce analogous invariants for *pairs* (X, D), which will coincide with Batyrev's *stringy E-function* and *stringy Euler number* for klt pairs [B3].

3.10 Definition-Proposition

Let X be a $\mathbb{Q}$-Gorenstein variety and W a subvariety of X. Let D be an effective $\mathbb{Q}$-Cartier divisor on X such that $X_{\mathrm{sing}} \subset$ supp D. Take $r \in \mathbb{N} \setminus \{0\}$ with rK_X and rD Cartier.

(i) We put

$$E_W\big((X, D)\big) := \lim_{s \to -1} Z_W(X, D, s) \in \mathbb{Q}(u^{1/r}, v^{1/r}) \cup \{\infty\}.$$

(ii) Let $h\colon Y \to X$ be a log resolution of supp D. Using the notation of 3.2, let a_i, $i \in T$, denote the log discrepancy of E_i with respect to the pair (X, D). Then, if $a_i \neq 0$ for all $i \in T$, we have

$$E_W\big((X, D)\big) = \frac{1}{(uv)^d} \sum_{I \subset T} H(E_I^\circ \cap h^{-1}W) \prod_{i \in I} \frac{uv - 1}{(uv)^{a_i} - 1}.$$

Remark By $\lim_{s \to -1}$ we mean taking the limit $(uv)^{-s/r} \to (uv)^{1/r}$.

Proof If $\nu_i - N_i \neq 0$ for all $i \in T$, then, because of the formula for $Z_W(X, D, s)$ in 3.3 (ii) this limit procedure just means substituting $(uv)^{1/r}$ for the variable $(uv)^{-s/r}$. Clearly we obtain the stated formula for $E_W\big((X, D)\big)$ since $a_i = \nu_i - N_i$ for $i \in T$. $\blacksquare$

3.11

When the pair (X, D) is klt and for $W = X$ Batyrev introduced in [B3] the same
invariant as the *stringy E-function* of (X, D), denoted by $E_{\mathrm{st}}(X, D)$. (We do not re-
cover his invariant completely as a special case of $E_W\big((X, D)\big)$ because Batyrev only
requires $K_X + D$ to be $\mathbb{Q}$-Cartier.) Analogously the invariant $e_X\big((X, D)\big)$ below was
baptized *stringy Euler number* by Batyrev and denoted by $e_{\mathrm{st}}(X, D)$.

3.12 Definition-Proposition

Let $W \subset X$ and D be as in 3.10. We define

$$e_W\big((X, D)\big) := \lim_{s \to -1} z_W(X, D, s) \in \mathbb{Q} \cup \{\infty\}.$$

Let $h\colon Y \to X$ be a log resolution of supp D for which $\nu_i - N_i \neq 0$ for all $i \in T$. Then,
denoting by a_i the log discrepancy of E_i with respect to the pair (X, D), we have

$$e_W\big((X, D)\big) = \sum_{I \subset T} \chi(E_I^\circ \cap h^{-1}W) \prod_{i \in I} \frac{1}{a_i}.$$

3.13

In Definition-Proposition 3.3, and hence in all subsequent constructions, we required
the effective divisor D to satisfy $X_{\mathrm{sing}} \subset$ supp D. We needed this to assure that for a
resolution $h\colon Y \to X$ the divisor $nh^*D + K_{Y|X}$ would be effective for $n \gg 0$. However,
using $A5$ in the appendix, it is in fact sufficient to require that supp D contains the
locus of log canonical singularities $\mathrm{LCS}(X)$ of X.

3.14 Definition-Theorem

Let X be a $\mathbb{Q}$-Gorenstein variety of dimension d and W a subvariety of X. Let D be
an effective $\mathbb{Q}$-Cartier divisor on X such that $\mathrm{LCS}(X) \subset$ supp D. Take $r \in \mathbb{N} \setminus \{0\}$
with rK_X and rD Cartier.

(i) The zeta function $Z_W(D, s) = Z_W(X, D, s)$ is the unique rational function in the
     variable $(uv)^{-s/r}$ and with coefficients in (the fraction field of) $\mathbb{Z}[u, v][(uv)^{1/r}]$
     such that for $n \gg 0$

$$Z_W(D, n) = E_{h^{-1}W}(Y, nh^*D + K_{Y|X}),$$

     where $h\colon Y \to X$ is a *log resolution* for supp D.
(ii) With the notation of 3.2 for h we have that

$$Z_W(D, s) = \frac{1}{(uv)^d} \sum_{I \subset T} H(E_I^\circ \cap h^{-1}W) \prod_{i \in T} \frac{uv - 1}{(uv)^{\nu_i + sN_i} - 1}.$$
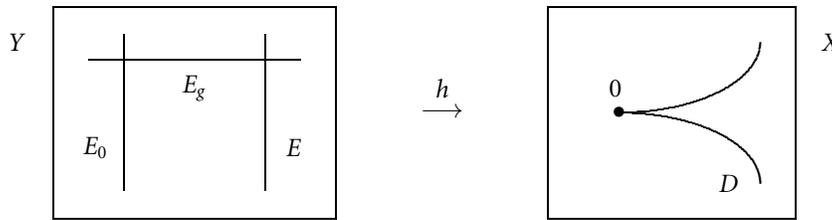
Figure 1

Proof We proceed analogously as in the proof of 3.3, but now working only with log resolutions $h\colon Y \to X$ of supp D. Then for $n \gg 0$ the coefficients $d_i = nN_i + \nu_i - 1$ of $nh^*D + K_{Y|X}$ all satisfy $d_i > -1$. Indeed any exceptional component E_i of h for which $\nu_i \leq 0$ satisfies $h(E_i) \subset \mathrm{LCS}(X) \subset$ supp D, and hence $N_i > 0$ for such an E_i. So in this case the invariant $E_{h^{-1}W}(Y, nh^*D + K_{Y|X})$ is well defined by A5 and Theorem A6. ∎

Remark In the formula above the 'denominators' $\nu_i + sN_i$ are thus always nonzero since either $\nu_i > 0$ or $N_i > 0$.

3.15

We can also extend all invariants which we considered in 3.4–3.12, *i.e.*, $z_W(D, s)$, $E_W(D)$, $e_W(D)$, $E_W\big((X, D)\big)$ and $e_W\big((X, D)\big)$, to the case that only $\mathrm{LCS}(X) \subset$ supp D.

