

A NOTE ON A RESULT OF RUZSA

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Abstract

Let $\sigma_A(n) = |\{(a, a') \in A^2 : a + a' = n\}|$, where $n \in \mathbb{N}$ and A is a subset of \mathbb{N} . Erdős and Turán conjectured that, for any basis A of \mathbb{N} , $\sigma_A(n)$ is unbounded. In 1990, Ruzsa constructed a basis $A \subset \mathbb{N}$ for which $\sigma_A(n)$ is bounded in the square mean. In this paper, based on Ruzsa's method, we show that there exists a basis A of \mathbb{N} satisfying $\sum_{n \leq N} \sigma_A(n)^2 \leq 1\,449\,757\,928N$ for large enough N .

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1. Introduction

For a set A of integers and $n \in \mathbb{Z}$ write

$$\begin{aligned}\sigma(n) &= \sigma_A(n) = |\{(a, a') \in A^2 : a + a' = n\}|, \\ \delta(n) &= \delta_A(n) = |\{(a, a') \in A^2 : a - a' = n\}|.\end{aligned}$$

A subset A of \mathbb{N} is called a basis of \mathbb{N} if $\sigma_A(n) \geq 1$ for $n \geq n_0$. In 1941, Erdős and Turán [2] formulated the following attractive conjecture.

ERDŐS–TURÁN CONJECTURE. *If $A \subset \mathbb{N}$ is a basis of \mathbb{N} , then $\sigma_A(n)$ cannot be bounded:*

$$\limsup_{n \rightarrow +\infty} \sigma_A(n) = +\infty.$$

This harmless looking conjecture proved to be extremely difficult. In 1954, using probabilistic methods, Erdős [1] proved the existence of a basis of \mathbb{N} for which $\sigma(n)$ satisfies

$$c_1 \log n < \sigma(n) < c_2 \log n, \tag{1}$$

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for all n with certain positive constants c_1, c_2 . It is still a challenging problem to give a constructive proof of (1). In 1990, Ruzsa [6] constructed a basis of \mathbb{N} for which $\sigma(n)$ is bounded in the square mean. In 2003, Grekos *et al.* [3] proved that if A is a basis of \mathbb{N} , then $\max_{n \in \mathbb{N}} \sigma_A(n) \geq 6$. In 2005, Borwein *et al.* [5] improved this result. They showed that the maximum number of representations of any basis is at least eight. For other related problems, see [4, 7, 8].

Based on Ruzsa’s method, we obtain the following result.

THEOREM. *There exists a set A of non-negative integers that forms a basis of \mathbb{N} , and satisfies $\sum_{n \leq N} \sigma_A(n)^2 \leq 1\,449\,757\,928N$ for large enough N .*

Throughout this paper, let p be an odd prime, \mathbb{Z}_p be the set of residue classes mod p and $G = \mathbb{Z}_p^2$. Denote $Q_k = \{(u, ku^2) : u \in \mathbb{Z}_p\} \subset G$ and for a finite set A , let

$$D(A) = \sum_{-\infty}^{+\infty} \sigma_A(n)^2 = |\{(a, b, c, d) \in A^4 : a + b = c + d\}|.$$

φ is a mapping

$$\varphi : G \rightarrow \mathbb{Z}, \quad \varphi(a, b) = a + 2pb,$$

where we identify the residues (mod p) with the integers $0 \leq j \leq p - 1$.

2. Proofs

LEMMA 1 (Tang and Chen [7, Lemma 4]). *Let p be prime for which $p > 5$ and $p \equiv 5 \pmod{8}$. Put $B = Q_3 \cup Q_4 \cup Q_6$ and $V = \varphi(B) + \{0, 2p^2 - p, 2p^2, 2p^2 + p\}$. Then $V \subset [0, 4p^2]$ is a set with $|V| \leq 12p$ and $[4p^2, 6p^2] \subseteq V + V$, $\sigma_V(n) \leq 256$ for all n .*

LEMMA 2. *For $g = (a, b) \in G$, and fixed $k, l \in \mathbb{Z}_p \setminus \{0\}$, consider the equation*

$$g = x - y, \quad x \in Q_k, y \in Q_l.$$

If $k - l \neq 0$, this equation is solvable unless

$$\left(\frac{(k - l)b + kla^2}{p} \right) = -1,$$

and it has at most two solutions. If $k - l = 0$, it has at most one solution except for $g = 0$, when it has p solutions.

