Canad. Math. Bull. Vol. **23** (2), 1980

ON THE CONTINUED FRACTIONS OF CONJUGATE QUADRATIC IRRATIONALITIES

BY
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1. **Introduction.** Let

$$x=[a_1, a_2, a_3, \cdots] \tag{1}$$

be the simple continued fraction (SCF) of an irrational number x. The partial quotients a_i which we shall sometimes refer to as the "terms" of the SCF are integers and, for $i \geq 2$, they are positive. If x is a quadratic irrationality then, by Lagrange's Theorem, the right side of (1) becomes periodic from some point on. In that case we shall use the notation

$$x=[a_1, a_2, \ldots, a_h, \overline{b_1, b_2, \ldots, b_k}], \tag{2}$$

where $a_1, a_2, \ldots, a_h$ is the non-periodic part ("Vorperiode") and $\overline{b_1, b_2, \ldots, b_k}$ is the period. It will be important to make the value of h unique; for this purpose we shall assume throughout the paper that, in case $h \geq 1$, the inequality

$$a_h \neq b_k \tag{3}$$

should always be satisfied. For the quadratic irrationality conjugate to x we shall use the notation y, that is, if $x=(\sqrt{D}+P)/Q$ then $y=(-\sqrt{D}+P)/Q$.

It was shown by Serret [4; Theorem III, p. 49] that the period of the SCF for y is the inverse of that for x, namely, $\overline{b_k, b_{k-1}, \ldots, b_1}$ or a cyclic permutation thereof if we wish to maintain the principle of inequality (3).

Now x is purely periodic or $h=0$ if and only if x is a reduced quadratic irrationality, that is, $x>1$ and $-1<y<0$. (See [3; Theorems 3.3, 3.4, pp. 73, 74].) In the case of mixed periodicity or $h \geq 1$, a number of results concerning the signs of x and y have been obtained by Serret [4; Corollary, p. 47], Kraitchik [2; pp. 13, 14] and Elte [1]. These results can be summarized as follows: In the case $h=1$, if $x>0$ then x and y are of equal signs if $a_1>b_k$, of opposite signs if $a_1<b_k$. In the case $h=2$, if $x>1$ then x and y are always of the same sign; if $0<x<1$ then x and y are of the same sign if $a_2>b_k$, of opposite signs if $a_2<b_k$. And in the case $h \geq 3$ if $x>0$ then x and y are always of the same sign. That in these statements the hypothesis restricting the range of x cannot be dispensed with is seen from the first three Examples in §2 below.

Received by the editors September 30, 1977 and, in revised form, November 30, 1978.

https://doi.org/10.4153/CMB-1980-027-x Published online by Cambridge University Press

In this paper we investigate in a more general way the relations between the SCF's of x and y. As our main result, we obtain in §2 completely general *formulas which give in all cases the SCF for y directly in terms of the partial quotients of the SCF for x.* (Theorem 1). In this manner a variety of problems, more general than those mentioned in the preceding paragraph, concerning pairs of conjugate quadratic irrationalities, can be dealt with. We remark that the formulas of Theorem 1 were originally obtained by the use of formulas for the SCF for $-x$ (see [3; p. 56, lines 2, 3]) and similar formulas for the SCF for $1/x$. But it is easier to prove the formulas of Theorem 1 directly by verification. Also in §2 several examples are given to illustrate those formulas.

In §3 three Theorems are proved that are direct consequences of the formulas of Theorem 1. In Theorem 2 we show, for all values of h, the most general inequalities that exist between $\alpha=[x]$ and $\beta=[y]$. In Theorem 3 the above-mentioned question as to when x and y have the same or opposite signs is completely answered, but without any restrictions on the range of x. Finally in Theorem 4 it is shown that the lengths of the non-periodic parts of x and y differ from each other by at most 2, except in a single case where they are 0 and 3, respectively.