In particular when X is log terminal and $D = 0$, then our invariants $E_X(0)$ and $e_X(0)$ are precisely the stringy E-function $E_{\mathrm{st}}(X; u, v)$ and stringy Euler number $e_{\mathrm{st}}(X)$ of Batyrev [B2].

4 Examples

In this section we present a number of examples, first in dimension two and then in higher dimension, for which we compute the invariants introduced above. Recall (see (0.4 (a))) that in dimension two we can consider more generally Weil divisors instead of Cartier divisors.

4.1

Let $0 \in X$ be a normal surface germ with minimal resolution $h\colon Y \to X$ such that $h^{-1}\{0\} = E_0 \cup E_g$, where E_0 and E_g are nonsingular curves of genus 0 and $g \geq 2$, respectively, intersecting transversely. So h is already a log resolution of X. (This singularity is quasihomogeneous.) Let E be a nonsingular curve (germ) in Y intersecting E_g transversely in one point and disjoint from E_0. Denote $D = h(E)$; so D is a prime Weil divisor on X through 0 and h is also a log resolution of D. See Figure 1.

Let $-\kappa_0$ and $-\kappa_g$ denote the self-intersection number of E_0 and E_g on Y, respec-

tively; we have that $\kappa_0 \geq 2$ and $\kappa_g \geq 1$. We will treat the germs $0 \in X$ for which $N := 2g - \kappa_g - 1 > 0$ in order to compute $z_0(ND, s)$ and $e_0(ND)$ for the effective Weil divisor ND on X.

We denote as usual $h^*ND = NE + N_0E_0 + N_gE_g$ and $K_Y = h^*K_X + (\nu_0 - 1)E_0 + (\nu_g - 1)E_g$. The following relations are well known (see for example [V3, Lemma 2.3]):

$$\begin{cases} \kappa_0 N_0 = N_g \\ \kappa_0 \nu_0 = \nu_g + 1 \end{cases} \quad \text{and} \quad \begin{cases} \kappa_g N_g = N_0 + N \\ \kappa_g \nu_g = (\nu_0 - 1) + 2 - 2g. \end{cases}$$

A short computation yields the expression for N_0, ν_0, N_g and ν_g in terms of our data κ_0, κ_g and g:

$$\begin{cases} N_0 = \frac{N}{\kappa_0 \kappa_g - 1} = \frac{2g - \kappa_g - 1}{\kappa_0 \kappa_g - 1} \\ \nu_0 = \frac{-2g + \kappa_g + 1}{\kappa_0 \kappa_g - 1} \end{cases} \quad \text{and} \quad \begin{cases} N_g = \frac{\kappa_0 N}{\kappa_0 \kappa_g - 1} = \frac{\kappa_0(2g - \kappa_g - 1)}{\kappa_0 \kappa_g - 1} \\ \nu_g = \frac{\kappa_0(1 - 2g) + 1}{\kappa_0 \kappa_g - 1}. \end{cases}$$

Remark that $\nu_0 + N_0 = 0$ (which, as you can guess, is forced by our choice of N); nevertheless $e_0(ND)$ will be a rational number. We have by definition that

$$z_0(ND, s) = \frac{1}{\nu_0 + sN_0} + \frac{1}{(\nu_0 + sN_0)(\nu_g + sN_g)} + \frac{-2g}{\nu_g + sN_g} + \frac{1}{(\nu_g + sN_g)(1 + sN)}$$

$$= \frac{1 + (\kappa_0 - 2g)(1 + sN)}{(\nu_g + sN_g)(1 + sN)}.$$

The fact that $\nu_0 + sN_0$ cancels in the denominator is a general fact; see [V3, 2.2]. Plugging in the expression for ν_g and N_g yields

$$z_0(ND, s) = \frac{(\kappa_0 \kappa_g - 1)[1 + (\kappa_0 - 2g)(1 + sN)]}{(1 - 2\kappa_0 g + \kappa_0(1 + sN))(1 + sN)} \quad \text{(with } N = 2g - \kappa_g - 1\text{)}$$

and

$$e_0(ND) = \lim_{s \to 1} z_0(ND, s) = \frac{(2g - \kappa_0)(2g - \kappa_g) - 1}{2g - \kappa_g}.$$

One can analogously compute $Z_0(ND, s)$ and $E_0(ND)$.

4.2

Let $0 \in X$ and $h \colon Y \to X$ be as above with $g = 1$ (instead of $g \geq 2$). Now let E' be a nonsingular curve germ in Y intersecting E_0 transversely in one point and disjoint from E_1, and denote $D' = h(E')$. See Figure 2.

One easily computes (see [V3, 2.5]) that

$$z_0(ND', s) = -\frac{\kappa_0 \kappa_1 - 1}{1 + sN} \quad \text{and thus} \quad e_0(ND') = -\frac{\kappa_0 \kappa_1 - 1}{1 + N}.$$

Now choose $N = \kappa_0 - 1$. It is easy to verify that then $\nu_1 + N_1 = 0$; so as in 4.1 we could not have defined $e_0\big((\kappa_0 - 1)D'\big)$ by the usual formula. However in this example our definition on the level of Hodge polynomials yields $E_0\big((\kappa_0 - 1)D'\big) = \infty$.
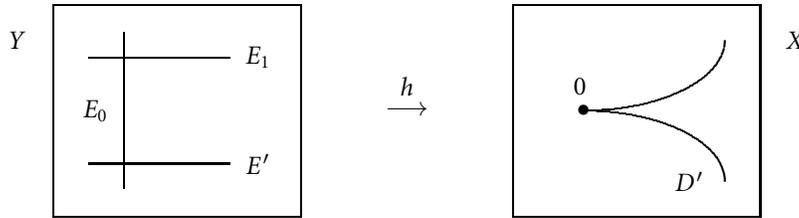
Figure 2

4.2.1   Remark

One could argue whether in Definition-Proposition 3.8 (and analogously in 3.12) it is more appropriate to introduce $e_W(D)$ as $\lim_{u,v \to 1} E_W(D)$. When $E_W(D) \neq \infty$ this amounts to the same, but when $E_W(D) = \infty$ we then would miss some interesting values of $e_W(D)$ as in 4.2 above.

4.3

Let X be the quadric hypersurface $\{xy - zw = 0\}$ in $\mathbb{A}^4$. The origin 0 is the only singular point of X. Blowing up 0 yields a log resolution $h_1 \colon Y_1 \to X$ of X, which is an isomorphism outside $h^{-1}\{0\}$ and with $E_1 = h^{-1}\{0\} \cong (\{xy - zw = 0\} \subset \mathbb{P}^3) \cong \mathbb{P}^1 \times \mathbb{P}^1$.