PROOF. Let $g = (a, b)$. Consider the system of equations

$$a = u - v, \tag{2}$$

$$b = ku^2 - lv^2. \tag{3}$$

Substituting the value of u from (2) into (3), we obtain the equation

$$b = (k - l)v^2 + 2kav + ka^2. \tag{4}$$

CASE 1. $k - l \neq 0$. Then we have

$$((k - l)v + ka)^2 = kla^2 + (k - l)b.$$

This is an equation of degree two; it is solvable unless the right-hand side is a quadratic non-residue mod p , that is,

$$\left(\frac{(k - l)b + kla^2}{p}\right) = -1,$$

and it has at most two solutions.

CASE 2. $k - l = 0$. Then (4) is an equation of degree one. If $a \neq 0$, (4) has one solution. If $a = b = 0$, (4) has p solutions. If $a = 0, b \neq 0$, (4) has no solution.

This completes the proof of Lemma 2. □

LEMMA 3. *Let p be prime for which $p > 5$ and $p \equiv 5 \pmod{8}$. Put $B = Q_3 \cup Q_4 \cup Q_6$ and let $B - B = \{b_1 - b_2 : b_1, b_2 \in B\}$. Then $B - B = G, \delta_B(g) \leq 11$ for all $g \neq 0$.*

PROOF. Suppose that there exists a $g = (a, b) \in G, g \notin Q_4 - Q_3, g \notin Q_6 - Q_4$. By Lemma 2, we have

$$\left(\frac{b + 12a^2}{p}\right) = -1, \quad \left(\frac{2b + 24a^2}{p}\right) = -1.$$

Thus

$$1 = \left(\frac{(b + 12a^2)(2b + 24a^2)}{p}\right) = \left(\frac{2}{p}\right) = -1.$$

Hence, $G = (Q_4 - Q_3) \cup (Q_6 - Q_4)$, which is stronger than the required $B - B = G$.

For any $g = (a, b) \in G (g \neq 0)$, by $p > 5$ we know that $b = 12a^2$ and $b = -12a^2$ cannot hold at the same time. Now we consider the following three cases.

CASE 1. $b \neq 12a^2$ and $b \neq -12a^2$. Then we have $g \notin (Q_3 - Q_4) \cap (Q_4 - Q_6)$ and $g \notin (Q_4 - Q_3) \cap (Q_6 - Q_4)$.

Indeed, if $g \in Q_3 - Q_4$ and $g \in Q_4 - Q_6$, by $b \neq 12a^2$, we have

$$\left(\frac{-b + 12a^2}{p}\right) = 1, \quad \left(\frac{-2b + 24a^2}{p}\right) = 1.$$

Thus

$$1 = \left(\frac{(-b + 12a^2)(-2b + 24a^2)}{p}\right) = \left(\frac{2}{p}\right) = -1.$$

Similarly, by $b \neq -12a^2$, we can show that $g \notin (Q_4 - Q_3) \cap (Q_6 - Q_4)$.

CASE 2. $b = 12a^2$ and $b \neq -12a^2$. Then $g \notin (Q_4 - Q_3) \cap (Q_6 - Q_4)$ and $g \notin Q_3 - Q_6$.

Indeed, if $g \in Q_4 - Q_3$ and $g \in Q_6 - Q_4$, then

$$\left(\frac{24a^2}{p}\right) = \left(\frac{b + 12a^2}{p}\right) = 1, \quad \left(\frac{48a^2}{p}\right) = \left(\frac{2b + 24a^2}{p}\right) = 1.$$

By $p \equiv 5 \pmod{8}$,

$$1 = \left(\frac{24a^2 \times 48a^2}{p}\right) = \left(\frac{2}{p}\right) = -1.$$

Thus, $g \notin (Q_4 - Q_3) \cap (Q_6 - Q_4)$.

Further, since

$$\left(\frac{-3b + 18a^2}{p}\right) = \left(\frac{-18a^2}{p}\right) = \left(\frac{-2}{p}\right) = -1,$$

by Lemma 2, we have $g \notin Q_3 - Q_6$.

