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THE HEAT EQUATION FOR THE $\bar{\partial}$-NEUMANN PROBLEM, II

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Let Ω be a compact complex $n + 1$-dimensional Hermitian manifold with smooth boundary M. In [**2**] we proved the following.

THEOREM 1. *Suppose $\bar{\Omega}$ satisfies condition $Z(q)$ with $0 \leqq q \leqq n$. Let $\Box_{p,q}$ denote the $\bar{\partial}$-Laplacian on (p, q) forms on $\bar{\Omega}$ which satisfy the $\bar{\partial}$-Neumann boundary conditions. Then as $t \rightarrow 0+$,*

$$(0.1)\quad \operatorname{tr} \exp(-t\Box_{p,q}) \sim t^{-n-1}\left\{c_0 + \sum_{j\geqq 1} (c_j + c_j' \log t)t^{(1/2)j}\right\}.$$

(If $q = n + 1$, the $\bar{\partial}$-Neumann boundary condition is the Dirichlet boundary condition and the corresponding result is classical.)

Theorem 1 is a version for the $\bar{\partial}$-Neumann problem of results initiated by Minakshisundaram and Pleijel [**8**] for the Laplacian on compact manifolds and extended by McKean and Singer [**7**] to the Laplacian with Dirichlet or Neumann boundary conditions and by Greiner [**5**] and Seeley [**9**] to elliptic boundary value problems on compact manifolds with boundary. McKean and Singer go on to show that the coefficients in the trace expansion are integrals of local geometric invariants. In this paper we prove that under suitable hypotheses on $\bar{\Omega}$ and on the metric the coefficients in (0.1) are integrals of local invariants of the Hermitian geometry and the complex structure.

The results needed from [**2**] are summarized in Section 1. In Section 2 we discuss the hypotheses we require for $\bar{\Omega}$ and the metric. The main result, Theorem 3.1, is proved in Section 3.

1. Background. Let P be the fundamental solution of the heat equation for the $\bar{\partial}$-Neumann problem on (p, q)-forms on $\bar{\Omega}$. Then

$$\operatorname{tr} \exp(-t\Box_{p,q}) = \operatorname{tr}(P|_t).$$

We proved Theorem 1 by constructing P in an appropriate class of operators. In this section, we summarize the results we need from that construction.

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We denote by Δ'' the formal $\bar{\partial}$-Laplacian on (p, q) forms, and extend Δ'' as a positive self-adjoint elliptic operator on a bundle on the double Ω' of Ω. Then

$$(1.1) \quad P = E - G' - H = G - H$$

where E is the fundamental solution of the heat equation for Δ'' on Ω' and $G = E - G'$ is the Green's operator: the fundamental solution of the heat equation for the Dirichlet realization of Δ'' on $\bar{\Omega}$. The operator G and its trace are well understood [**5**], [7], [**9**], but it involves very little extra work to discuss G here.

In [**2**, Sections 2, 7, 8] we showed that E, G', and H can be constructed locally, modulo errors which do not affect the asymptotic expansion (0.1). Let U be a coordinate neighborhood on $\bar{\Omega}$ with coordinates x and let $\{Z_j\}_{j=1}^{n+1}$ be an orthonormal frame for $T^{1,0}(U)$ with

$$Z_j = X_j - iX_{n+1+j},$$

X_k real. We denote the dual variables to $(x, t) \in U \times \mathbf{R}$ by (ξ, τ) and let σ_j be the symbol of the vector field $-iX_j$. The localization of E to U is a pseudodifferential operator with symbol belonging to

$$S_{\mathrm{cp}}^{-2}(U \times \mathbf{R}, \mathbf{C}^N), \quad N = \binom{n+1}{p}\binom{n+1}{q}.$$

