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Martin's Axiom and some classical constructions

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We display the relevance of Martin's Axiom to suitable forms of some long-known results in classical measure and category theory (for example, outer-measure-preserving partitions of sets of positive outer measure). While our theorems are at best fragmentarily new, the proofs are very simple, and, we think, instructive as regards the force of the axiom.

1. Introduction

"Martin's Axiom" ([2], [7], [8]) is a rather powerful set-theoretical principle which, though weaker than the continuum hypothesis, proves many of the same analytical and topological theorems formerly known only on pain of assuming $2^{\aleph_0} = \aleph_1$. (For an excellent survey of the topological uses of the axiom, see [8].) In the present, essentially expository, paper, we shall illustrate both types of application more-or-less simultaneously, in relation to some well known classical results. Although the applications to be discussed represent only minor and fairly obvious extensions of what is already explicit in the literature, we think they constitute interesting propaganda on behalf of the new (or at any rate still youthful) axiom. We shall express the axiom in its usual partial-order formulation. Let $\langle P, \leq\rangle$ be a partial order, and let $Q \subseteq P$. Q is defined to be *dense* if and only if $(\forall p \in P)\ (\exists q \in Q)\ [p \leq q]$. Q is defined to be *open* if and only if $(\forall q \in Q)\ (\forall p \in P)\ [q \leq p \Rightarrow p \in Q]$. A set $G \subseteq P$ is defined to be *internally compatible* if and only if

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$$(\forall g_1 \in G)\ (\forall g_2 \in G)\ (\exists g_3 \in G)\ [g_1 \le g_3\ \&\ g_2 \le g_3]\ .$$

If F is a collection of subsets of P, and $G \subseteq P$, then G is defined to *meet* F if and only if $(\forall F \in F)\ [G \cap F \ne \emptyset]$. $\langle P, \le\rangle$ is defined to *have countable cellularity* if and only if

$$(\forall Q \subseteq P)\ \big[(\forall x)(\forall y)\ [(x \in Q\ \&\ y \in Q\ \&\ x \ne y) \Rightarrow \text{there is no element } z \text{ of } P \text{ such that } x \le z\ \&\ y \le z] \Rightarrow \text{card}(Q) \le \aleph_0\big]\ .$$

Martin's Axiom, which is shown in [7] to be consistent with both $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_0} > \aleph_1$, is formulated thus:

MARTIN'S AXIOM. *If* $\langle P, \le\rangle$ *is a partial order having countable cellularity and* F *is a family of fewer than* $2^{\aleph_0}$ *dense open subsets of* P, *then there exists an internally compatible set* $G \subseteq P$ *which meets* F.

What we shall need for our examples is not Martin's Axiom itself, but rather the following twin consequences of it (see [2, §4])[1].

COROLLARY 1. *Let* T *be a second-countable topological space in which points are closed but not open, and assume Martin's Axiom. Then the union of fewer than* $2^{\aleph_0}$ *first-category subsets of* T *is again of first category.*

COROLLARY 2. *Let* T *be a second-countable space, and let* μ *be a complete bi-regular Borel measure defined in* T. *Assume Martin's Axiom. Then the union of fewer than* $2^{\aleph_0}$ *subsets of* T *each of which is of* μ*-measure* 0 *is again of* μ*-measure* 0.

2. Some definitions

Let it be a standing assumption that all spaces to which we refer are T_1-spaces; that is, points are closed. To say that μ is a bi-regular Borel measure defined in some space T means, here, that μ is countably additive and defined for all Borel subsets of the space, and that μ is induced (in the usual way) from outer and inner measures μ^* and μ_* each of which is defined (via infs over open supersets and sups over closed

[1] Very minor modifications are needed in the proof given in [2] for, say, the proof of Corollary 1.

