ON A QUESTION OF SEIDEL CONCERNING HOLOMORPHIC FUNCTIONS BOUNDED ON A SPIRAL

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Introduction. Let S be a spiral contained in $D = \{|z| < 1\}$ such that S tends to $C = \{|z| = 1\}$. For the sake of brevity, by "f is bounded on S" we shall mean that f is holomorphic in D, unbounded, and bounded on S. The existence of such functions was first discussed by Valiron (**9**; **10**); see also (**1**; **3**; **8**). Valiron also proved that any function that is "bounded on a spiral" must have the asymptotic value ∞ (**10**, p. 432). Functions that are bounded on a spiral may also have finite asymptotic values (**1**, p. 1254). In view of the above, Seidel has raised the question (oral communication): "Does there exist a function bounded on a spiral that has *only* the asymptotic value ∞?". The following theorem answers this question affirmatively.

1. Statement and proof of the main theorem.

THEOREM. *Let S be any spiral in D whose equation, in polar coordinates $z = re^{i\theta}$, is $r = \lambda(\theta)$, where $\lambda(\theta)$ is continuous and strictly increasing in $0 \leqq \theta < \infty$ and satisfies*

$$\lambda(0) = 0 \quad \textit{and} \quad \lim_{\theta\to\infty} \lambda(\theta) = 1.$$

Then there exists a function $w(z)$, holomorphic and unbounded in D, such that $w(z)$ is bounded on S and $w(z)$ has only the asymptotic value ∞.

Remark. The proof is based on an adaptation of techniques used previously in (**4**; **5**).

Proof. First we need some notation. Let

$$C[a, r] = \{z\colon |z - a| = r\},$$
$$D[a, r] = \{z\colon |z - a| < r\},$$
$$A[a, r_1, r_2] = \{z\colon r_1 \leqq |z - a| \leqq r_2\},$$
$$T[a, \theta_1, \theta_2] = \{z\colon \theta_1 \leqq \arg(z - a) \leqq \theta_2\}$$

(note that $T[a, \theta_1, \theta_1]$ is a half ray emanating from the point a).

Next we shall make some technical, but straightforward, definitions which while fairly numerous can all be kept in mind quite easily by referring to Figures 1, 2, and 3. After the definitions, a short, intuitive summary of the proof will be given before going into the proof in detail.

All definitions are for $n = 0, 1, 2, \ldots$.

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Definition 1. The points b_n: Let $\{b_n\}$ denote the set of points at which S intersects the positive real axis. We choose the notation so that

$$0 = b_0 < b_1 < b_2 < \ldots \uparrow 1.$$

Definition 2. The points c_n and the points d_n: Let

$$c_n = b_{2n}, \qquad d_n = b_{2n+1}.$$

Definition 3. The points a_n and the semi-circular arcs σ_n: Let

$$a_n = (c_n + d_n)/2, \qquad \sigma_n = C[a_n, |d_n - c_n|/8] \cap T[a_n, -\pi, 0].$$

(See Figure 1.)