Here S_{cp}^{-2} denotes the symbol class of [**2**, Definition 2.7]. Thus the terms in the asymptotic expansion of the symbol $\sigma(E)$ are homogeneous of degree $\leqq -2$ with respect to parabolic dilations in (ξ, τ) and holomorphic for $\operatorname{Im} \tau < 0$. By (2.27) of [**2**], they are sums of terms of the form

$$(1.2) \quad (|\sigma|^2 + i\tau)^{-r} g(x, \xi)$$

where g is a homogeneous polynomial in ξ whose coefficients are polynomials in the derivatives of the coefficients of the vector fields X_j and in the inverse of the determinant of the coefficients of the $\{X_j\}$. Thus the terms of (1.2) are *uniform* in the sense of Definition 4.12 of [**1**]: in suitable coordinates at $x_0 \in U$, the terms evaluated at x_0 are universal polynomials in the derivatives of the coefficients of the $\{X_j\}$. The coefficients of these polynomials are universal functions of the Fourier transform variables ξ alone. (For suitable coordinates, we may take any coordinates so that X_j coincides with $\partial/\partial x^j$ at x_0.) Let $e(x, y, t)$ denote the kernel of E. Taking the inverse Fourier transform of the symbol $\sigma(e)$ and using the homogeneity of the terms in (1.2) we see that

$$(1.3) \quad \operatorname{tr} e(x, x, t) \sim t^{-(n+1)} \sum_{j=0}^{\infty} a_j(x) t^j \quad \text{as } t \to 0+,$$

where the coefficients a_j in suitable coordinates are universal polynomials in the derivatives of the coefficients of the $\{X_j\}$. (Here we have used the fact that (1.2) is homogeneous under $(\xi, \tau) \to (\lambda\xi, \lambda^2\tau)$ for $\lambda < 0$ as well as $\lambda > 0$ to eliminate half-integer powers of t.)

Modulo terms vanishing to infinite order as $t \to 0+$, the traces of G' and H are the same as the traces of pseudodifferential operators G_0' and H_0' on the boundary $M \times \mathbf{R}$; see [**2**], Proposition 7.12 and the discussion following Corollary 9.22. To describe the operators G_0' and H_0 we require some notation. Fix $x_0 \in M$ and choose a coordinate neighborhood U of $x_0 \in \bar{\Omega}$ with coordinates $(x^0, \dots, x^{2n}, \rho)$ which identify U with $V \times [0, \epsilon) \subset \mathbf{R}^{2n+2}$, where ρ is the distance to M and $(x^0, \dots, x^{2n}, 0)$ are the coordinates of the point on M closest to the point with coordinates $(x^0, \dots, x^{2n}, \rho)$. We choose an orthonormal frame $\{Z_j\}$ for $T^{1,0}(U)$ with

$$(1.4) \quad Z_{n+1} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\rho} + iT\right), \quad T = -J_0\frac{\partial}{\partial\rho}.$$

Here J_0 is the almost complex structure. Then $Z_1, \dots, Z_n$, and T are tangent to $\{\rho = c\}$ for $0 \leqq c < \epsilon$. In particular $\{Z_1, \dots, Z_n\}$ is an orthonormal frame for $T^{1,0}(V)$. Let

$$(1.5) \quad X_j = \mathrm{Re}\, Z_j, \quad X_{n+j} = -\mathrm{Im}\, Z_j, \quad 1 \leqq j \leqq n,$$
$$X_0 = T.$$

We denote the dual variables to $(x, t) \in V \times \mathbf{R}$ by (ξ, τ) and set

$$(1.6) \quad \sigma_j = \sigma(-iX_j), \quad 0 \leqq j \leqq 2n.$$

Let Δ be the square root

$$(1.7) \quad \Delta(x, \xi, \tau)^2 = \sigma_0^2(x, 0, \xi) + 2\sum_{j=1}^{2n} \sigma_j^2(x, 0, \xi) + 2i\tau$$

with $\mathrm{Re}\, \Delta \geqq 0$. Then G_0' has symbol

$$\sigma(G_0') \in S_{\mathrm{cp}}^{-2}(V \times \mathbf{R}, \mathbf{C}^N).$$

Each term in the asymptotic expansion of $\sigma(G_0')$ has the form

$$(1.8) \quad \Delta(x, \xi, \tau)^{-s} p(x, \xi)$$

where each entry of p is a homogeneous polynomial in ξ whose coefficients are polynomials in the derivatives of the coefficients of the $\{X_j\}$ and of the inverse of the determinant of those coefficients. Thus the terms in (1.8) are uniform; for suitable coordinates at $x_0 \in M$ we choose any coordinates $(x^0, \dots, x^{2n}, \rho)$ as above for which X_j coincides with $\partial/\partial x^j$ at x_0. Let $g_0'(x, y, t)$ denote the kernel of G_0'. Taking the inverse Fourier transform of the symbol $\sigma(G_0')$ we see that

$$(1.9) \quad \mathrm{tr}\, g_0'(x, x, t) \sim t^{-(n+1/2)} \sum_{j=0}^{\infty} k_j(x) t^{(1/2)j}, \quad t \to 0+.$$

Here $k_j(x_0)$ is a universal polynomial in the derivatives at x_0 of the coefficients of the $\{X_j\}$ in suitable coordinates depending on x_0.