(a) Consider the divisor $D = E + E'$ on X, where E and E' are the zero sets of the functions $z - w$ and y on X, respectively. Remark that E is irreducible and that E' consists of two irreducible components. We want to compute $z_0(D, s)$. In this example we will use the same notation for divisors and their strict transforms by blowing-ups.

In Figure 3 we present the intersection configuration of E_1, E and E' on Y_1. The variety Y_1 is naturally covered by 4 affine charts, each isomorphic to $\mathbb{A}^3$. In the 'main chart' the exceptional surface E_1 and the strict transforms E and E' are given in affine coordinates x, z, w by

$$E_1 : x = 0,$$
$$E : z - w = 0,$$
$$E' : z \cdot w = 0$$

(in the other charts E and E' do not intersect).

We obtain a log resolution h of D by composing h_1 with the blowing-up $h_2 \colon Y_2 \to Y_1$ of the curve $E \cap E' (\cong \mathbb{A}^1)$ in Y_1. The exceptional variety E_2 of h_2 is isomorphic to $\mathbb{A}^1 \times \mathbb{P}^1$; the intersection configuration of E_2, E_1, E and E' is presented in Figure 4.

Denoting as usual $h^*D = E + E' + N_1 E_1 + N_2 E_2$ and $K_Y = h^*K_X + (\nu_1 - 1)E_1 + (\nu_2 - 1)E_2$, one easily verifies that $(\nu_1, N_1) = (2, 2)$ and $(\nu_2, N_2) = (2, 3)$. The contributors to $z_0(D, s)$ are E_1°, $(E_1 \cap E_2)^\circ$, $(E_1 \cap E)^\circ$, $(E_1 \cap E')^\circ$, $E_1 \cap E_2 \cap E$ and
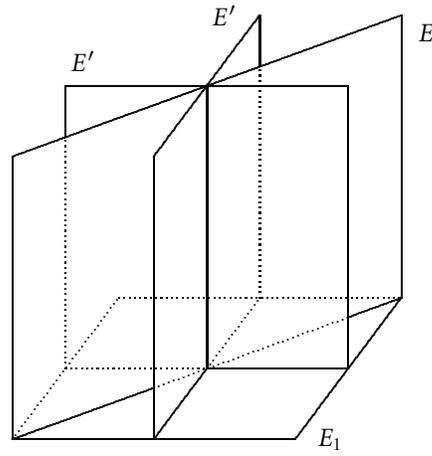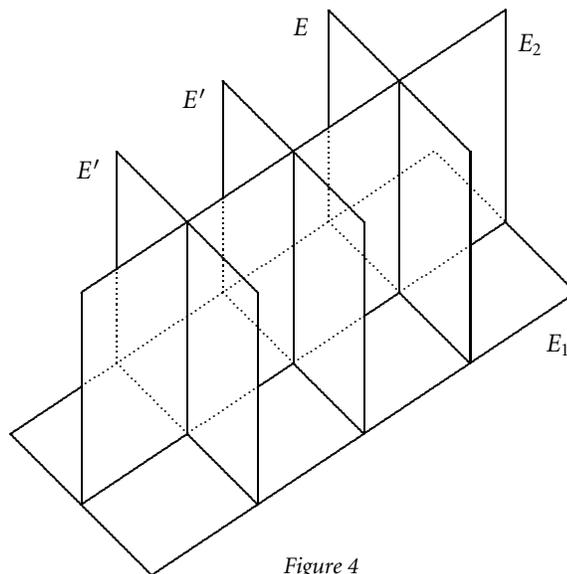
Figure 3

Figure 4

$E_1 \cap E_2 \cap E'$. Now $\chi(E_1^\circ) = 0$ and the other Euler characteristics are obvious; then

$$z_0(D,s) = \frac{1}{\nu_1 + sN_1}\left(\frac{-1}{\nu_2 + sN_2} + \frac{3}{1+s} + \frac{3}{(\nu_2 + sN_2)(1+s)}\right)$$

$$= \frac{4}{(2+3s)(1+s)}.$$

Also $e_0(D) = \lim_{s\to 1} z_0(D,s) = \frac{2}{5}$ and $e_0\big((X,D)\big) = \lim_{s\to -1} z_0(D,s) = \infty$.

(b) Now consider the $\mathbb{Q}$-divisor $D = NE + N'E'$ with $N > 0$, $N' > 0$, $N \neq 1$, $N' \neq 1$ and $N + N' = 2$. The morphism $h\colon Y_2 \to X$ in (a) is of course still a log resolution of D. The only difference with the data in (a) is that here $h^*D = NE + NE' + N_1E_1 + N_2E_2$ with $N_1 = N + N' = 2$ and $N_2 = N + 2N' = 2 + N'$. So

$$z_0(D,s) = \frac{1}{\nu_1 + sN_1}\Big(\frac{-1}{\nu_2 + sN_2} + \frac{1}{1+sN} + \frac{2}{1+sN'} + \frac{1}{(\nu_2 + sN_2)(1+sN)}$$

$$+ \frac{2}{(\nu_2 + sN_2)(1+sN')}\Big)$$

$$= \frac{1}{2+2s}\cdot\frac{8 + 16s + 8s^2}{\big(2 + s(2+N')\big)(1+sN)(1+sN')}$$

$$= \frac{4(1+s)}{\big(2 + s(2+N')\big)(1+sN)(1+sN')}.$$

And then $e_0(D,s) = \frac{8}{(4+N')(1+N)(1+N')}$ and $e_0\big((X,D)\big) = 0$.

4.4

Fix $d \in \mathbb{N}$, $d \geq 3$. Take a homogeneous polynomial F in $d + 1$ variables of degree $a \geq 2$ such that $\{F = 0\} \subset \mathbb{P}^d$ is nonsingular.

Let X be the hypersurface in $\mathbb{A}^{d+1}$ given by the zero set of F; so X is the affine cone over $\{F = 0\} \subset \mathbb{P}^d$ and the origin is the only singular point of X. Let D be the intersection of X with a general hyperplane through the origin in $\mathbb{A}^{d+1}$. The blowing-up $h\colon Y \to X$ of the origin yields a log resolution of X, which is moreover a log resolution of D. We denote the strict transform of D by E, and the exceptional variety of h by E_1. Notice that E_1 is isomorphic to $\{F = 0\} \subset \mathbb{P}^d$. We try to give an impression of this situation in Figure 5.