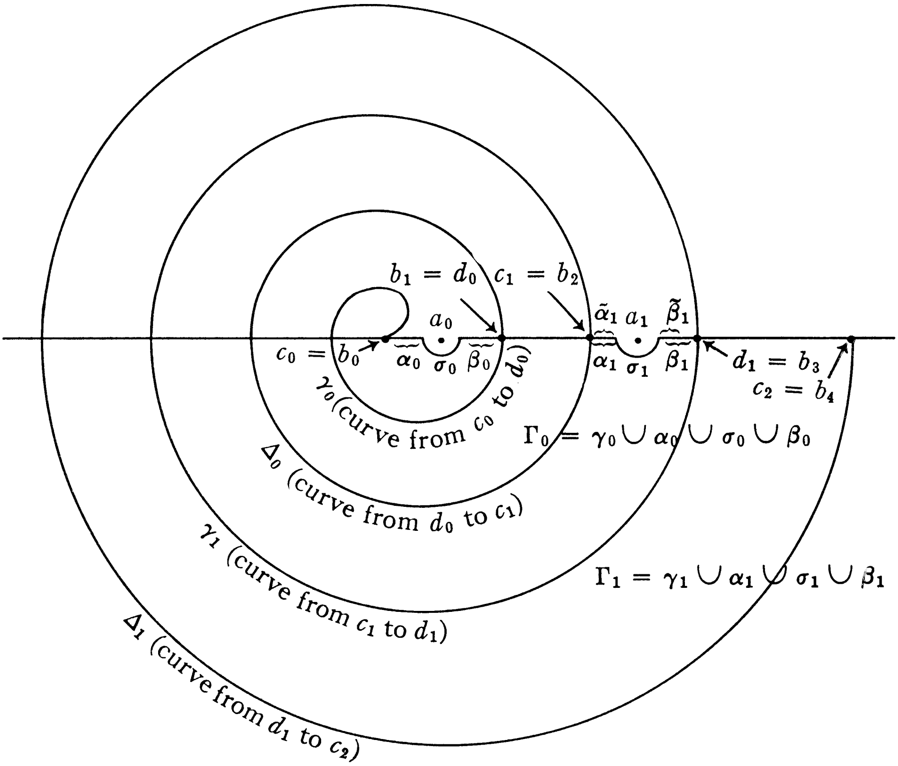

FIGURE 1

Definition 4. The segments α_n, $\tilde{\alpha}_n$, β_n, $\tilde{\beta}_n$: Let

$$\alpha_n = T[0, 0, 0] \cap A[0, c_n, c_n + (3/8) \cdot (d_n - c_n)],$$

$$\tilde{\alpha}_n = \alpha_n \cap A[0, c_n + (1/4) \cdot (d_n - c_n), c_n + (3/8) \cdot (d_n - c_n)],$$

$$\beta_n = T[0, 0, 0] \cap A[0, c_n + (5/8) \cdot (d_n - c_n), d_n],$$

$$\tilde{\beta}_n = \beta_n \cap A[0, c_n + (5/8) \cdot (d_n - c_n), c_n + (3/4) \cdot (d_n - c_n)].$$

(See Figure 1.)

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Definition 5. The arcs γ_n, Δ_n, and Γ_n: Let γ_n (Δ_n) be the subarc of S with c_n and d_n (d_n and c_{n+1}) as its endpoints. Let

$$\Gamma_n = \gamma_n \cup \alpha_n \cup \beta_n \cup \sigma_n.$$

(See Figure 2.)

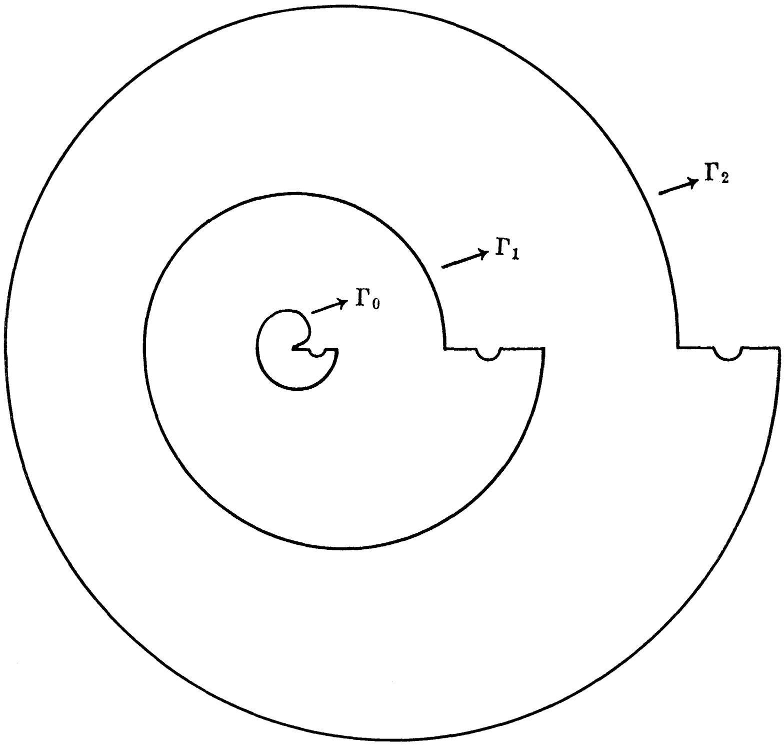

FIGURE 2

Definition 6. The sets F_n, G_n, and H_n: Recall that the spiral S was defined as

$$S = \{z\colon z = \lambda(\theta)e^{i\theta},\ 0 \leqq \theta < \infty\}.$$

Let

$$F_n = \{z\colon z = re^{i\theta},\ \rho(\theta) < r < \tau(\theta),\ 2n(2\pi) < \theta < \infty\},$$

where

$$\rho(\theta) = \lambda(\theta) + (7/16)\cdot[\lambda(\theta + 2\pi) - \lambda(\theta)],$$
$$\tau(\theta) = \lambda(\theta) + (9/16)\cdot[\lambda(\theta + 2\pi) - \lambda(\theta)]$$