Similarly, $\sigma(H_0)$ has locally an asymptotic expansion whose terms have entries of the form

(1.10) $p(x, \xi, \tau)q(x, \xi, \tau)$

where p is homogeneous of degree $j \leqq 0$ with respect to the parabolic dilations and is holomorphic for $\operatorname{Im} \tau < 0$, while q is homogeneous of degree $k \leqq -2$ with respect to the non-isotropic parabolic dilations

(1.11) $(\xi, \tau) \rightarrow (\lambda^2\xi_0, \lambda\xi_1, \ldots, \lambda\xi_{2n}, \lambda^2\tau)$

and is also holomorphic for $\operatorname{Im} \tau < 0$. Thus, $\sigma(H_0)$ belongs to the symbol class $S_{\text{hcp}}^{-2,0}(V \times \mathbf{R}, \mathbf{C}^N)$ of Definition 5.10 of [**2**].

We assume now that the induced metric on M is a Levi metric. Then the symbols p and q of (1.10) are uniform: see [**2**], Propositions 3.27, 7.18, and 7.26, Definition 7.23, Remark 6.20, and Corollary 9.22. As suitable coordinates we take (x, ρ) as above and assume in addition that (x) are anti-symmetric x_0 coordinates on V in the sense of Section 5 of [**1**]. Thus in suitable local coordinates $\sigma(H_0)$ has an asymptotic expansion whose terms have entries of the form

(1.12) $f(x)p(\xi, \tau)q(\xi, \tau)$

where p and q are as in (1.10) and are independent of x, while $f(x_0)$ is a universal polynomial in the derivatives of the coefficients of the $\{X_j\}$ at x_0. Let h_0 denote the kernel of H_0. Then $\operatorname{tr} h_0(x, x, t)$ is asymptotically a sum of terms of the form

(1.13) $f(x)(pq)^{\vee}(0, t)$.

By Theorem 10.17 of [**2**] it follows that

(1.14) $$\operatorname{tr} h_0(x, x, t) \sim t^{-(n+1)} \times \left\{k_0'(x) + \sum_{j=1}^{\infty} [k_j'(x) + k_j''(x) \log t]t^{(1/2)j}\right\} \text{ as } t \rightarrow 0+.$$

Here again k_j' and k_j'' are universal polynomials in derivatives of the coefficients of the $\{X_j\}$, pointwise in suitable coordinates.

2. Metrics. In Section 1 we required that the metric on $\bar{\Omega}$ induce a Levi metric on M, in order to get a good expression for the coefficients of the expansion of $\operatorname{tr} H$ or equivalently of $\operatorname{tr} H_0$. To obtain a geometric interpretation of the coefficients in the trace expansion (0.1) we need a canonical connection on M which preserves both the metric and the CR structure. We assume that M is strictly pseudoconvex, i.e., that at each point of M the manifold $\bar{\Omega}$ is either strictly pseudoconvex or strictly pseudoconcave. Then the Webster-C. M. Stanton connection [**10**], [**11**] has the desired properties. In addition we shall have to differentiate coefficients of the $\{X_j\}$ with respect to normal as well as tangential variables, so we want a metric on $\bar{\Omega}$ which is Kähler in a neighborhood of the boundary.

The following result shows that the pseudoconvexity condition is the only limitation on an embedded manifold $\bar{\Omega}$.

PROPOSITION 2.1. *Suppose $\bar{\Omega}$ is embedded in a complex $n + 1$-dimensional manifold $\tilde{\Omega}$ without boundary and suppose that at each point of its boundary M, $\bar{\Omega}$ is either strictly pseudoconvex or strictly pseudoconcave. Then $\bar{\Omega}$ admits a Hermitian metric which is Kähler in a neighborhood of M and which induces a Levi metric on M.*

Proof. Let h_1 be a Hermitian metric on $\bar{\Omega}$. Suppose there is a neighborhood U of M in $\bar{\Omega}$ and a Kähler metric h_0 on U which induces a Levi metric on M. Choose a non-negative cut-off function $\chi \in C^\infty(\bar{\Omega})$ with support in U such that $\chi \equiv 1$ in a neighborhood of M in $\bar{\Omega}$. Then

$$h = \chi h_0 + (1 - \chi)h_1$$

is a Hermitian metric with the desired properties.