As usual we denote $K_Y = h^*K_X + (\nu_1 - 1)E_1$ and $h^*(ND) = NE + N_1E_1$ for $N \in \mathbb{Q}$, $N > 0$. One can verify that $\nu_1 = d + 1 - a$ and $N_1 = N$.

To compute $z_X(ND, s)$ we need the Euler characteristics of the varieties Y°, E°, E_1° and $E \cap E_1$ (which stratify Y). Since X and D are affine cones we have that

$$\chi(E^\circ) = \chi(D \setminus \{0\}) = 0 \quad \text{and} \quad \chi(Y^\circ) = \chi(X \setminus D) = 0.$$

Now E_1 is a nonsingular hypersurface of degree a in $\mathbb{P}^d$, yielding

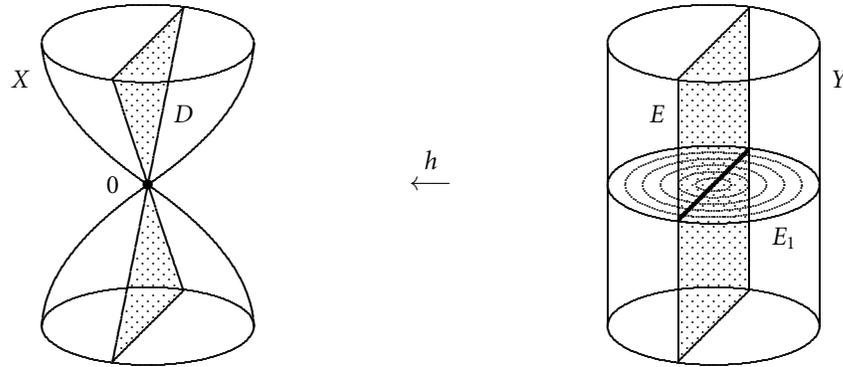$$\chi(E_1) = (1-a)\left(\frac{(1-a)^d - 1}{a}\right) + d$$

Figure 5

(see for example [Hirz]). And because D was chosen to be general we have moreover
that $E \cap E_1$ is a nonsingular hypersurface of degree a in $\mathbb{P}^{d-1}$; so

$$\chi(E \cap E_1) = (1-a)\left(\frac{(1-a)^{d-1} - 1}{a}\right) + d - 1.$$

Then finally $\chi(E_1^\circ) = \chi(E_1) - \chi(E \cap E_1) = -(1-a)^d + 1$ and

$$z_X(ND, s) = z_0(ND, s) = \frac{\chi(E_1^\circ)}{\nu_1 + sN_1} + \frac{\chi(E \cap E_1)}{(\nu_1 + sN_1)(1 + sN)}$$

$$= \frac{-(1-a)^d + 1}{d + 1 - a + sN} + \frac{(1-a)\left(\frac{(1-a)^{d-1} - 1}{a}\right) + d - 1}{(d + 1 - a + sN)(1 + sN)}$$

$$= \frac{(1-a)\left(\frac{(1-a)^d - 1}{a}\right) + d + s\left(1 - (1-a)^d\right)N}{(d + 1 - a + sN)(1 + sN)}.$$

A (not very exciting) calculation shows that there is no cancellation in this expression,
except when $d = 3$ and $a = 2$ or 3, in which case $z_X(ND, s)$ is

$$\frac{2}{1 + sN} \quad \text{and} \quad \frac{9}{1 + sN},$$

respectively. Taking limits we obtain

$$e_X(ND) = \frac{(1-a)\left(\frac{(1-a)^d - 1}{a}\right) + d + \left(1 - (1-a)^d\right)N}{(d + 1 - a + N)(1 + N)} \quad \text{if } d + 1 + N \neq a$$

and

$$e_X\left((X, ND)\right) = \frac{(1-a)\left(\frac{(1-a)^d - 1}{a}\right) + d + \left((1-a)^d - 1\right)N}{(d + 1 - a - N)(1 - N)} \quad \text{if } \begin{cases} d + 1 \neq a + N \\ N \neq 1. \end{cases}$$

5 Zeta Functions Associated to Divisors and Differential Forms

5.1

In the *p*-adic theory of Igusa's local zeta functions one also associates this invariant to both polynomials and differential forms, see *e.g.* [L, III3.5]. Let $f \in \mathbb{Q}_p[x] = \mathbb{Q}_p[x_1, \ldots, x_d]$ and $w \in \Omega^d_{\mathbb{A}^d}$, *i.e.*, $w = g\,dx$ where $g \in \mathbb{Q}_p[x]$ and $dx = dx_1 \wedge \cdots \wedge dx_d$. Then, with the notation of 1.3.1,

$$Z_p(f, w, s) := \int_{\mathbb{Z}_p^d} |f(x)|^s \, |g(x)| \, |dx|.$$

With the notation of 1.4 let $\nu'_i - 1$ be the multiplicity of E_i in the divisor of h^*w. Then (for $f \in \mathbb{Q}[x]$) the same formula as in 1.4.1 is valid when we replace ν_i by ν'_i.

We also want to introduce this notion on the level of the Grothendieck ring of algebraic varieties as in 1.3. Our motivation in this paper is that we will use it to construct on a $\mathbb{Q}$-Gorenstein variety X an invariant $\mathcal{Z}_W(X, D, s)$, generalizing $Z_W(X, D, s)$ in 3.3, on the level of the Grothendieck ring. Furthermore we will need this notion in future work.

5.2

We fix notations for this section. Let X be an irreducible nonsingular variety of dimension d and W a subvariety of X. Let D be an effective divisor on X and $J \subset \Omega^d_X$ an invertible subsheaf of the sheaf of regular differential d-forms Ω^d_X on X.

We will only consider the situation where $\operatorname{supp} J \subset \operatorname{supp} D$; we motivate this below.