CASE 3. $b = -12a^2$ and $b \neq 12a^2$. Then $g \notin (Q_3 - Q_4) \cap (Q_4 - Q_6)$ and $g \notin Q_6 - Q_3$.

Hence, there are at most four sub-equations for the equation

$$g = x - y, \quad x \in Q_i, y \in Q_j (i, j \in \{3, 4, 6\}, i \neq j)$$

and three sub-equations for the equation

$$g = x - y, \quad x, y \in Q_i (i = 3, 4, 6).$$

By Lemma 2, we have $\delta_B(g) \leq 11$ for all $g \neq 0$.

This completes the proof of Lemma 3. □

LEMMA 4. Let p be prime for which $p > 5$ and $p \equiv 5 \pmod{8}$, $B = Q_3 \cup Q_4 \cup Q_6$ and $B' = \varphi(B)$. Then $\delta_{B'}(n) \leq 11$ for all $n \neq 0$.

PROOF. Let $g = (a, b)$, $g' = (a', b')$, $h = (c, d)$, $h' = (c', d') \in B$.

If $\varphi(g) - \varphi(g') = \varphi(h) - \varphi(h')$, then

$$2p|(b + d' - b' - d)| = |c + a' - c' - a|;$$

thus, $b - b' = d - d'$, $a - a' = c - c'$.

Hence, $\varphi(g) - \varphi(g') = \varphi(h) - \varphi(h')$ is possible only if $g - g' = h - h'$. This shows that φ cannot increase the value of δ . By Lemma 3, we have $\delta_{B'}(n) \leq 11$ for all $n \neq 0$.

This completes the proof of Lemma 4. □

LEMMA 5. *Let p be prime for which $p > 5$ and $p \equiv 5 \pmod{8}$. Put $B = Q_3 \cup Q_4 \cup Q_6$ and $V = \varphi(B) + \{0, 2p^2 - p, 2p^2, 2p^2 + p\}$. Then $V \subset [0, 4p^2)$ is a set with $|V| \leq 12p$ and $\delta_V(n) \leq 176$ for all n with at most 11 exceptions.*

PROOF. By the Proof of [7, Lemma 4], we have $V \subset [0, 4p^2)$ and $|V| \leq 12p$. Note that

$$V = \varphi(B) + \{0, 2p^2 - p, 2p^2, 2p^2 + p\},$$

$$V - V = B' - B' + \{0, \pm(2p^2 - p), \pm 2p^2, \pm(2p^2 + p), \pm p, \pm 2p\}.$$

By Lemma 4,

$$\delta_V(n) \leq 16 \times \max \delta_{B'}(n) \leq 16 \times 11 = 176,$$

unless $n = 0, \pm(2p^2 - p), \pm 2p^2, \pm(2p^2 + p), \pm p, \pm 2p$.

This completes the proof of Lemma 5. □

The following Lemma 6 belongs to Ruzsa [6, Lemma 4.1]; here we give a stronger version by explicit computation.

LEMMA 6. *Let X be a finite set of integers and p be a prime for which $p > 5$ and $p \equiv 5 \pmod{8}$. There is a set Y such that*

$$Y \subset \left(\frac{p^2}{2}, 5p^2\right), \quad |Y| \leq 12p, \quad [6p^2, 7p^2) \subset Y + Y, \tag{5}$$

and

$$D(X \cup Y) < D(X) + \frac{24}{p}|X|^3 + 928|X|^2 + 6672p|X| + 73\,728p^2. \tag{6}$$

PROOF. Let V be the set of Lemma 5, and put $Y = V + t$ with an integer $t \in ((p^2/2), p^2]$. Equation (5) holds for any choice of t ; we show that (6) holds for a suitable choice.

Let $Z = X \cup Y$. $D(Z)$ is the number of quadruples (z_1, z_2, z_3, z_4) of elements of Z satisfying

$$z_1 + z_2 = z_3 + z_4. \tag{7}$$

We split equation (7) into the following five classes.

- (a) All four unknowns are from X . This gives the term $D(X)$.
- (b) One comes from Y , three from X . Equation (7) can be written as

$$t = x_1 + x_2 - x_3 - v, \quad v \in V.$$

Let S_t be the number of solutions; so we have

$$\sum S_t \leq 12p|X|^3,$$

thus

$$\left(\left\lceil \frac{p^2}{2} \right\rceil + 1 \right) \min S_t \leq 12p|X|^3,$$

and hence

$$\min S_t \leq \frac{24|X|^3}{p}.$$

(c) Two come from Y , two come from X .