To construct such a metric h_0 we use an approximate solution of the Monge-Ampère equation. Let r be a smooth defining function for Ω: $\Omega = \{r < 0\}$, $M = \{r = 0\}$, and $dr \neq 0$ on M. Then $i\partial\bar{\partial}r$ is a definite form on $TM \cap J_0TM$, so $(\partial\bar{\partial}r)^n \neq 0$ on $TM \cap J_0TM\backslash\{0\}$. Thus

$$\partial r \wedge \bar{\partial}r \wedge (\partial\bar{\partial}r)^n \neq 0 \quad \text{on } M.$$

On M there is a smooth function φ_0 such that

$$(2.2) \qquad (\partial\bar{\partial}r)^{n+1} = -2(n + 1)\varphi_0\partial r \wedge \bar{\partial}r \wedge (\partial\bar{\partial}r)^n \quad \text{on } M.$$

Extend φ_0 to a smooth function φ on a neighborhood of M and let $u = r + \varphi r^2$. Then

$$(2.3) \qquad \partial\bar{\partial}u = \partial\bar{\partial}r + 2\varphi\partial r \wedge \bar{\partial}r \quad \text{on } M,$$

so (2.2) implies

$$(2.4) \qquad (\partial\bar{\partial}u)^{n+1} = 0 \quad \text{on } M,$$

i.e., u solves the homogeneous Monge-Ampère equation on M. With the appropriate choice of sign on each component of M, (2.4) implies $\pm i\partial\bar{\partial}u$ is nonnegative on $T\tilde{\Omega}|_M$. Let $f = e^{\pm u}$, with the appropriate sign on each component of M. Then

$$(2.5) \qquad i\partial\bar{\partial}f = \pm ie^{\pm u}(\partial\bar{\partial}u \pm \partial u \wedge \bar{\partial}u)$$

is positive definite on $T\tilde{\Omega}|_M$. Therefore the form (2.5) is positive definite on a neighborhood U of M and is the Kähler form for a Kähler metric on U.

Remark 2.6. Our proof of (2.4) is the first step in a construction by Bedford and Burns, who showed that when M is real analytic there is a solution of (2.4) in a neighborhood of M with $u = 0$ on M and $du \neq 0$ on M; see Proposition 1.5 of [**3**]. If $\bar{\Omega}$ is a bounded strictly convex domain in $\mathbf{C}^{n+1}$, Lempert showed that for any $z_0 \in \Omega$ there is a defining function u for $\bar{\Omega}$ which satisfies (2.4) on $\Omega \backslash \{z_0\}$; see Theorem 4 of [**6**].

3. Geometry. Our goal is the following.

THEOREM 3.1. *Let $\bar{\Omega}$ be a compact complex $n + 1$-dimensional manifold with smooth boundary M. Suppose that at each point of M, $\bar{\Omega}$ is either strictly pseudoconvex or strictly pseudoconcave. Suppose also that $\bar{\Omega}$ is equipped with a Hermitian metric which is Kähler in a neighborhood of the boundary M and which induces a Levi metric on M. Then for $1 \leqq q \leqq n - 1$,*

$$(3.2) \quad \operatorname{tr} \exp(-t\Box_{p,q}) \sim t^{-n-1}\left\{ \sum_{j=0}^{\infty} b_j t^j + c_0 + \sum_{j=1}^{\infty} (c_j + c_j' \log t) t^{(1/2)j} \right\} \text{ as } t \to 0+.$$

If $\bar{\Omega}$ is strictly pseudoconvex (pseudoconcave) then (3.2) *also holds for $q = n$ ($q = 0$). The coefficients b_j are integrals over Ω of universal polynomials in the components of the curvature and torsion of the Hermitian connection and their covariant derivatives with respect to this connection. The coefficients c_j and c_j' are integrals over M of universal polynomials in the components of the second fundamental form of M, the curvature and torsion of the Webster-C. M. Stanton connection, and their covariant derivatives with respect to this connection, as well as the components of the Hermitian curvature of $\bar{\Omega}$ and its Hermitian covariant derivatives on M.*

Remark 3.3. By the Hermitian connection we mean the unique type (1, 0) connection which preserves the metric. Because the metric is Kähler in a neighborhood of the boundary, this is the Riemannian connection in this neighborhood.