(7/16 and 9/16 specifically chosen so that F_n does not intersect the α_k's and β_k's)

$$G_n = F_n \cup D[a_n,\ (1/32)\cdot(d_n - c_n)],$$
$$H_n = F_n \cup D[a_n,\ (1/4)\cdot(d_n - c_n)].$$

(See Figure 3.)

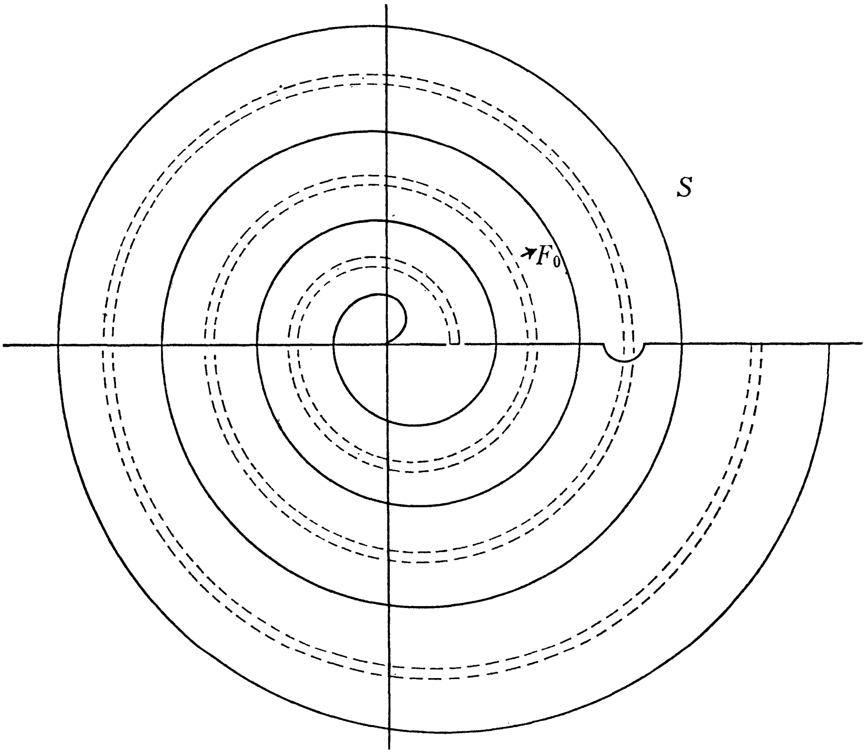

FIGURE 3

The idea of the construction is as follows. Using Mergelyan's theorem (**6**, p. 3), we construct a function f such that f is bounded on S, non-zero in $D[0, 1]$, approximately 1 on $\gamma_{2n} \cup \alpha_{2n} \cup \beta_{2n}$ and approximately -1 on $\gamma_{2n+1} \cup \alpha_{2n+1} \cup \beta_{2n+1}$. Using functions of the form $1/[A(z - a)]^n$ and sweeping the poles out to $C[0, 1]$ through the "channels" G_n, we construct a holomorphic function g $(= 1 + \sum_{n=0}^{\infty} g_n)$ such that:

(i) g is approximately 1 on

$$S \cup \left(\bigcup_{n=0}^{\infty} [(\alpha_n - \tilde{\alpha}_n) \cup (\beta_n - \tilde{\beta}_n)] \right),$$

(ii) Re $g \geqq 1$ and Im g is small on $\bigcup_{n=0}^{\infty} (\tilde{\alpha}_n \cup \tilde{\beta}_n)$,

(iii) g is so large on σ_n that

$$\lim_{n\to\infty} \min_{z\in\sigma_n} |f(z) \cdot g(z)| = \infty.$$

Then it follows that $w = f \cdot g$ is bounded on S and that its only possible asymptotic value is ∞.