5.3

First we rephrase the definition of $\mathcal{Z}_W(D, s)$ in terms of the *motivic volume* μ of [DL3, 3.2] or [DL4]. Denote by $\mathcal{C}$ the family of subsets of $\mathcal{L}(X)$ of the form $\pi_n^{-1}A_n$ for some $n \in \mathbb{N}$ and constructible subset A_n of $\mathcal{L}_n(X)$. We call these *cylindrical* subsets as in [B2] or [DL4]. There exists a unique additive measure $\mu\colon \mathcal{C} \to \mathcal{M}_L$ satisfying $\mu(\pi_n^{-1}A_n) = \frac{[A_n]}{L^{(n+1)d}}$ for A_n as above. (In fact this map is denoted by $\bar{\mu}$ in [DL3] and there μ is a map from the more complicated family of semi-algebraic subsets of $\mathcal{L}(X)$ to $\hat{\mathcal{M}}$.) For A in $\mathcal{C}$ and $\alpha\colon A \to \mathbb{N}$ a bounded function with cylindrical fibres one defines the integral

(5.3.1) $$\int_A L^{-\alpha} \, d\mu := \sum_{n \in \mathbb{N}} L^{-n} \mu(\alpha^{-1}\{n\}) \in \mathcal{M}_L.$$

Now re-examining the definition of $\mathcal{Z}_W(D, s)$ in 1.3 we have, with the notation introduced there, that $\mu(Y_{n,D,W}) = [X_{n,D,W}]L^{-(n+1)d}$ and hence

$$\mathcal{Z}_W(D, s) = \sum_{n \in \mathbb{N}} \mu(Y_{n,D,W})L^{-ns} \in \mathcal{M}_L[[L^{-s}]].$$

5.4

The following construction is a special case of [DL3, 3.5]. To the sheaf J is associated as follows a measure μ_J on $\mathcal{C}$, such that $\mu_{\Omega_X^d} = \mu$.

For $P \in X$ let dx and $g_P dx$ be local generators of Ω_X^d and J, respectively, around P. Denote then by $\operatorname{ord}_t J \colon \mathcal{L}(X) \to \mathbb{N} \cup \{\infty\}$ the function assigning to φ in $\mathcal{L}(X)$ the order of the power series given by $g_{\pi_0(\varphi)} \circ \varphi$. For A in $\mathcal{C}$ we define

$$\mu_J(A) := \int_A L^{-\operatorname{ord}_t J} d\mu = \sum_{\ell \in \mathbb{N}} L^{-\ell} \mu(A \cap \{\operatorname{ord}_t J = \ell\}).$$

Indeed the sets $\{\operatorname{ord}_t J = \ell\}$ are cylindrical. For arbitrary A the right hand side above is only defined as an element in $\widehat{\mathcal{M}}$; however we will only consider sets A for which the sum over ℓ is finite and then $\mu_J(A) \in \mathcal{M}_L$. Replacing μ by μ_J we can consider analogous integrals as in (5.3.1).

The following change of variables formula is a special case of [DL3, 3.5.2]. (It follows immediately from [DL3, 3.3] of which Theorem 1.10 is a special case.)

Proposition 5.5 *Let X' be another irreducible smooth variety and $\rho \colon X' \to X$ a proper birational morphism. For A in $\mathcal{C}$ and $\alpha \colon A \to \mathbb{N}$ a bounded function with cylindrical fibres we have that*

$$\int_A L^{-\alpha} d\mu_J = \int_{\rho^{-1}A} L^{-\alpha \circ \rho} d\mu_{\rho^* J}.$$

Definition 5.6 To the data of 5.2 we associate the *motivic zeta function*

$$\mathcal{Z}_W(D, J, s) = \mathcal{Z}_W(X, D, J, s) := \sum_{n \in \mathbb{N}} \mu_J(Y_{n,D,W}) L^{-ns}$$

$$= \sum_{n \in \mathbb{N}} \left(\sum_{\ell \in \mathbb{N}} L^{-\ell} \mu(Y_{n,D,W} \cap \{\operatorname{ord}_t J = \ell\}) \right) L^{-ns} \in \mathcal{M}_L[[L^{-s}]].$$

We explain why the sum over ℓ is finite. For any fixed n we have that $Y_{n,D,W} = \coprod_{\ell \in \mathbb{N} \cup \{\infty\}} (Y_{n,D,W} \cap \{\operatorname{ord}_t J = \ell\})$. But our condition $\operatorname{supp} J \subset \operatorname{supp} D$ implies that

$$\{\operatorname{ord}_t J = \infty\} = \mathcal{L}(\operatorname{supp} J) \subset \mathcal{L}(\operatorname{supp} D) = \{\operatorname{ord}_t D = \infty\},$$

hence we have that $Y_{n,D,W} \cap \{\operatorname{ord}_t J = \infty\} = \varnothing$ and so $Y_{n,D,W}$ is the countable union of the *cylindrical* sets $Y_{n,D,W} \cap \{\operatorname{ord}_t J = \ell\}$, $\ell \in \mathbb{N}$. Then this union is finite by [B2, Theorem 6.6].

Theorem 5.7 *Let X' be another irreducible smooth variety and $\rho \colon X' \to X$ a proper birational morphism. Then*

$$\mathcal{Z}_W(X, D, J, s) = \mathcal{Z}_{\rho^{-1}W}(X', \rho^* D, \rho^* J, s).$$

Proof This is a consequence of Proposition 5.5. ∎

Theorem 5.8 *Let $h\colon Y \to X$ be a log resolution of* supp D. *Denote as usual the irreducible components of $h^{-1}(\mathrm{supp}\, D)$ by E_i, $i \in T$. We set $h^*D = \sum_{i \in T} N_i E_i$ and $\mathrm{div}(h^*w) = \sum_{i \in T}(\nu_i' - 1)E_i$, where w is a local generator of J. Then*

$$\mathcal{Z}_W(D, J, s) = L^{-d} \sum_{I \subset T} [E_I^\circ \cap h^{-1}W] \prod_{i \in I} \frac{L - 1}{L^{\nu_i' + sN_i} - 1}.$$

Remark Let as in 1.4 dx be a local generator of Ω_X^d and $\mathrm{div}(h^*dx) = \sum_{i \in T}(\nu_i - 1)E_i$. Say $w = g\,dx$ and $\mathrm{div}(h^*g) = \sum_{i \in T} M_i E_i$. Then $\nu_i' = \nu_i + M_i$ for $i \in T$.

Proof One can adapt the proof of [DL2, Theorem 2.2.1] completely to this more general setting with the sheaf J. ∎

5.9

The notion introduced above is sufficient to introduce zeta functions on the level of the Grothendieck ring for Gorenstein varieties. To cover the case of $\mathbb{Q}$-Gorenstein varieties we need 'sheaves of multivalued differential forms'. We briefly describe this generalization.