Proof of Theorem 3.1. By the results of Section 1, the coefficients are given by

$$(3.4) \quad b_j = \int_\Omega a_j(x) dV, \quad c_j = \int_M \{k_j(x) + k_j'(x)\} dV, \quad c_j' = \int_M k_j''(x) dV$$

where a_j, k_j, k_j', and k_j'' are as in (1.3), (1.9), and (1.14).

As noted in connection with (1.3), given $x_0 \in \Omega$ we may choose an orthonormal frame $\{Z_j\}$ and coordinates (x) near x_0 with $X_j = \partial/\partial x^j$ at

x_0, and then $a_j(x_0)$ is obtained by evaluating a universal polynomial in the derivatives of the coefficients of the $\{X_j\}$. We obtain such a frame and such coordinates by choosing Z_j at x_0 and letting (x) be normal coordinates at x_0 for the corresponding frame $\{X_j|_{x_0}\}$, with respect to the Hermitian connection. Let X_j be obtained from $X_j|_{x_0}$ by parallel transport along the geodesics through x_0. By the argument of [**1**, Section 7] (see also [**4**]), the derivatives of the coefficients of the $\{X_j\}$ evaluated at x_0 are universal polynomials in the curvature and torsion of the Hermitian connection and their covariant derivatives. Thus a_j is such a polynomial.

For $x_0 \in M$ we choose an orthonormal frame $\{Z_j\}$ for $T^{1,0}_{x_0}(\bar{\Omega})$ with

$$(3.5)\qquad Z_{n+1} = \frac{1}{\sqrt{2}}(n_0 - iJ_0 n_0),$$

where n_0 is the unit inward normal to M at x_0. As in (1.5) we define a frame for $T_{x_0}(M)$ by

$$(3.6)\qquad X_j = \operatorname{Re} Z_j, \quad X_{n+j} = -\operatorname{Im} Z_j, \quad 1 \leqq j \leqq n,$$
$$X_0 = -J_0 n_0.$$

Let $(x) = (x^0, \dots, x^{2n})$ be normal coordinates for the Webster-Stanton connection on M with respect to this frame, on a neighborhood U_0 of x_0 in M. We extend the frame $\{X_j\}$ to U_0 by the parallel transport along the geodesics through x_0. By Remark 7.29 of [**1**], the coordinates are anti-symmetric x_0-coordinates. For small $\epsilon > 0$ we let

$$(3.7)\qquad U = \{\exp_w \rho n_0 : w \in U_0, 0 \leqq \rho < \epsilon\}.$$

Here $n_0 = n_0(w)$ is the unit inward normal to M at w and exp is the Riemannian exponential map. We coordinatize U by letting (x, ρ) be the coordinates of $\exp_w \rho n_0$ when (x) are the normal coordinates of $w \in U_0$. The frame $\{X_j\}$ is now extended to U by parallel transport with respect to the Riemannian connection along geodesics perpendicular to M. For small enough ϵ, the metric is Kähler in U. On U_0 we have

$$X_{n+j} = J_0 X_j, \quad 1 \leqq j \leqq n,$$

since the Webster-Stanton connection preserves the CR structure. Because the metric is Kähler, the same relation carries over to U. Moreover, $\partial/\partial\rho$ is the unit tangent to geodesics perpendicular to M and

$$X_0 = -J_0 \frac{\partial}{\partial \rho}.$$

Both the Webster-Stanton connection and the Riemannian connection preserve the metric, so $\{Z_j\}$ is an orthonormal frame for $T^{1,0}(U)$, where

$$(3.8)\qquad Z_j = X_j - iX_{n+j}, \quad 1 \leqq j \leqq n,$$
$$Z_{n+1} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \rho} + iX_0\right).$$

Thus we have constructed a frame as in (1.5). It follows that the coefficients k_j, k_j', and k_j'' in (1.9) and (1.14) are universal polynomials in the derivatives of the coefficients of the $\{X_j\}$, pointwise in suitable coordinates. The proof of Theorem 3.1 will be complete when we have proved the following.

