DINI'S THEOREM FOR ALMOST PERIODIC FUNCTIONS

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The object of this note is to extend Dini's theorem about (monotonic) sequences of continuous functions on a compact topological space to the case where the underlying domain is an abstract group which is free from topological restrictions. Continuous functions are replaced by almost periodic real-valued functions and the main result may be stated as follows: If a monotonically increasing sequence (f_n) of almost periodic real-valued functions on a group G converges pointwise to an almost periodic function f on G, then the sequence converges to f uniformly. The basic idea of the present (elementary) proof is due to v. Kampen [2] and A. Weil [4], i.e., every almost periodic function on a group induces a kind of compact topology in it, relative to which the function is continuous. We modify this idea with the aid of the mean-value of an almost periodic function and obtain a pseudo-metric topology. This topology facilitates convergence proofs greatly. Moreover, it turns out to be equivalent with the previous one (Lemma 1). No use will be made of the theory of bounded matrix representations. This is significant as any use of the "Approximation Theorem" [3, p. 66, see also p. 226] would violate the claim of an elementary proof.

Let $G = \{a, b, \cdots, x, y, \cdots\}$ be an arbitrary group and $f(x)$ a real-valued function on G. Following W. Maak [3, p. 26], we define: $f(x)$ is almost periodic (a.p.) if there is to every $\varepsilon > 0$ a finite number of subsets $A_i \subset G$, $i = 1, 2, \cdots, n$, which cover G, such that $a, b \in A_i$, $i = 1, 2, \cdots, n$, implies $|f(xay) - f(xby)| < \varepsilon$, for all $x, y \in G$.

We choose a non-trivial ($\neq$ const), real-valued and a.p. function $f(x)$ and introduce the following real-valued function on the group $G \times G$:

$$(\mathrm{M}_\delta) \qquad \delta(a, b) = \sup_{x, y \in G} |f(xay) - f(xby)|.$$

It can be easily shown that the function $|f(xay) - f(xby)|$ is a.p. on $G \times G$, whence we can define [3, pp. 43—44]:

$$(\mathrm{M}_\sigma) \qquad \sigma(a, b) = \int_{G\times G} |f(xay) - f(xby)| \,^{1}).$$

[1] $\int$ denotes the mean-value operation.

δ and σ share the following properties:

$$\text{(M)}\qquad \begin{array}{ll} \text{(i)}\ \delta(a, b) \geqq 0; & \text{(ii)}\ \delta(a, a) = 0; \\ \text{(iii)}\ \delta(a, b) = \delta(b, a); & \text{(iv)}\ \delta(a, b) \leqq \delta(a, c) + \delta(b, c); \\ \text{(v)}\ \delta(xay\ xby) = \delta(a, b); & \text{(vi)}\ \delta(a^{-1}, b^{-1}) = \delta(a, b). \end{array}$$

(i)—(vi) are fairly obvious w.r.t. δ, and follow, in the case of σ, immediately from the basic properties of the mean-value operation. (M) shows in particular that δ and σ are G-invariant pseudo-metrics in G. Moreover, G is totally bounded w.r.t. δ[3, p. 54]. From $\sigma(a, b) \leqq \delta(a, b)$ we infer that G is also totally bounded w.r.t. σ.

We notice that the "kernels" $N_\delta = \{a \in G | \delta(a, 1) = 0\}$ (1 denotes the identity in G) and $N_\sigma = \{a \in G | \sigma(a, 1) = 0\}$ are identical [2]) invariant subgroups. $f(x)$ is constant on the cosets modulo N_δ of G, and the factor group G/N_δ may be regarded as a metric group w.r.t. δ and σ. We wish to prove