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subsets) for *all* subsets of the space; this of course entails that if A is a μ-measurable subset of T then there are Borel sets B_1 and B_2 (called, respectively, a *Borel cover* and a *Borel kernel* for A) such that

$$B_2 \subseteq A \subseteq B_1 \;\&\; \mu(B_2) = \mu_*(A) = \mu(A) = \mu^*(A) = \mu(B_1) \;.$$

By "μ is non-atomic" we mean that points have μ-measure 0. As noted, for example in [1], there is a direct category-theoretic analogue of the notion of *Borel cover*: given a subset A of a space T, there exists a Borel set B (in fact, an F_σ set) such that $A \subseteq B$ & $B - C$ is of first category whenever C is a superset of A such that C has the Baire property. Call such a set B a *Baire cover* for A. This gives rise to a notion of two subsets of T having the same "Baire character" (in restricted analogy to a pair of sets having the same outer measure): let $A_1 \approx_B A_2$ mean that A_1 and A_2 have Baire covers which differ by a set of first category.

Next, we recall (see [3]) the definitions of *Lusin set* and *Sierpiński set*: classically, a Lusin set is a set of reals of cardinality $2^{\aleph_0}$ each uncountable subset of which is of second category; a Sierpiński set is a set of reals of cardinality $2^{\aleph_0}$ each uncountable subset of which has positive outer measure in the sense of Lebesgue. The existence of such sets, given $2^{\aleph_0} = \aleph_1$, is well known ([3], [4]). Martin and Solovay noted in [2, §5] that if one assumes Martin's Axiom plus $2^{\aleph_0} > \aleph_1$ *without altering the above definition of Lusin set*, then (on account of Corollary 1) no such sets exist; a corresponding remark (based on Corollary 2) applies to Sierpiński sets. They then went on to observe, however, that if one reverses the loading of cardinal fine structure from the *bottom* of the $\leq 2^{\aleph_0}$ - hierarchy to the *top* of it, by substituting "of cardinality $2^{\aleph_0}$" for "uncountable", then the existence of Lusin sets *follows from Martin's Axiom* (because of Corollary 1!); again, the corresponding observation holds for Sierpiński sets. Our definitions run as follows. If T is a given topological space, we shall say that a subset L of T is defined to be *Lusin* if and only if

$$\text{card}(L) = \text{card}(T) \ \& \ (\forall X)\ \big[\big(X \subseteq L \ \& \ \text{card}(X) = \text{card}(T)\big) \Rightarrow X \text{ is of second category}\big] .$$

We shall define L to be *strongly Lusin* if and only if L is Lusin and

$$(\forall X)\ \big[\big(X \subseteq T \ \& \ \text{Card}(X) = \text{Card}(T) \ \& \ X \text{ is a second-category set having the Baire property}\big) \Rightarrow L \cap X \text{ is Lusin}\big] .$$

Let μ be a measure defined in T, and let μ^* be an outer measure, defined for all subsets of T, such that μ^* extends μ. Let $Y \subseteq T$. We shall say that Y is defined to be (μ^*, μ)*-Sierpiński* if and only if

$$\text{card}(Y) = \text{card}(T) \ \& \ (\forall X)\ \big[\big(X \subseteq Y \ \& \ \text{card}(X) = \text{card}(T)\big) \Rightarrow \mu^*(X) > 0\big] .$$

Y is defined to be *strongly* (μ^*, μ)*-Sierpiński* if and only if Y is (μ^*, μ)-Sierpiński and

$$(\forall X)\ \big[\big(X \subseteq T \ \& \ \text{card}(X) = \text{card}(T) \ \& \ X \text{ is } \mu\text{-measurable with } \mu(X) > 0\big) \Rightarrow Y \cap X \text{ is } (\mu^*, \mu)\text{-Sierpiński}\big] .$$

3. A set-theoretical lemma

The observations to be made in §4 will hinge upon a partitioning property which arises as a straightforward consequence of the following well known fact from pure set theory.