2. **The main result.** We are going to use the following elementary properties of continued fractions. If x is given by (1) and if $x_{n+1}=[a_{n+1}, a_{n+2}, \ldots]$ is the $(n+1)$st complete quotient of (1) then

$$x=[a_1, a_2, \ldots, a_n, x_{n+1}]. \tag{4}$$

(See [3; IIA, p. 23].) Also

$$[0, a_2, a_3, \ldots]=\frac{1}{[a_2, a_3, \ldots]} \tag{5}$$

and, if g is any integer,

$$g+[a_1, a_2, a_3, \ldots]=[g+a_1, a_2, a_3, \ldots]. \tag{6}$$

The only cyclic permutations of the inverse period $\overline{b_k, b_{k-1}, \ldots, b_2, b_1}$ (see §1) which will occur in Theorem 1 below are the following three for which we shall use the indicated notations:

$$\begin{aligned} P_1&=\overline{b_{k-1}, b_{k-2}, \ldots, b_1, b_k},\\ P_2&=\overline{b_{k-2}, b_{k-3}, \ldots, b_1, b_k, b_{k-1}},\\ P_3&=\overline{b_{k-3}, b_{k-4}, \ldots, b_1, b_k, b_{k-1}, b_{k-2}}.\end{aligned} \tag{7}$$

We remark that, for $k\le 3$, whenever in (7) or in Columns 3 and 4 of Table I below a term b_i occurs whose index does not lie in the range $1\le i\le k$ the correct interpretation of such a term can be obtained by changing the index

https://doi.org/10.4153/CMB-1980-027-x Published online by Cambridge University Press

modulo k. This amounts to the same thing as replacing the primitive period in (2) temporarily by an imprimitive one of appropriate length and in the end returning to the primitive period. Thus for $k=1$, $P_1=P_2=P_3=\overline{b_1}$; for $k=2$, $P_2=\overline{b_2, b_1}$, $P_3=\overline{b_1, b_2}$; and for $k=3$, $P_3=\overline{b_3, b_2, b_1}$.

Each case listed in Table I is identified by a symbol (see Column 2) whose first component represents the value of h. Where the letter h itself is used it is to represent any integer greater than or equal to 4.

It will be noted that, in all cases listed in Column 4 of Table I, the principle of inequality (3) is followed. It is for this reason that the cases $2C$ and $3C$ are subdivided into two subcases each. In the proof, however, $2C'$ and $3C'$ can be treated as special cases of $2C''$ and $3C''$, respectively.

THEOREM 1. *If x is a quadratic irrationality whose simple continued fraction is given by* (2) *then, depending on the conditions stated in Column* 3 *of Table* I, *the simple continued fraction of y, the conjugate of x, is given in Column* 4 *of Table* I. (For the definition of the periods P_i, see (7).)

TABLE I

h	Case	Conditions	Simple Continued Fraction for y
0	$0A$	$b_k=1$	$[-1, b_{k-1}+1, P_2]$
0	$0B$	$b_k\geq 2$	$[-1, 1, b_k-1, P_1]$
1	$1A$	$b_{k-1}=1$	$[a_1-b_k-1, b_{k-2}+1, P_3]$
1	$1B$	$b_{k-1}\geq 2$	$[a_1-b_k-1, 1, b_{k-1}-1, P_2]$
2	$2A$	$a_2\leq b_k-2$	$[a_1-1, 1, -a_2+b_k-1, P_1]$
2	$2B$	$a_2=b_k-1$	$[a_1-1, b_{k-1}+1, P_2]$
2	$2C'$	$a_2=b_k+1, b_{k-1}=1, a_1=-1$	$[P_2]$
2	$2C''$	$a_2=b_k+1, b_{k-1}=1, a_1\neq -1$	$[a_1+b_{k-2}+1, P_3]$
2	$2D$	$a_2=b_k+1, b_{k-1}\geq 2$	$[a_1+1, b_{k-1}-1, P_2]$
2	$2E$	$a_2\geq b_k+2, b_{k-1}=1$	$[a_1, a_2-b_k-1, b_{k-2}+1, P_3]$
2	$2F$	$a_2\geq b_k+2, b_{k-1}\geq 2$	$[a_1, a_2-b_k-1, 1, b_{k-1}-1, P_2]$
3	$3A$	$a_3\leq b_k-2, a_2=1$	$[a_1+1, -a_3+b_k-1, P_1]$
3	$3B$	$a_3\leq b_k-2, a_2\geq 2$	$[a_1, a_2-1, 1, -a_3+b_k-1, P_1]$
3	$3C'$	$a_3=b_k-1, a_2=1, a_1=-1$	$[P_1]$
3	$3C''$	$a_3=b_k-1, a_2=1, a_1\neq -1$	$[a_1+b_{k-1}+1, P_2]$
3	$3D$	$a_3=b_k-1, a_2\geq 2$	$[a_1, a_2-1, b_{k-1}+1, P_2]$
3	$3E$	$a_3=b_k+1, b_{k-1}=1$	$[a_1, a_2+b_{k-2}+1, P_3]$
3	$3F$	$a_3=b_k+1, b_{k-1}\geq 2$	$[a_1, a_2+1, b_{k-1}-1, P_2]$
3	$3G$	$a_3\geq b_k+2, b_{k-1}=1$	$[a_1, a_2, a_3-b_k-1, b_{k-2}+1, P_3]$
3	$3H$	$a_3\geq b_k+2, b_{k-1}\geq 2$	$[a_1, a_2, a_3-b_k-1, 1, b_{k-1}-1, P_2]$
≥ 4	hA	$a_h\leq b_k-2, a_{h-1}=1$	$[a_1,\ldots, a_{h-3}, a_{h-2}+1, -a_h+b_k-1, P_1]$
≥ 4	hB	$a_h\leq b_k-2, a_{h-1}\geq 2$	$[a_1,\ldots, a_{h-2}, a_{h-1}-1, 1, -a_h+b_k-1, P_1]$
≥ 4	hC	$a_h=b_k-1, a_{h-1}=1$	$[a_1,\ldots, a_{h-3}, a_{h-2}+b_{k-1}+1, P_2]$
≥ 4	hD	$a_h=b_k-1, a_{h-1}\geq 2$	$[a_1,\ldots, a_{h-2}, a_{h-1}-1, b_{k-1}+1, P_2]$
≥ 4	hE	$a_h=b_k+1, b_{k-1}=1$	$[a_1,\ldots, a_{h-2}, a_{h-1}+b_{k-2}+1, P_3]$
≥ 4	hF	$a_h=b_k+1, b_{k-1}\geq 2$	$[a_1,\ldots, a_{h-2}, a_{h-1}+1, b_{k-1}-1, P_2]$
≥ 4	hG	$a_h\geq b_k+2, b_{k-1}=1$	$[a_1,\ldots, a_{h-1}, a_h-b_k-1, b_{k-2}+1, P_3]$
≥ 4	hH	$a_h\geq b_k+2, b_{k-1}\geq 2$	$[a_1,\ldots, a_{h-1}, a_h-b_k-1, 1, b_{k-1}-1, P_2]$