Now let $J \subset (\Omega_X^d)^{\otimes m}$ be an invertible subsheaf of the m-fold tensor product of Ω_X^d, still satisfying supp $J \subset$ supp D. We define

$$\mu_{J^{1/m}}(A) := \int_A L^{-\frac{\mathrm{ord}_t J}{m}}\, d\mu = \sum_{\ell \in \mathbb{N}} L^{-\ell/m} \mu(A \cap \{\mathrm{ord}_t J = \ell\}) \in \mathcal{M}_L[L^{1/m}]$$

for the sets A in $\mathcal{C}$ for which the last sum is finite. Then the motivic zeta function is

$$\mathcal{Z}_W(D, J^{1/m}, s) = \mathcal{Z}_W(X, D, J^{1/m}, s)$$

$$:= \sum_{n \in \mathbb{N}} \mu_{J^{1/m}}(Y_{n,D,W}) L^{-ns} \in \mathcal{M}_L[L^{1/m}][[L^{-s}]].$$

Theorem 5.8 easily generalizes to this setting, but now the $\nu_i' \in \frac{1}{m}(\mathbb{N} \setminus \{0\})$.

5.10

Finally as in 1.5 we can generalize further to $\mathbb{Q}$-divisors. Now if D is an effective $\mathbb{Q}$-divisor on X, such that rD is a divisor for an $r \in \mathbb{N} \setminus \{0\}$, we define $\mathcal{Z}_W(D, J^{1/m}, s) := \mathcal{Z}_W(rD, J^{1/m}, s/r)$. Again Theorem 5.8 generalizes, with now the $N_i \in \frac{1}{r}(\mathbb{N} \setminus \{0\})$.

6 Singular Varieties; On the Level of the Grothendieck Ring

6.1

In this section we generalize the zeta function of 3.3 to the level of the Grothendieck ring. In order to focus on the main idea we first treat the essential case, being an effective Cartier divisor D on a Gorenstein variety X. For a normal variety V we denote its canonical sheaf (corresponding to K_V) by ω_V; we have that ω_V is invertible or $\omega_V^{\otimes m}$ is invertible for some $m \in \mathbb{N} \setminus \{0\}$ precisely when V is Gorenstein or $\mathbb{Q}$-Gorenstein, respectively.

Also in the sequel $\mathfrak{I}(F)$ denotes the sheaf of ideals associated to an effective divisor F on a nonsingular variety.

6.2 Definition-Proposition

Let X be a Gorenstein variety of dimension d and W a subvariety of X. Let D be an effective Cartier divisor on X such that $X_{\text{sing}} \subset \text{supp}\, D$.

(i) The *motivic zeta function*

$$\mathcal{Z}_W(D,s) = \mathcal{Z}_W(X,D,s) := \mathcal{Z}_{h^{-1}W}\left(Y, h^*D, h^*\omega_X \otimes \mathfrak{I}(ah^*D), s-a\right)$$

where $h\colon Y \to X$ is a log resolution of $\text{supp}\, D$ and $a \in \mathbb{N}, a \gg 0$.

(ii) Let $h\colon Y \to X$ be a log resolution of $\text{supp}\, D$. With the notation of 3.2 we have that

$$\mathcal{Z}_W(D,s) = L^{-d} \sum_{I \subset T} [E_I^\circ \cap h^{-1}W] \prod_{i \in I} \frac{L-1}{L^{\nu_i + s N_i} - 1}.$$

Proof We first explain the right hand side of our definition. Since $X_{\text{sing}} \subset \text{supp}\, D$ we have that $\text{supp}(h^*\omega_X) \subset \text{supp}(h^*D)$, yielding for $a \gg 0$ that $h^*\omega_X \otimes \mathfrak{I}(ah^*D)$ is an invertible subsheaf of Ω_Y^d. So to this sheaf and the effective divisor h^*D we can associate the motivic zeta function of 5.6. The substitution '$s-a$ instead of s' means replacing the variable L^{-s} by $L^a(L^{-s})$.

Now we show independency of the chosen resolution; it is sufficient to consider another log resolution $h'\colon Y' \to X$ that factors as $h'\colon Y' \xrightarrow{\varphi} Y \xrightarrow{h} X$. By Theorem 5.7 we indeed have that

$$\mathcal{Z}_{h^{-1}W}\left(Y, h^*D, h^*\omega_X \otimes \mathfrak{I}(ah^*D), s-a\right)$$
$$= \mathcal{Z}_{\varphi^{-1}(h^{-1}W)}\left(Y', \varphi^*h^*D, \varphi^*(h^*\omega_X) \otimes \varphi^*\left(\mathfrak{I}(ah^*D)\right), s-a\right)$$
$$= \mathcal{Z}_{h'^{-1}W}\left(Y', h'^*D, h'^*\omega_X \otimes \mathfrak{I}(ah'^*D), s-a\right).$$

Let η and f be local generators of ω_X and $\mathfrak{I}(D)$, respectively. Then $(h^*f)^a(h^*\eta)$ is a local generator of $h^*\omega_X \otimes \mathfrak{I}(ah^*D)$ and its divisor of zeroes is $\sum_{i \in T}\left((\nu_i - 1) + aN_i\right)E_i$. Hence Theorem 5.8 (with $h = \text{Id}_Y$) yields the stated formula for $\mathcal{Z}_W(D,s)$, which also proves independency of the number a. ∎

6.3

Now let X be $\mathbb{Q}$-Gorenstein and say that mK_X is Cartier for some $m \in \mathbb{N} \setminus \{0\}$. We define $\mathcal{Z}_W(X, D, s)$ just as in 6.2, interpreting the expression $h^* \omega_X \otimes \mathcal{I}(ah^*D)$ as an abbreviation of $\left(h^*(\omega_X^{\otimes m}) \otimes \mathcal{I}(mah^*D) \right)^{1/m}$ (see 5.9). Now $\mathcal{Z}_W(X, D, s)$ lives in a localization of $\mathcal{M}_L[L^{1/m}][L^{-s}]$ and is given by the same formula as in 6.2 (with now the $\nu_i \in \mathbb{Q}$).

When D is an effective $\mathbb{Q}$-Cartier divisor we set as usual $\mathcal{Z}_W(D, s) := \mathcal{Z}_W(rD, s/r)$ if rD is Cartier for an $r \in \mathbb{N} \setminus \{0\}$. Then in full generality we have the following.