CASE 1. The two y are on the same side. Equation (7) can be written as

$$y_1 + y_2 = x_1 + x_2, \quad y_i \in Y, \quad x_i \in X.$$

By Lemma 1, for every pair x_1, x_2 , there are at most 256 solutions which give a total of $256|X|^2$. According to the position of the y 's in (7), the contribution of this term is at most $2 \times 256|X|^2 = 512|X|^2$.

CASE 2. The y are on different sides, that is,

$$y_1 - y_2 = x_1 - x_2, \quad y_i \in Y, \quad x_i \in X.$$

By Lemma 5, if $x_1 - x_2$ is none of the 11 exceptional numbers, then the contribution of this term is at most $2 \times 176|X|^2 = 352|X|^2$; if $x_1 - x_2$ is one of the 11 exceptional numbers, then, after fixing the value of $x_1 - x_2$, the numbers x_1 and y_1 determine x_2 and y_2 uniquely; thus the contribution of this term is at most $4 \times 11 \times |X| \times |Y| \leq 528p|X|$.

(d) Three come from Y , one comes from X . Equation (7) can be written as

$$y_1 + y_2 = y_3 + x, \quad y_i \in Y, \quad x \in X.$$

In this case, the contribution of this term is at most $2 \times 256 \times |X| \times 12p = 6144p|X|$.

(e) Four unknowns are from Y . The contribution of this term is at most $2 \times 256 \times (12p)^2 = 73\,728p^2$.

Hence

$$D(X \cup Y) < D(X) + \frac{24}{p}|X|^3 + 864|X|^2 + 6672p|X| + 73\,728p^2.$$

This completes the proof of Lemma 6. □

PROOF OF THEOREM. By the Prime number theorem in arithmetic progression, there exists an M such that if $x > M$, there is a prime p for which $1.08x < p < \sqrt{7/6}x$. Thus we can take a sequence p_1, p_2, \dots of primes such that $p \equiv 5 \pmod{8}$

and $1.08 < p_{i+1}/p_i < \sqrt{7/6}$ for all i . This ensures that the intervals $[6p_i^2, 7p_i^2)$ overlap and together cover $[6p_1^2, +\infty)$. Apply Lemma 6 to $p = p_i$, we obtain the set Y_i . Let $X_0 = [0, 6p_1^2]$ and $X_i = X_{i-1} \cup Y_i$. Then the set $A = \bigcup_{i=0}^{\infty} X_i$ will be a basis of \mathbb{N} .

For large enough $N (> (7/12)(6p_1^2 + 1)^4)$, there exists $i > 1$ such that $p_i^2 < 2N < p_{i+1}^2$, so

$$\begin{aligned} |X_{i-1}| &\leq |X_0| + 12(p_1 + p_2 + \cdots + p_{i-1}) \\ &= |X_0| + 12p_i \left(\frac{25}{27} + \cdots + \left(\frac{25}{27} \right)^{i-1} \right) \\ &< 151p_i. \end{aligned}$$

By Lemma 6,

$$\begin{aligned} D(X_i) &= D(X_{i-1} \cup Y_i) \\ &< D(X_{i-1}) + \frac{24}{p_i} |X_{i-1}|^3 + 864 |X_{i-1}|^2 + 6672 p_i |X_{i-1}| + 73\,728 p_i^2 \\ &< D(X_{i-1}) + 103\,412\,088 p_i^2. \end{aligned}$$

By induction,

$$\begin{aligned} D(X_i) &< D(X_0) + 103\,412\,088(p_i^2 + \cdots + p_1^2) \\ &= D(X_0) + 103\,412\,088 p_i^2 \left(1 + \left(\frac{25}{27} \right)^2 + \cdots + \left(\frac{25}{27} \right)^{2i-2} \right) \\ &< (6p_1^2 + 1)^4 + 724\,878\,963 p_i^2 \\ &< 724\,878\,964 p_i^2. \end{aligned}$$

Therefore,

$$\sum_{n \leq N} \sigma(n)^2 \leq D(X_i) < 724\,878\,964 p_i^2 \leq 1\,449\,757\,928 N. \quad \square$$

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