THEOREM 3.1 ([5, p. 455]). *Let A be an infinite set of cardinality λ, and let F be a collection of subsets of A such that* $\text{card}(F) = \lambda \ \& \ (\forall F \in F)\ [\text{card}(F) = \lambda]$. *Then there is a family G of mutually disjoint subsets of A such that*

$$\text{card}(G) = \lambda \ \& \ (\forall F \in F)\ (\forall G \in G)\ [\text{card}(F \cap G) = \lambda] .$$

We shall carry out some very simple manipulations involving Theorem 3.1, in order to obtain the desired partitioning result. Suppose that A, λ are as in Theorem 3.1, and let F' be a collection of sets (not necessarily subsets of A, and not necessarily of cardinality λ) such that $\text{card}(F') \le \lambda$. Let $F'' = \{F \in F' \mid F \subseteq A \ \& \ \text{card}(F) = \lambda\}$, and let $\{H_\alpha \mid \alpha < \lambda\}$ be a disjoint family, of cardinality λ, of subsets of A such that $\text{card}(H_\alpha) = \lambda$ holds for all $\alpha < \lambda$. Let $F = F'' \cup \{H_\alpha \mid \alpha < \lambda\}$; then A, λ, and F are as in Theorem 3.1. Hence, there exists a family G, of cardinality λ, consisting of mutually disjoint subsets of A and having the property that $(\forall G \in G)\ (\forall F \in F)\ [\mathrm{card}(G \cap F) = \lambda]$. Let $\langle G_\tau \rangle_{\tau<\lambda}$ be a (nonrepetitive) well-ordering of G. Setting $A_0 = G_0 \cup \{x \mid x \in A - \bigcup G\}$ and $A_\tau = G_\tau$ for $0 < \tau < \lambda$, we obtain a sequence $\langle A_\tau \rangle_{\tau<\lambda}$ which, clearly, enjoys the following three properties:

(1) $A = \bigcup_{\tau<\lambda} A_\tau$;

(2) $\mathrm{card}(A_\tau) = \lambda$ for all $\tau < \lambda$; and

(3) $(\forall \tau < \lambda)\ (\forall F \in F)\ [(F \subseteq A\ \&\ \mathrm{card}(F) = \lambda) \Rightarrow \mathrm{card}(A_\tau \cap F) = \lambda]$.

Now set $F_1 = \{A-F \mid F \in F'\}$, and repeat the above manipulations with F_1 in place of F'. In view of (1), (2), and (3), we thereby obtain a proof of

LEMMA 3.2. *Let A be an infinite set of cardinality λ, and let F be a collection of sets such that $\mathrm{card}(F) \le \lambda$. Then there is a collection $\{A_\tau \mid \tau < \lambda\}$ of mutually disjoint subsets of A (that is, $\tau_1 < \tau_2 < \lambda \Rightarrow A_{\tau_1} \cap A_{\tau_2} = \emptyset$) such that $A = \bigcup_{\tau<\lambda} A_\tau$, $(\forall \tau < \lambda)\ [\mathrm{card}(A_\tau) = \lambda]$, and $(\forall F \in F)\ (\forall \tau < \lambda)\ [A_\tau \subseteq F \Rightarrow \mathrm{card}(A-F) < \lambda]$.*

We remark that a direct, constructive proof of Lemma 3.2, via transfinite recursion, is easy. (To see how such a proof would go, the reader need only look at the proof of Theorem 3.1 given in [5].) We remark further that (as is obvious) we can strengthen the assertion that $(\forall F \in F)\ (\forall \tau < \lambda)\ [A_\tau \subseteq F \Rightarrow \mathrm{card}(A-F) < \lambda]$; thus,

$$(\forall F \in F)\ (\forall \tau < \lambda)\ [\mathrm{card}(A_\tau - F) < \lambda \Rightarrow \mathrm{card}(A-F) < \lambda] .$$

In what follows, however, we only need the weaker assertion.