LEMMA 1. *δ and σ are equivalent pseudo-metrics.*

PROOF. According to our last remark we may assume that $N_\delta = N_\sigma = \{1\}$. We embed G in its metric completion $\bar{G}_\delta$ w.r.t. δ. $f(x)$ is δ-continuous on G in the sense that to every $\varepsilon > 0$ there exists an $\eta > 0$ such that $\delta(a, b) < \eta$ implies $|f(a) - f(b)| < \varepsilon$. This follows at once from the inequality $|f(a) - f(b)| \leqq \delta(a, b)$. The (unique) prolongation of $f(x)$ onto $\bar{G}_\delta$ is also δ-continuous and therefore a.p. ($\bar{G}_\delta$ is compact). It is a straightforward matter to show that

$$\delta(\bar{a}, \bar{b}) = \sup_{\bar{x}, \bar{y} \in \bar{G}_\delta} |f(\bar{x}\bar{a}\bar{y}) - f(\bar{x}\bar{b}\bar{y})| \quad (\bar{a}, \bar{b} \in \bar{G}_\delta).$$

Moreover,

$$\bar{\sigma}(\bar{a}, \bar{b}) = \int_{\bar{G}_\delta \times \bar{G}_\delta} |f(\bar{x}\bar{a}\bar{y}) - f(\bar{x}\bar{b}\bar{y})| \quad (\bar{a}, \bar{b} \in \bar{G}_\delta)$$

is easily seen to be a prolongation of σ onto $\bar{G}_\delta$. From the inequality $\bar{\sigma}(\bar{a}, \bar{b}) \leqq \delta(\bar{a}, \bar{b})$ and the compactness of $\bar{G}_\delta$ we infer that $\bar{G}_\delta$ is homeomorphic to $\bar{G}_\sigma$. This finishes the proof.

The space $\mathscr{A}$ of all a.p. and real-valued functions on G becomes a normed vector-lattice $\mathscr{A}^1$ if we introduce the norm

$$\text{(N)}\qquad ||f|| = \int_G |f|, \quad f \in \mathscr{A}.$$

We recall that $\mathscr{A}^1$ is in general not a Banach space.

If (f_n) is a sequence of a.p. functions, let (δ_n) and (σ_n) denote the correspondent sequences of pseudo-metrics as defined by (M_δ) and (M_σ),

[2] This depends on the fact that a non-negative a.p. function with a zero mean-value is identically zero [3, p. 42, Satz 9].

respectively. We require

LEMMA 2. If $f_n \in \mathscr{A}$, $n = 1, 2, \cdots$, $f \in \mathscr{A}$, *and if* $\lim_{n\to\infty} ||f_n - f|| = 0$, *then*

$$\lim_{n\to\infty} |\sigma_n(a, b) - \sigma(a, b)| = 0, \text{ uniformly.}$$

PROOF. To $\varepsilon > 0$ we choose an index N so that $||f_n - f|| < \varepsilon/2$, for $n > N$. Then

$$\begin{aligned} &\big|||f_n(xay) - f_n(xby)|| - ||f(xay) - f(xby)||\big| \\ &\leqq ||f_n(xay) - f(xay)|| + ||f(xby) - f_n(xby)|| < \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

for $n > N$, $a \in G$, $b \in G$, and $y \in G$. On the other hand,

$$\begin{aligned} &\big|||f_n(xay) - f_n(xby)|| - ||f(xay) - f(xby)||\big| \\ &= \left|\int_{x\in G} |f_n(xay) - f_n(xby)| - \int_{x\in G} |f(xay) - f(xby)|\right|. \end{aligned}$$

Therefore

$$\left|\int_{y\in G}\int_{x\in G} |f_n(xay) - f_n(xby)| - \int_{y\in G}\int_{x\in G} |f(xay) - f(xby)|\right| < \varepsilon,$$

and thus $|\sigma_n(a, b) - \sigma(a, b)| < \varepsilon$, for $n > N$, uniformly.

LEMMA 3. *If* σ_n, $n = 0, 1, 2, \cdots$, *is a sequence of pseudo-metrics, relative to which* G *is totally bounded, and if* $\lim_{n\to\infty} \sigma_n(a, b) = \sigma_0(a, b)$, *uniformly, then* G *is equi-totally bounded w.r.t.* $\sigma_0, \sigma_1, \sigma_2, \cdots$, *viz., to every* $\varepsilon > 0$ *there exists a finite subset* $\{a_1, a_2, \cdots, a_m\} \subset G$ *such that, for every* $a \in G$, *there is a* $\mu = 1, 2, \cdots, m$, *such that* $\sigma_n(a_\mu, a) < \varepsilon$, $n = 0, 1, 2, \cdots$.