For the proof of the formulas of Theorem 1 we first have to show that all terms in the last column of Table I are integers, which is obvious, and that from the second term on in each Case they are positive, which follows at once from the conditions for that Case. Secondly, in order to show that the SCF in Column 4 is, indeed, equal to y, the conjugate of x, we first note that, by (4), $x=[a_1, a_2, \ldots, a_h, \xi]$, where $\xi=[\overline{b_1, b_2, \ldots, b_k}]$. If η denotes the conjugate of ξ, we have

$$y=[a_1, a_2, \ldots, a_h, \eta]. \tag{8}$$

Now, by [3; Theorem 3.6, p. 76],

$$\lambda_1=[\overline{b_k, \ldots, b_2, b_1}]=-\frac{1}{\eta}. \tag{9}$$

It follows from (9) by the use of (5) and (6) that, in the notation (7),

$$\lambda_2=[P_1]=[b_{k-1}, \ldots, b_1, b_k]=-\frac{\eta}{b_k\eta+1}. \tag{10}$$

Now we conclude from (6), (7), (9) and (10) that for integral g

$$[g+b_k, P_1]=g+\lambda_1, \qquad [g+b_{k-1}, P_2]=g+\lambda_2. \tag{11}$$

The formulas (11) thus enable us to write every SCF in Column 4 of Table I which involves P_1 or P_2 in the form of a finite continued fraction whose last term (which is a complete quotient) is of the form $g+\lambda_i$. Thus, in view of (8), the proof of the corresponding formula is reduced to the verification of an identity in which both sides are rational functions of the variables a_i, b_j and η all of which we may, for the remainder of the proof, consider as real variables.

We apply the principle outlined above to the three Cases $0B$, $1B$ and $2A$ which will turn out to be sufficient for the proof of all twenty-eight Cases. In these three Cases the identities to be verified are

$$\eta=[-1, 1, \lambda_1-1], \tag{12}$$

$$\begin{aligned}[a_1, \eta]&=[a_1-b_k-1, 1, \lambda_2-1],\\ [a_1, a_2, \eta]&=[a_1-1, 1, \lambda_1-a_2-1],\end{aligned} \tag{13}$$

respectively, λ_1 and λ_2 being given by (9) and (10). Evidently by (6), we may, without loss of generality, take $a_1=0$ in (13) thus reducing (13) to

$$[0, \eta]=[-b_k-1, 1, \lambda_2-1], \qquad [0, a_2, \eta]=[-1, 1, \lambda_1-a_2-1]. \tag{14}$$

Now it is easily verified that both sides of the three equations in (12) and (14) are equal to η, $1/\eta$ and $\eta/(a_2\eta+1)$, respectively.