6.4 Definition-Proposition

Let X be a $\mathbb{Q}$-Gorenstein variety of dimension d and W a subvariety of X. Let D be an effective $\mathbb{Q}$-Cartier divisor on X (with rD Cartier for an $r \in \mathbb{N} \setminus \{0\}$) such that $X_{\text{sing}} \subset \text{supp } D$. The *motivic zeta function*

$$\mathcal{Z}_W(D, s) = \mathcal{Z}_W(X, D, s) := \mathcal{Z}_{h^{-1}W}\left(Y, h^*(rD), h^*\omega_X \otimes \mathcal{I}(arh^*D), s/r - a \right)$$

where $h \colon Y \to X$ is a log resolution of $\text{supp } D$ and $a \in \mathbb{N}, a \gg 0$. We have the same formula as in 6.2.

Of course $\mathcal{Z}_W(D, s)$ specializes to the zeta function $Z_W(D, s)$ of 3.3.

6.5

Finally we consider for arbitrary $\mathbb{Q}$-Gorenstein varieties X 'Kontsevich' invariants $\mathcal{E}_W(D)$ and $\mathcal{E}_W\big((X, D)\big)$ on the level of the Grothendieck ring, which specialize to $E_W(D)$ and $E_W\big((X, D)\big)$ of 3.6 and 3.10, respectively. Notice first that in 6.4 we have, by the formula for $\mathcal{Z}_W(D, s)$ in terms of a log resolution, that it already belongs to the localization of a polynomial ring $\mathcal{M}_L[L^{1/r}][L^{-s/r}]$ with respect to $(1 - L^{-\alpha - \beta s})_{\alpha \in \mathbb{Q}, \beta \in \mathbb{Q}_{>0}}$. Morally we again take limits for $s \to 1$ and $s \to -1$ to define $\mathcal{E}_W(D)$ and $\mathcal{E}_W\big((X, D)\big)$, respectively.

Definition 6.6 Let X be a $\mathbb{Q}$-Gorenstein variety of dimension d and W a subvariety of X. Let D be an effective $\mathbb{Q}$-Cartier divisor on X such that $X_{\text{sing}} \subset \text{supp } D$. Take $r \in \mathbb{N} \setminus \{0\}$ with rK_X and rD Cartier.

(i) If $\mathcal{Z}_W(D, s)$ belongs to the localization of $\mathcal{M}_L[L^{1/r}][L^{-s/r}]$ with respect to $(1 - L^{-\alpha - \beta s})_{\alpha \in \mathbb{Q}, \beta \in \mathbb{Q}_{>0}, \alpha + \beta \neq 0}$, then we put

$$\mathcal{E}_W(D) = \mathcal{E}_W(X, D) := \mathcal{Z}_W(D, s)|_{s=1}.$$

Otherwise we put $\mathcal{E}_W(D) = \mathcal{E}_W(X, D) := \infty$.

(ii) If $\mathcal{Z}_W(D, s)$ belongs to the localization of $\mathcal{M}_L[L^{1/r}][L^{-s/r}]$ with respect to $(1 - L^{-\alpha - \beta s})_{\alpha \in \mathbb{Q}, \beta \in \mathbb{Q}_{>0}, \alpha \neq \beta}$, then we put

$$\mathcal{E}_W\big((X, D)\big) := \mathcal{Z}_W(D, s)|_{s=-1}.$$

Otherwise we put $\mathcal{E}_W\big((X, D)\big) := \infty$.

Here the evaluations $s = 1$ and $s = -1$ mean substituting the variable $L^{-s/r}$ by $L^{-1/r}$ and $L^{1/r}$, respectively, yielding a well defined element in $\hat{\mathcal{M}}[L^{1/r}]$.

Proposition 6.7 *Consider the same data as is 6.6.*

(i) *Suppose there is a log resolution $h\colon Y \to X$ of supp D for which $\nu_i + N_i \neq 0$ for all $i \in T$ (using the notation of 3.2). Then*

$$\mathcal{E}_W(D) = L^{-d} \sum_{I \subset T} [E_I^\circ \cap h^{-1}W] \prod_{i \in I} \frac{L-1}{L^{\nu_i+N_i}-1}.$$

(ii) *Suppose there is a log resolution $h\colon Y \to X$ of supp D for which all log discrepancies a_i, $i \in T$, with respect to the pair (X, D) satisfy $a_i \neq 0$ (using the notation of 3.2). Then*

$$\mathcal{E}_W\big((X,D)\big) = L^{-d} \sum_{I \subset T} [E_I^\circ \cap h^{-1}W] \prod_{i \in I} \frac{L-1}{L^{a_i}-1}.$$

Appendix

A1

In this appendix let X be a smooth irreducible variety of dimension d and W a subvariety of X.

In 1.7–1.10 we described the Kontsevich invariant $\mathcal{E}_W(D) \in \hat{\mathcal{M}}$, associated to an effective divisor D on X, and we mentioned its important properties. Here we will generalize this notion to effective $\mathbb{Q}$-divisors; if rD is a divisor for an $r \in \mathbb{N} \setminus \{0\}$ we obtain an invariant $\mathcal{E}_W(D)$ in a finite extension $\hat{\mathcal{M}}[L^{1/r}]$ of $\hat{\mathcal{M}}$, and we treat analogous properties. We also introduce this invariant for a $\mathbb{Q}$-divisor $D = \sum_i d_i D_i$ (with the D_i irreducible) such that all $d_i > -1$ and supp $D = \bigcup_i D_i$ is a divisor with strict normal crossings. This is used in 3.14.

A2

First we describe the ring $\hat{\mathcal{M}}[L^{1/r}]$. Consider the integral ring extension $\mathcal{M}_L \hookrightarrow \mathcal{M}_L[L^{1/r}] := \frac{\mathcal{M}_L[X]}{(X^r - L)}$, where $L^{1/r}$ is the class of X in this quotient. Each element $a \in \mathcal{M}_L[L^{1/r}]$ has a unique expression of the form $a = \sum_{i=0}^{r-1} a_i L^{i/r}$ or $a = \sum_{i=0}^{r-1} a_i' L^{-i/r}$ with $a_i, a_i' \in \mathcal{M}_L$.

We extend the decreasing filtration $(F_m)_{m \in \mathbb{Z}}$ on $\mathcal{M}_L$, introduced in 1.7, to the ring $\mathcal{M}_L[L^{1/r}]$. Let F'_m, $m \in \mathbb{Z}$, be the subgroup of $\mathcal{M}_L[L^{1/r}]$ generated by

$$\left\{ \sum_{i=0}^{r-1} \frac{[A_i]}{L^{n_i}} L^{-i/r} \ \middle| \ \dim A_i - n_i \leq -m \text{ for } i = 0, \dots, r-1 \right\}.$$

(So indeed $F_m = \mathcal{M}_L \cap F'_m$.) We take the completion $\hat{\mathcal{M}}'$ of $\mathcal{M}_L[L^{1/r}]$ with respect to this filtration $(F'_m)_{m \in \mathbb{Z}}$; then we have an injection $\hat{\mathcal{M}} \hookrightarrow \hat{\mathcal{M}}'$.