We first outline the construction of the function f. The construction is only a very slight variation of a now standard technique of Bagemihl and Seidel (**2**);

the interested reader may refer to (**2**) for the details of the construction. Following (**2**), by Mergelyan's theorem (**6**, p. 3), we can find a polynomial $p_1(z)$ such that ($\approx$ means "is approximately"):

(i) $p_1(z) \approx 0$ for $|z| \leqq d_0$,

(ii) $p_1(z) \approx \pi i$ for $z \in \gamma_1 \cup \alpha_1 \cup \beta_1$,

(iii) $|p_1(z)| < 4$ for $z \in \Delta_0$ (4 chosen here since $|\pi i| < 4$).

We next choose $p_2(z)$ so that:

(i) $p_2(z) \approx 0$ for $|z| \leqq d_1$,

(ii) $p_1(z) + p_2(z) \approx 0$ for $z \in \gamma_2 \cup \alpha_2 \cup \beta_2$,

(iii) $|p_1(z) + p_2(z)| < 4$ for $z \in \Delta_1$.

In general, pick $p_n(z)$ such that

(i) $p_n(z) \approx 0$ for $|z| \leqq d_{n-1}$,

(ii) $\sum_{k=1}^{n} p_k(z) \approx (\pi i) \cdot (1 + (-1)^{n+1})/2$ for $z \in \gamma_n \cup \alpha_n \cup \beta_n$,

(iii) $|\sum_{k=1}^{n} p_k(z)| < 4$ for $z \in \Delta_{n-1}$.

It follows by the Weierstrass M-test that $p(z) = \sum_{n=1}^{\infty} p_n(z)$ is holomorphic in $D[0, 1]$ and that $f = e^p$ has the following properties:

(i) f is bounded on S,

(ii) $f \neq 0$ in $D[0, 1]$,

(iii) Given any $\epsilon > 0$ we *could* have constructed f so that $|f - 1| < \epsilon$ for $z \in \gamma_{2n} \cup \alpha_{2n} \cup \beta_{2n}$ and $|f + 1| < \epsilon$ for $z \in \gamma_{2n+1} \cup \alpha_{2n+1} \cup \beta_{2n+1}$.

(We might note here that, in this case as in many other cases where Mergelyan's theorem is commonly used, a much older theorem of Walsh (**11**, p. 47, Theorem 15) is more than powerful enough to obtain the desired results.)

We shall next construct $g(z)$. The function g will be defined as

$$g = 1 + \sum_{n=0}^{\infty} g_n,$$

and we shall now define the g_n's inductively. Let $\{\epsilon_n\}$ be chosen so that $\epsilon_n > 0$ and $\sum_{n=0}^{\infty} \epsilon_n = \epsilon < 1/4$. By proper choice of m_0, θ_0, and A_0 the function

$$h_0(z) = 1/[A_0 e^{i\theta_0}(z - a_0)]^{m_0}$$

has the properties (Ω denotes the Riemann sphere):

(i) $|h_0(z)| < \epsilon_0/2$ in $\Omega - D[a_0, (1/4) \cdot (d_0 - c_0)]$,

(ii) $h_0(z)$ is real and positive on $\tilde{\alpha}_0 \cup \tilde{\beta}_0$,

(iii) $\min_{z\in\sigma_0}|h_0(z)| \geqq 2/\delta_0^2$ (where $\delta_0 = \min\{1, \min_{z\in\sigma_0}|f(z)|\}$).

Note that $\delta_0 > 0$ since $f(z) \neq 0$ in $D[0, 1]$. Next, using an analogous technique to that used in (**7**), we sweep the pole of h_0 at a_0 out to $C[0, 1]$ through the channel G_0. The idea is to approximate $h_0(z)$ in

$$\Omega - ((G_0 - G_1) \cup D[a_1, (1/32) \cdot (d_1 - c_1)])$$

by a rational function $h_{0,1}(z)$ with pole at a_1, then approximate $h_{0,1}$ in

$$\Omega - ((G_1 - G_2) \cup D[a_2, (1/32) \cdot (d_2 - c_2)])$$

by a rational function $h_{0,2}(z)$ with pole at a_2, etc. In this way we obtain a function $g_0(z)$ ($= \lim_{k\to\infty} h_{0,k}(z)$), holomorphic in $D[0, 1]$, with the properties:

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(i) $|g_0(z)| < \epsilon_0$ in $D[0, 1] - H_0$,
(ii) $\operatorname{Re} g_0(z) > 0$ and $|\operatorname{Im} g_0(z)| < \epsilon_0$ on $\tilde{\alpha}_0 \cup \tilde{\beta}_0$,
(iii) $\min_{z \in \sigma_0} |g_0(z)| \geqq 1/\delta_0^2$.