From the identities in (13) we can obtain further identities by the following procedure: replace each a_i by a_{i+1} and then add the extra term a_1 as the first term of the continued fraction on both sides of the identity. If this procedure is applied repeatedly we obtain from (13) the following two chains of implications:

$$1B \Rightarrow 2F \Rightarrow 3H \Rightarrow 4H \Rightarrow \cdots \Rightarrow hH \Rightarrow h+1, H \Rightarrow \cdots,$$
$$2A \Rightarrow 3B \Rightarrow 4B \Rightarrow \cdots \Rightarrow hB \Rightarrow h+1, B \Rightarrow \cdots.$$

We have, therefore, at this point proved the following eight Cases

$$0B, 1B, 2A, 2F, 3B, 3H, hB, hH. \tag{15}$$

Finally, as is seen from an inspection of Column 3 of Table I, each of the remaining twenty Cases results from one of the Cases listed in (15) by subjecting one or more of the variables a_i, b_j to the equation listed in Column 3. The resulting continued fraction in Column 4 will in all these Cases have a term equal to zero, and we then apply the identity

$$\begin{aligned}&[c_1, c_2, \ldots, c_{p-1}, c_p, 0, c_{p+1}, c_{p+2}, \ldots]\\ &=[c_1, c_2, \ldots, c_{p-1}, c_p + c_{p+1}, c_{p+2}, \ldots],\end{aligned}$$

which is an immediate consequence of (5) and (6). Moreover in those Cases in which the resulting zero term is followed directly by P_i, it will be necessary to replace P_1 by b_{k-1}, P_2 and P_2 by b_{k-2}, P_3 or, symbolically,

$$g+[P_1]=[g+b_{k-1}, P_2], \qquad g+[P_2]=[g+b_{k-2}, P_3].$$

In this manner we obtain the following chains of implications:

$$0B \Rightarrow 0A, \quad 1B \Rightarrow 1A, \quad 2A \Rightarrow 2B, \quad 2F \Rightarrow 2D,$$
$$2F \Rightarrow 2E \Rightarrow 2C'' \Rightarrow 2C', \quad 3B \Rightarrow 3D \Rightarrow 3C'' \Rightarrow 3C',$$
$$3B \Rightarrow 3A, \quad 3H \Rightarrow 3F, \quad 3H \Rightarrow 3G \Rightarrow 3E,$$
$$hB \Rightarrow hA, \quad hB \Rightarrow hD \Rightarrow hC,$$
$$hH \Rightarrow hF, \quad hH \Rightarrow hG \Rightarrow hE.$$

This proves the remaining twenty Cases and completes the proof of Theorem 1.

We give below a number of examples which illustrate some of the formulas of Theorem 1. It should be emphasized that, since x is also the conjugate of y, each example illustrates two formulas. In fact, it is possible to pair the Cases of Table I into pairs of "Conjugate Cases". These pairs are listed in Table II, where the letter h again stands for any integer greater than or equal to 4.

TABLE II
PAIRS OF CONJUGATE CASES

$0A$; $2C'$	$2A$; $3A$	$3B$; $4A$	$h+1, A$; hB
$0B$: $3C'$	$2B$; $2D$	$3D$; $3F$	$h+2, C$; hH
$1A$; $2C''$	$2E$; $3E$	$3G$; $4E$	hD; hF
$1B$; $3C''$	$2F$; $4C$	$3H$; $5C$	$h+1, E$; hG

The five pairs of examples given below illustrate the pairs of Cases indicated.

EXAMPLE 1. Cases $1B$ and $3C''$.

$$x=\frac{\sqrt{17}-9}{4}=[-2, \overline{1, 3, 1}]; \qquad y=\frac{-\sqrt{17}-9}{4}=[-4, 1, 2, \overline{1, 1, 3}].$$

EXAMPLE 2. Cases $2C''$ and $1A$.

$$x=\frac{-\sqrt{5}-1}{2}=[-2, 2, \overline{1}]; \qquad y=\frac{\sqrt{5}-1}{2}=[0, \overline{1}].$$

EXAMPLE 3. Cases $3A$ and $2A$.

$$x=\frac{-\sqrt{99}+2}{19}=[-1, 1, 1, \overline{2, 1, 1, 3}]; \qquad y=\frac{\sqrt{99}+2}{19}=[0, 1, \overline{1, 1, 2, 3}].$$