PROPOSITION 3.9. *Let $\{X_j\}$ and (x, ρ) be the frame and the coordinates just constructed, corresponding to a point $x_0 \in M$. Each Taylor coefficient at x_0 of each component of the $\{X_j\}$ with respect to the coordinate frame $\{\partial/\partial x^k, \partial/\partial\rho\}$ is a universal polynomial in the components of the second fundamental form of M, the curvature and torsion of the Webster-C. M. Stanton connection, and their Webster-Stanton covariant derivatives, together with the components of the Hermitian curvature and their Hermitian covariant derivatives.*

Proof. We systematically use Greek indices α, β, γ in the range

$$0 \leqq \alpha, \beta, \gamma \leqq 2n + 1$$

and Roman indices i, j, k in the range

$$0 \leqq i, j, k \leqq 2n.$$

Thus our frame for $T(U)$ is $\{X_\alpha\}$ while our frame for $T(U_0)$ is $\{X_j\}$. Let $\{\varphi^\alpha\}$ be the dual coframe for $T^*(U)$, and let $\{\psi^\alpha_\beta\}$ be the Riemannian connection form. Thus

$$(3.10) \quad d\varphi^\alpha + \psi^\alpha_\beta \wedge \varphi^\beta = 0$$

(with summation over repeated indices). Let $\{\Psi^\alpha_\beta\}$ be the Riemannian curvature form:

$$(3.11) \quad \Psi^\alpha_\beta = d\psi^\alpha_\beta + \psi^\alpha_\gamma \wedge \psi^\gamma_\beta.$$

Let (s_{ij}) be the matrix of the second fundamental form of M with respect to the frame $\{X_j\}$:

$$(3.12) \quad \psi_j^{2n+1}|_M = s_{jk}\varphi^k.$$

Now $X_{2n+1} = \partial/\partial x^{2n+1} = \partial/\partial\rho$, so

$$(3.13) \quad \frac{\partial}{\partial\rho} \lrcorner\, \varphi^\alpha = \delta^\alpha_{2n+1}.$$

LEMMA 3.14. *The restriction to M of any normal derivative of any component of the 1-forms φ^α can be expressed as a universal linear combination of restrictions to M of the components of the φ^α, the components of the ψ_α^{2n+1}, and normal derivatives of the curvature forms Ψ_α^{2n+1}.*

Proof. Write

$$x = (x^0, \dots, x^{2n}, \rho) = (x', \rho).$$

Given a 1-form a, write

$$a = a' + a'' \quad \text{where } \frac{\partial}{\partial \rho} \lrcorner\, a' = 0 = \frac{\partial}{\partial x^j} \lrcorner\, a'', 0 \leqq j \leqq 2n.$$

A Taylor series expansion around $\rho = 0$ gives

$$a \sim \sum_{k=0}^{\infty} \rho^k a_k \quad \text{as } \rho \to 0+,$$

with

$$a_k = a'_k + a''_k = \left\{ \sum_{j<2n+1} a_{k,j}(x')dx^j \right\} + a_{k,2n+1}(x')d\rho.$$

The k-th normal derivatives of the components of a restricted to M are just the components of the form $k!a_k$.

Let $R = \rho\partial/\partial\rho$. Then clearly

$$Ra \sim \sum_{k=0}^{\infty} \{\rho^k k a'_k + \rho^k(k + 1)a''_k\}$$

where we also use R to denote the Lie derivative on forms. Therefore normal derivatives of components of a on M can be expressed in terms of the components themselves ($k = 0$), together with normal derivatives of components of Ra on M. With this in mind we consider

$$\begin{aligned} (3.15) \quad R\boldsymbol{\varphi}^j &= R \lrcorner\, d\boldsymbol{\varphi}^j + d(R \lrcorner\, \boldsymbol{\varphi}^j) \\ &= R \lrcorner\, (\boldsymbol{\varphi}^\beta \wedge \psi^j_\beta) + 0 \\ &= \rho\psi^j_{2n+1} = -\rho\psi^{2n+1}_j, \quad j < 2n + 1. \end{aligned}$$