In order to construct $g_1(z)$ we first choose A_1, θ_1, and m_1 so that

$$h_1(z) = 1/[A_1 e^{i\theta_1}(z - a_1)^{m_1}$$

has the properties:

(i) $|h_1(z)| < \epsilon_1/2$ in $\Omega - D[a_1, (1/4) \cdot (d_1 - c_1)]$,
(ii) $h_1(z) > 0$ on $\tilde{\alpha}_1 \cup \tilde{\beta}_1$,
(iii) $\min_{z \in \sigma_1} |h_1(z)| - \omega_1 \geqq 2/\delta_1^2$ (where $\delta_1 = \min\{1, \min_{z \in \sigma_1} |f(z)|\}$ and $\omega_1 = \max_{z \in \sigma_1} |g_0(z)|$).

Again, in a manner analogous to the one used above, we sweep the poles out to $C[0, 1]$ and obtain a $g_1(z)$ that satisfies:

(i) $|g_1(z)| < \epsilon_1$ in $D[0, 1] - H_1$,
(ii) $\operatorname{Re} g_1(z) > 0$ and $|\operatorname{Im} g_1(z)| < \epsilon_1$ for $z \in \tilde{\alpha}_1 \cup \tilde{\beta}_1$,
(iii) $\min_{z \in \sigma_1} |g_1(z)| - \omega_1 > 1/\delta_1^2$.

In general, using the same techniques, we inductively construct $g_n(z)$, holomorphic in $D[0, 1]$, with the properties:

(i) $|g_n(z)| < \epsilon_n$ in $D[0, 1] - H_n$,
(ii) $\operatorname{Re} g_n(z) > 0$ and $|\operatorname{Im} g_n(z)| < \epsilon_n$ for $z \in \tilde{\alpha}_n \cup \tilde{\beta}_n$,
(iii) $\min_{z \in \sigma_n} |g_n(z)| - \omega_n \geqq 1/\delta_n^2$ (where $\delta_n = \min\{1/n, \min_{z \in \sigma_n} |f(z)|\}$ and $\omega_n = \max_{z \in \sigma_n} \sum_{k=0}^{n-1} |g_k(z)|$).

Note that $\delta_n > 0$ since $f \neq 0$ in $D[0, 1]$.

By the Weierstrass M-test, we see that

$$g = 1 + \sum_{n=0}^{\infty} g_n$$

is holomorphic in $D[0, 1]$. We claim that

$$w = f \cdot g$$

is the desired function. First we shall prove that w has the properties:

(i) w is bounded on S,
(ii) $\operatorname{Re} w \geqq 1/2$ on $\alpha_{2n} \cup \beta_{2n} \cup \gamma_{2n}$,
(iii) $\operatorname{Re} w \leqq -1/2$ on $\alpha_{2n+1} \cup \beta_{2n+1} \cup \gamma_{2n+1}$,
(iv) $|w| \geqq n$ for $z \in \sigma_{n+1}$,

and then we shall prove that (ii), (iii), and (iv) imply that the only possible asymptotic value of f is ∞.

First we note that w is bounded on S since

$$\begin{aligned} \max_{z \in S} |w(z)| &= \max_{z \in S} (|f(z)| \cdot |g(z)|) \\ &\leqq \max_{z \in S} \left(|f(z)| \cdot \left(1 + \sum_{n=0}^{\infty} |g_n(z)|\right)\right) \\ &\leqq e^4 \left(1 + \sum_{n=0}^{\infty} \epsilon_n\right) \\ &= e^4 (1 + \epsilon). \end{aligned}$$

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Next we shall prove (ii) and (iii). Suppose that

$$z \in \gamma_{2n} \cup (\alpha_{2n} - \tilde{\alpha}_{2n}) \cup (\beta_{2n} - \tilde{\beta}_{2n}) \qquad (n = 0, 1, 2, \ldots);$$

then we have (recalling that $\epsilon < 1/4$):

$$\operatorname{Re} w = (\operatorname{Re} f) \cdot (\operatorname{Re} g) - (\operatorname{Im} f) \cdot (\operatorname{Im} g) \geqq (1-\epsilon) \cdot (1-\epsilon) - (\epsilon \cdot \epsilon) \geqq 1/2.$$

