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A NOTE ON THE TORSION OF BERNSTEIN CURVES

BY
MARTIN E. PRICE

ABSTRACT. It is shown that Bernstein Polynomials do not diminish the total "twist" of space curves in contrast to their length and curvature diminishing properties. This phenomenon is shown to be related to the fact that Bernstein Polynomials of a plane curve may have more inflections than the curve possesses.

The Bernstein polynomial operator is known to possess many smoothing properties. In particular it is total variation, arc-length and total-curvature diminishing, [2], [3], [4]. For this reason, Bernstein approximations of vector functions have been used in applications in spite of their slow convergence.

In [3], we showed:

THEOREM 1. *Let $F(t)$ be a C^2 mapping of $[0, 1]$ to R^3 and $b_nF(t)$ be its n-th Bernstein approximation whose coordinates are defined by*

$$b_nf_i(t) = \sum_{r=0}^{n} f_i\left(\frac{r}{n}\right)\binom{n}{r}t^r(1-t)^{n-r}.$$

For any C^2 mapping G, let $T_G(t)$ be its unit tangent vector so that the total curvature is given by

$$KG = \int_0^1 \left|\frac{dT_G}{dt}\right| dt.$$

Then for all n, $Kb_nF < KF$.

In view of this it is natural to ask if the Bernstein operator also diminishes the total "twist" of a C^3 space curve. Let ΥF denote the total absolute torsion of a parametric C^3 space curve $F(t)$ on $[0, 1]$, i.e.,

$$\Upsilon F = \int_0^1 \left|\frac{dB_F}{dt}\right| dt = \int_0^1 \frac{|F' \cdot F'' \times F'''|}{|F' \times F''|^2} |F'| \, dt$$

where $B_F(t)$ is the binormal to F at t. Then is $\Upsilon b_nF \leq \Upsilon F$ for all n? The difficulty is that the proof of Theorem 1 depends on the fact that the total curvature of a space curve is the average of the plane total curvature of its

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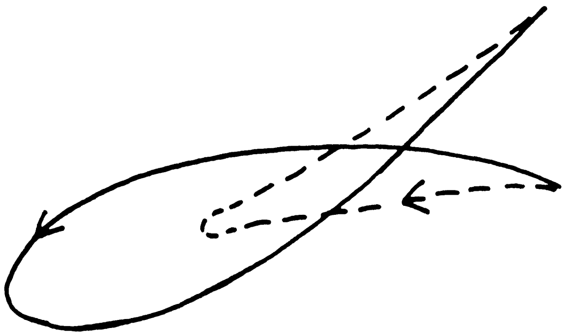

projections [1]. Since the torsion of any plane curve is zero, the corresponding statement for torsion is false.

We give a counterexample to the torsion conjecture which is based on properties of the Bernstein operator for plane vector curves. Define $F(t) = (\frac{3}{2}t^2 - \frac{3}{2}t, \frac{3}{2}t^3 - \frac{11}{6}t^2 + \frac{4}{9}t)$ for t on $[0, 1]$. The corresponding Bernstein cubic is $b_3F(t) = (t^2 - t, (t^3/3) - \frac{2}{9}t^2)$. The situation is shown in the Figure where the solid line is F and the dotted line is b_3F.

The curve F has been chosen so that it has a loop and no inflection points but such that B_3F has no loop and has two inflections (at $t = \frac{1}{3}$ and $\frac{2}{3}$). Note that counting the inflections amounts to counting the zeros of the curvature. Thus $(b_3f_1)'(b_3f_2)'' - (b_3f_2)'(b_3f_1)''$ has two zeros while $f_1'f_2'' - f_2'f_1''$ has none. We now create a space curve $\hat{F}$ by using the two components of F and setting $f_3(t) = \frac{9}{2}t^3 - \frac{9}{2}t^2 + t$ so that the corresponding $b_3f_3(t) = t^3$. Geometrically, this "pulls up" one endpoint of both curves in Figure 1.

Integration yields $\Upsilon\hat{F} = 1.96$ while $\Upsilon b_3\hat{F} = 3.08$. By comparison $K\hat{F} = 4.82$ and $Kb_3\hat{F} = 1.97$.

The connection between the inflections of the plane curve and the torsion of the space curve is explained by considering a theorem of Fary [1]:

THEOREM 2. *If G is a rectifiable space curve lying on the unit sphere S and D_x denotes a great circle of S where $\pm x$ are the unit vectors orthogonal to the plane of D_x, then the length of G is $LG = \frac{1}{4}\int_S n_G(x)\,dx$ where $n_G(x)$ is the cardinality of $G \cap D_x$.*

We prove a corollary:

COROLLARY 1. *If F is a C^3 space curve, then*

$$\Upsilon F = \frac{1}{4}\int_S Z[(A_x \circ F)_1'(A_x \circ F)_2'' - (A_x \circ F)_2'(A_x \circ F)_1'']\,dx \tag{1}$$

where A_x is the natural rotation sending x to the north pole, and $Z[\cdot]$ counts the zeros of the indicated function.

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Proof. Since $B_F(t)$ is a unit vector, its arclength is given by $\frac{1}{4}\int_S n_{B_F}(x)\,dx$. Since A_x is a rotation, $B_F \in D_x$ if and only if $A_x B_F$ lies on the equator of S. But $A_x B_F = B_{A_x F}$. Thus $n_{B_F}(x)$ is the number of zeros of the third component of $B_{A_x F}$ which is the integrand in (1).

Thus absolute integral torsion is an average over all rotations of the number of zeros of the curvature of the projection of the rotated space curve upon the equatorial plane. But the example shows that plane Bernstein curves can have zeros of the curvature when none exists in the original curve. Since for any projection P or rotation A_x, we have $Pb_n\hat{F} = b_n P\hat{F}$ and $A_x b_n \hat{F} = b_n A_x \hat{F}$ (because b_n commutes with any linear operator on R^3), it follows that for some values of x, the integrand in (1) for $b_n\hat{F}$ will exceed that of $\hat{F}$.

REMARK. The same phenomenon occurs for other approximation procedures, e.g., integral means.

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FRAMINGHAM STATE COLLEGE
FRAMINGHAM, MA 01701