EXAMPLE 4. Cases $0B$ and $3C'$.

$$x=\frac{\sqrt{21}+3}{6}=[\overline{1, 3}]; \qquad y=\frac{-\sqrt{21}+3}{6}=[-1, 1, 2, \overline{1, 3}].$$

EXAMPLE 5. Cases $6C$ and $4H$.

$$x=\frac{\sqrt{156}+275}{463}=[0, 1, 1, 1, 1, 1, \overline{3, 4, 1, 2}];$$

$$y=\frac{-\sqrt{156}+275}{463}=[0, 1, 1, 3, \overline{4, 3, 2, 1}].$$

3. **Some further theorems.** The following results are direct consequences of the formulas of Theorem 1.

THEOREM 2. *Let x, y and h be defined as in Theorem 1. Let $\alpha=[x]$, $\beta=[y]$, where $[\quad]$ denotes the bracket function. Then the following relations hold between α and β for the indicated values of h.*

If $h=0$, $\quad\alpha\geq 1,\quad \beta=-1$;
if $h=1$, $\quad\beta\leq\alpha-2$;
if $h=2$, $\quad\beta\geq\alpha-1$;
if $h=3$, $\quad\beta\geq\alpha$;
if $h\geq 4$, $\quad\beta=\alpha$.

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Furthermore, the above relations are the best possible ones in the sense that any given triple of integers h, α, β satisfying one of the five situations above can actually be realized.

The relations between α and β stated in Theorem 2 follow at once from the formulas of Theorem 1 in view of the fact that α and β are the first terms in the SCF's for x and y, respectively. That any triple h, α, β compatible with one of the five situations listed above can actually occur is shown as follows: First choose $a_1 = \alpha$ (in case $h = 0$, choose $b_1 = \alpha$). Then select the appropriate Case according to Column 4 of Table I; in most cases there is more than one such selection possible. Finally, in the case $h = 1$ choose the appropriate value of b_k; and in the cases $h = 2$ and $h = 3$, if the given values of α and β were such that $\beta \geq \alpha + 2$ (Cases $2C$ and $3C$), choose the appropriate value of b_{k-2} or b_{k-1}, respectively.

COROLLARY. *In the notation of Theorem* 2 *we have* $\beta \leq \alpha - 2$ *if and only if* $h \leq 1$ *and* $\beta \geq \alpha - 1$ *if and only if* $h \geq 2$.

THEOREM 3. *In the notation of Theorem* 1, *the conjugate root y is of the opposite sign as x in the Cases listed below in Table* III, *provided that the additional condition, if any, is satisfied. In all other circumstances y has the same sign as x.*

TABLE III

Cases	Additional Condition
$0A$, $0B$	*none*
$1A$, $1B$	$0 \leq a_1 \leq b_k - 1$
$2A$, $2B$	$a_1 = 0$
$2D$, $3A$	$a_1 = -1$
$2C$	$-b_{k-2} - 1 \leq a_1 \leq -1$
$3C$	$-b_{k-1} - 1 \leq a_1 \leq -1$

With $\alpha = [x]$ and $\beta = [y]$ it is clear that y has the opposite sign as x if and only if $\alpha \geq 0$ and $\beta < 0$ or $\alpha < 0$ and $\beta \geq 0$. In going through the various formulas in Column 4 of Table I, one confirms easily that $\alpha \geq 0$ and $\beta < 0$ in the situations corresponding to the first three lines of Table III and only under these circumstances. The same applies to the inequalities $\alpha < 0$, $\beta \geq 0$ and the last three lines of Table III.

THEOREM 4. *Let x and y be a pair of conjugate quadratic irrationalities, and let h and h' denote the number of terms in the non-periodic part of the simple continued fraction for x and y, respectively. Then in all cases* $|h - h'| \leq 3$.

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Moreover, the equation $|h-h'|=3$ *holds if and only if one of the roots, say* x, *is greater than* 1 *while* $-1/2<y<0$, *in which case* $h=0$ *and* $h'=3$.

The fact that $|h-h'|\leq 3$ is seen at once by an inspection of Table II. The only pair of conjugate cases with $|h-h'|=3$ is the pair $0B$; $3C'$. (For an example, see §2, Example 4.) In this case $x=[\overline{b_1, b_2, \ldots, b_k}]$ is a reduced quadratic irrationality with $b_k \geq 2$, so that $x>1$ and $-1/y=[\overline{b_k, b_{k-1}, \ldots, b_1}]>2$.

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