If $z \in \tilde{\alpha}_{2n} \cup \tilde{\beta}_{2n}$, we obtain

$$\operatorname{Re} w = (\operatorname{Re} f) \cdot (\operatorname{Re} g) - (\operatorname{Im} f) \cdot (\operatorname{Im} g) = (\operatorname{Re} f) \cdot \left(1 + \operatorname{Re} g_{2n} + \sum_{k=0}^{\infty}{}' \operatorname{Re} g_k\right) - (\operatorname{Im} f) \cdot \left(\operatorname{Im} g_{2n} + \sum_{k=0}^{\infty}{}' \operatorname{Im} g_k\right)$$

(where by $\sum_{k=0}^{\infty}{}'$ we mean the series $\sum_{k=0}^{\infty}$ with the $2n$th term omitted)

$$\geqq (1-\epsilon) \cdot \left(1 + 0 - \sum_{k=0}^{\infty}{}' \epsilon_k\right) - \epsilon \cdot \left(\epsilon_{2n} + \sum_{k=0}^{\infty} \epsilon_k\right) \geqq (1-\epsilon) \cdot (1-\epsilon) - (\epsilon \cdot \epsilon) \geqq 1/2.$$

Similarly, we can prove that:

$$\operatorname{Re} w \leqq -1/2 \text{ for } z \in \gamma_{2n+1} \cup \alpha_{2n+1} \cup \beta_{2n+1}.$$

Finally, to prove (iv), if $z \in \sigma_n$, we see that:

$$|w(z)| = |f(z)| \cdot |g(z)| = |f(z)| \cdot \left|1 + \left(\sum_{k=0}^{n-1} g_k\right) + g_n + \sum_{k=n+1}^{\infty} g_k\right|$$

$$\geqq \delta_n \cdot \left| |g_n| - 1 - \sum_{k=0}^{n-1} |g_k| - \sum_{k=n+1}^{\infty} |g_k| \right| \geqq \delta_n \cdot |1/\delta_n^2 + \omega_n - 1 - \omega_n - \epsilon|$$

$$\geqq \delta_n \cdot (1/\delta_n^2 - 1 - \epsilon) \geqq n - 1.$$

Recall that $\Gamma_n = \gamma_n \cup \alpha_n \cup \beta_n \cup \sigma_n$; we now observe that Γ_n is a closed Jordan curve contained in $D[0, 1]$, Γ_{n+1} separates Γ_n from $C[0, 1]$ in $D[0, 1]$, and given any $\epsilon > 0$ there exists an $N(\epsilon)$ such that for all $n \geqq N(\epsilon)$,

$$\Gamma_n \subset \{z\colon 1 - \epsilon < |z| < 1\}.$$

Let ψ be any curve that is not contained in any compact subset of $D[0, 1]$. It follows that ψ must intersect all but a finite number of the Γ_n. We shall prove that the only possible asymptotic value of f on ψ is ∞. This is clear if ψ intersects infinitely many of the σ_n. If ψ intersects only finitely many of the σ_n, then ψ must intersect infinitely many of the arcs $\gamma_{2n+1} \cup \alpha_{2n+1} \cup \beta_{2n+1}$ and infinitely many of the arcs $\gamma_{2n} \cup \alpha_{2n} \cup \beta_{2n}$. If we recall that

$$\operatorname{Re} w(z) \geqq 1/2 \quad \text{on } \gamma_{2n} \cup \alpha_{2n} \cup \beta_{2n}$$

and

$$\operatorname{Re} w(z) \leqq -1/2 \quad \text{on } \gamma_{2n+1} \cup \alpha_{2n+1} \cup \beta_{2n+1},$$

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we see that $w(z)$ can have no finite asymptotic values on ψ and the proof of the theorem is complete.

2. Remarks. In conclusion, recall that we assumed that the equation of the spiral S was of the form $z = \lambda(\theta)e^{i\theta}$, where $\lambda(\theta) \geqq 0$ and strictly increasing. It is clear from the proof that these assumptions, while helping to keep the technical details of the proof to a minimum, are not necessary and that with sufficient patience the proof could be carried through for almost any conceivable spiral.

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