One can verify that $\hat{\mathcal{M}}' \cong \hat{\mathcal{M}}[L^{1/r}]$, where the right hand side can be interpreted either as the subring of $\hat{\mathcal{M}}'$ generated by $\hat{\mathcal{M}}$ and $L^{1/r}$, or as $\frac{\hat{\mathcal{M}}[X]}{(X^r - L)}$.

A3

We will use the following notation. Let D be a prime divisor on X. Then $\operatorname{ord}_t D$: $\mathcal{L}(X) \to \mathbb{N} \cup \{\infty\}$ assigns to $\varphi \in \mathcal{L}(X)$ the order of the power series in t given by $f \circ \varphi$, where f is a local equation of D at $\pi_0(\varphi)$. For a $\mathbb{Q}$-divisor $D = \sum_i d_i D_i$ (with the D_i prime divisors) we then define $\operatorname{ord}_t D$: $\mathcal{L}(X) \to \mathbb{Q} \cup \{\infty\}$ by $\operatorname{ord}_t D :=$ $\sum_i d_i \operatorname{ord}_t D_i$.

Definition A4 Let D be a $\mathbb{Q}$-divisor on X and $r \in \mathbb{N} \setminus \{0\}$ such that rD is a divisor.

(i) If D is effective we define for $n \in \mathbb{N}$ the subscheme $Y_{n,D,w}$ of $\mathcal{L}(X)$ and the subscheme $X_{n,D,w}$ of $\mathcal{L}_n(X)$ as in 1.3 with only the following adaptation: now f is a local equation of the divisor rD (instead of D). Then we set

$$\mathcal{E}_W(D) = \mathcal{E}_W(X, D) := \sum_{n \in \mathbb{N}} \frac{[X_{n,D,W}]}{L^{(n+1)d}} L^{-n/r} \in \hat{\mathcal{M}}[L^{1/r}].$$

In terms of the motivic volume μ of 5.3 we can describe $\mathcal{E}_W(D)$ as

$$\mathcal{E}_W(D) = \int_{\pi_0^{-1} W} L^{-\operatorname{ord}_t D} \, d\mu := \sum_{n \in \mathbb{N}} \mu\left(\pi_0^{-1} W \cap \left\{\operatorname{ord}_t D = \frac{n}{r}\right\}\right) L^{-n/r}.$$

(ii) In general we say that $\operatorname{ord}_t D$: $\mathcal{L}(X) \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ is integrable on $\pi_0^{-1} W$ if

$$\int_{\pi_0^{-1} W} L^{-\operatorname{ord}_t D} \, d\mu := \sum_{n \in \mathbb{Z}} \mu\left(\pi_0^{-1} W \cap \left\{\operatorname{ord}_t D = \frac{n}{r}\right\}\right) L^{-n/r}$$

converges in $\hat{\mathcal{M}}[L^{1/r}]$; we then denote this invariant again by $\mathcal{E}_W(D)$.

A5

An important case of this last definition occurs when $D = \sum_{i=1}^{k} d_i D_i$ with the D_i irreducible, all $d_i > -1$, and $\operatorname{supp} D = \bigcup_{i=1}^{k} D_i$ a divisor with strict normal crossings. For $J \subset \{1, \ldots, k\}$ denote $D_J^\circ := (\bigcap_{j \in J} D_j) \setminus (\bigcup_{\ell \notin J} D_\ell)$ and $M_J := \{(m_1, \ldots, m_k) \in \mathbb{N}^k \mid m_j > 0 \Leftrightarrow j \in J\}$. Then one can compute that

$$\int_{\pi_0^{-1} W} L^{-\operatorname{ord}_t D} \, d\mu = L^{-d} \sum_{J \subset \{1, \ldots, k\}} (L - 1)^{|J|} [D_J^\circ \cap W] \sum_{(m_1, \ldots, m_k) \in M_J} L^{-\sum_{j \in J}(d_j+1)m_j},$$

which converges in $\hat{\mathcal{M}}[L^{1/r}]$ since all $d_j + 1 > 0$. See [B2, Theorem 6.28] and [C, Theorem 1.17].

Theorem A6 *Let also X' be a smooth irreducible variety and $\rho \colon X' \to X$ a proper birational morphism. Let D be a $\mathbb{Q}$-divisor on X. Then $\operatorname{ord}_t D$ is integrable on $\pi_0^{-1} W$ if and only if $\operatorname{ord}_t(\rho^* D + K_{X'|X})$ is integrable on $\pi_0^{-1}(\rho^{-1} W)$; and in this case*

$$\mathcal{E}_W(X, D) = \mathcal{E}_{\rho^{-1} W}(X', \rho^* D + K_{X'|X}).$$

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Proof The proof of [DL3, Lemma 3.3], based on the crucial and difficult [DL3, Lemma 3.4], can be adapted to this setting. See [B2, Theorem 6.27] for an analogous statement and proof when $W = X$. We also remark that when D is an *effective* $\mathbb{Q}$-divisor (implying that both functions are integrable), then one can prove the stated equality *using* the equality in [DL3, Lemma 3.3]. ∎

Theorem A7 *Let D be a $\mathbb{Q}$-divisor on X (with rD a divisor for an $r \in \mathbb{N} \setminus \{0\}$) such that $\mathrm{ord}_t D$ is integrable on $\pi_0^{-1}W$. Using the notation of 1.4 we have the following formula for $\mathcal{E}_W(D)$ in terms of a log resolution $h: Y \to X$ of supp D:*

$$\mathcal{E}_W(D) = L^{-d} \sum_{I \subset T} [E_I^\circ \cap h^{-1}W] \prod_{i \in I} \frac{L-1}{L^{\nu_i + N_i} - 1} \quad in \quad \hat{\mathcal{M}}[L^{1/r}].$$

In particular $\mathcal{E}_W(D)$ belongs to the image of $\mathcal{M}_L[(1 - L^{-n/r})^{-1}]_{n \in \mathbb{N} \setminus \{0\}}$ in $\hat{\mathcal{M}}[L^{1/r}]$.

Proof This follows from A5 and Theorem A6. One can also adapt [DL3, (6.5)]. ∎

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