PROOF. To $\varepsilon > 0$ we choose an index N such that $|\sigma_n(a, b) - \sigma_0(a, b)| < \varepsilon/3$, for $n > N$ and all $a, b \in G$. There exists a finite number of group elements $a_1, a_2, \cdots, a_m$ such that, for every $a \in G$, there is a $\mu = 1, 2, \cdots, m$, such that $\sigma_l(a_\mu, a) < \varepsilon/3$, $l = 1, 2, \cdots, N$. Then we have

$$\sigma_0(a_\mu, a) \leqq |\sigma_0(a_\mu, a) - \sigma_N(a_\mu, a)| + \sigma_N(a_\mu, a) < 2\varepsilon/3$$

and

$$\sigma_n(a_\mu, a) \leqq |\sigma_n(a_\mu, a) - \sigma_0(a_\mu, a)| + \sigma_0(a_\mu, a) < \varepsilon,$$

if $n > N$, which proves the lemma.

LEMMA 4. *If* $f_n \in \mathscr{A}$, $n = 1, 2, \cdots$, *is a monotonically increasing sequence that converges pointwise to a function* $f \in \mathscr{A}$, *then* $\lim_{n\to\infty} ||f_n - f|| = 0$.

PROOF. One can prove readily that the completion $\bar{\mathscr{A}}^1$ of $\mathscr{A}^1$ w.r.t. the norm (N) is an abstract L-space [1, p. 254]. By assumption $(f - f_n)$ is a monotonically decreasing sequence of non-negative functions which converges pointwise to the zero-function. But this implies by a standard result about L-spaces that $\lim_{n\to\infty} ||f - f_n|| = 0$ [1, p. 249, Theorem 12].

We are now in the position to furnish a proof of our main result. Let (f_n) be a monotonically increasing sequence of a.p. functions that converges pointwise to $f_0 \in \mathscr{A}$. By Lemma 4, $\lim_{n\to\infty}||f_n - f_0|| = 0$, and therefore, by Lemma 2, $\lim_{n\to\infty}|\sigma_n(a, b) - \sigma_0(a, b)| = 0$, uniformly. There exists a constant $C > 0$ so that $\sigma_n(a, b) < C$, $n = 0, 1, 2, \cdots$, for all $a, b \in G$. We put

$$(\mathrm{M}_{\sigma^*}) \qquad \sigma^*(a, b) = \sup_{0 \leqq i} \sigma_i(a, b),$$

and verify readily that σ^* is a pseudo-metric for which

$$(\mathrm{P}) \qquad \sigma_n(a, b) \leqq \sigma^*(a, b), \quad n = 0, 1, 2 \cdots,$$

holds, for all $a, b \in G$. Furthermore, by Lemma 3, G is totally bounded w.r.t. σ^*.

f_k is δ_k-continuous and, by virtue of Lemma 1, also σ_k-continuous, $k = 0, 1, 2, \cdots$. It is clear from (P) that f_k, $k = 0, 1, 2, \cdots$, is σ^*-continuous. If N_{σ^*} denotes the kernel of $\sigma^* (N_{\sigma^*} = \bigcap_{\nu=0}^{\infty} N_{\sigma_\nu})$, let G^* be the (compact) metric completion of the factor group G/N_{σ^*}. The sequence (f_n) can be extended to a sequence (f_n^*) of continuous functions on G^* such that the original assumptions about (f_n) are preserved. Using Dini's theorem, we conclude

THEOREM. *If a monotonically increasing sequence (f_n) of almost periodic (real-valued) functions on a group G converges pointwise to an almost periodic function f, then the sequence converges to f uniformly.*

References

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