Here we have used (3.13) and also the fact that the parallel transport in the direction $\partial/\partial\rho$ implies

$$(3.16) \quad R \lrcorner\, \psi^\alpha_\beta = 0.$$

Now $\boldsymbol{\varphi}^{2n+1} = d\rho$. To complete the proof, therefore, we need only consider normal derivatives of positive order of the components of the ψ^{2n+1}_j. But

$$\begin{aligned} (3.17) \quad R\psi^{2n+1}_j &= R \lrcorner\, d\psi^{2n+1}_j + d(R \lrcorner\, \psi^{2n+1}_j) \\ &= R \lrcorner\, (\Psi^{2n+1}_j - \psi^{2n+1}_\alpha \wedge \psi^\alpha_j) + 0 \\ &= R \lrcorner\, \Psi^{2n+1}_j. \end{aligned}$$

Thus for these higher derivatives we need only normal derivatives of the components of Ψ^{2n+1}_j.

Remark 3.18. By (3.15) and (3.17) only normal derivatives of order $\leqq k - 2$ of Ψ_j^{2n+1} enter the k-th normal derivative of φ^j. By (3.15) the components of $\psi_j^{2n+1}|_M$ only enter the first normal derivative of φ^j.

LEMMA 3.19. *Any derivative at x_0 of any component of the 1-forms $\{\varphi^\alpha\}$ can be expressed as a universal polynomial in the derivatives (with respect to the chosen coordinates) of the components of the $\{\Psi_j^{2n+1}\}$, and tangential derivatives of components of the Webster-Stanton curvature and torsion tensors, and tangential derivatives of the components of the $\{s_{jk}\}$.*

Proof. In view of Lemma 3.14 we only need to examine tangential derivatives of the components of the $\{\varphi^\alpha\}$ and the $\{\psi_j^{2n+1}\}$. For $\{\varphi^j\}$, $j \leqq 2n$, the result is in [**1**], Section 7, while $\varphi^{2n+1} = d\rho$. We are left with ψ_j^{2n+1}, and (3.12) gives the desired result.

Proof of Proposition 3.9, *completed.* Let us say that a term is "acceptable" if it can be expressed in the form indicated in the statement of Proposition 3.9. Let $A = (a_\beta^\alpha)$ be the matrix of coefficients of the coframe $\{\varphi^\alpha\}$ with respect to the coordinate coframe:

$$\varphi^\alpha = a_\beta^\alpha dx^\beta.$$

Then A^{-1} is the matrix of coefficients of the $\{X_\alpha\}$ with respect to the coordinate frame, and we want to show that the entries of A^{-1} are acceptable at x^0. Now $A = I$ at x^0, so it is enough to show that the a_β^α are acceptable. Components of the (s_{jk}) are polynomials in the a_j^i, det A^{-1} and components of the second fundamental form. Similarly, components of Ψ_j^{2n+1} are polynomials in the a_β^α, det A^{-1} and the components of the Riemannian (= Hermitian) curvature tensor. Thus, by Remark 3.18, Lemma 3.19 remains true if we replace $\{\Psi_j^{2n+1}\}$ by the Riemannian curvature tensor and $\{s_{jk}\}$ by the second fundamental form. Conversion of tangential derivatives of components of the Webster-Stanton curvature and torsion and of the second fundamental form to Webster-Stanton covariant derivatives is carried out in Section 7 of [**1**], so these are acceptable terms. To complete the proof, we induce on the order of derivatives; $a_\beta^\alpha(x_0) = \delta_\beta^\alpha$ is acceptable. Suppose derivatives of order $\leqq k$ of the $\{a_\alpha^\beta\}$ are acceptable. The metric tensor has components $G = (g_\beta^\alpha) = A^tA$, so the components of the Riemannian connection form with respect to the $\{\partial/\partial x^\alpha\}$, $\{dx^\alpha\}$, i.e., the Christoffel symbols, are polynomials in the a_β^α, det A^{-1} and first-order derivatives of the a_β^α. Thus the Christoffel symbols and their derivatives of order $\leqq k - 1$ evaluated at x_0 are acceptable. Hence, a derivative of order $k - 1$ of a component of the Riemannian curvature tensor is a component of a $k - 1$ *st* covariant derivative of the tensor, modulo acceptable terms. By Remark 3.18, derivatives of order $k + 1$ of the a_β^α are acceptable.

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