4. Partitions which simultaneously preserve Baire character and outer measure

We shall give an easy proof, in pure set theory, of a slight generalization of a classical result due to Lusin and Sierpiński [6]; essentially, Lemma 3.2 is all that is needed beyond some manipulation of definitions.[2]

THEOREM 4.1. *Let M be a separable, complete metric space of cardinality $2^{\aleph_0}$, such that M has no isolated points, and let μ be a complete, non-atomic, countably additive, bi-regular Borel measure defined in M. Let A be a subset of M such that $\mathrm{card}(A) = 2^{\aleph_0}$, and let κ be a cardinal number such that $2 \leq \kappa \leq 2^{\aleph_0}$. Then there exists a partition $\{A_\alpha\}_{\alpha<\kappa}$ of A into κ pieces, such that the following two assertions hold:*

(a) *if A is a second-category set having the Baire property, then A_τ (for each $\tau < \kappa$) fails to have the Baire property;* in fact,

$$A_\tau \approx_{\mathcal{B}} A \ \& \ (\forall B)\ \big[\big(B \subseteq A_\tau \ \& \ B \text{ has the Baire property}\big) \Rightarrow B \text{ is of first category}\big] ;$$

(b) *if A is μ-measurable and $\mu(A) > 0$, then, for each $\tau < \kappa$, $\mu^*(A_\tau) = \mu^*(A) = \mu(A)$ and $\mu_*(A_\tau) = 0$.*

Proof. Let $\mathcal{F}$ be the class of all Borel subsets of M; since M is separable metric with $2^{\aleph_0}$ points, $\mathrm{card}(\mathcal{F}) = 2^{\aleph_0}$. Set $\lambda = 2^{\aleph_0}$ and apply Lemma 3.2; the result is a partition $\left\{A_\tau \mid \tau < 2^{\aleph_0}\right\}$ of A such that $(\forall\tau < \lambda)\ \left[\mathrm{card}(A_\tau) = 2^{\aleph_0}\right]$ and

$$(\forall F \in \mathcal{F})\ (\forall \tau < \lambda)\ \left[A_\tau \subseteq F \Rightarrow \mathrm{card}(A-F) < 2^{\aleph_0}\right] .$$

It is clear that if we set $A'_\alpha = A_\alpha$ for $\alpha < \kappa - 1$ and $A'_{\kappa-1} = \bigcup_{\kappa-1\leq\alpha<\lambda} A_\alpha$, then we obtain a new partition of A, *with* κ *cells*, which still satisfies Lemma 3.2 relative to $\mathcal{F}$. ("κ-1" is understood to denote κ if κ is infinite; if $\kappa = 2^{\aleph_0}$, we do not define $A'_{\kappa-1}$.) We now verify each of *(a)*, *(b)*, relative to this partition $\{A'_\alpha \mid \alpha < \kappa\}$.

[2] I am indebted to Steven Brock for observations which led to my awareness of the Lusin-Sierpiński Theorem.

(a) Suppose $B \subseteq A'_\alpha$, where B has the Baire property. Since B has the Baire property, $B = G \cup K$ where G is a G_δ set, K is of first category, and $G \cap K = \emptyset$. Since we are in a separable complete metric space of cardinality $2^{\aleph_0}$, either $\operatorname{card}(G) \le \aleph_0$ or $\operatorname{card}(G) = 2^{\aleph_0}$ ([1, Chapter III, §37]). It follows that if B is of second category then $\operatorname{card}(G) = 2^{\aleph_0}$. (Points are of first category in M, since M has no isolated points.) But then $M - G$ is a Borel set such that $M - G \supseteq A'_\beta$ holds for some β (any β satisfying $\beta < \kappa$ & $\beta \neq \alpha$ will serve) and $\operatorname{card}\big(A-(M-G)\big) = 2^{\aleph_0}$; a contradiction. We conclude that B is of first category. (The alert reader will notice that what we have shown, at this point, is that the sets A'_α *are Bernstein sets relative to* A; that is, if K is any uncountable Borel subset of A then A'_α intersects both K and $A - K$.) Finally, we want to verify $A'_\alpha \approx_B A$. Let C be a Baire cover for A, and let C' be a Baire cover for A'_α. Since $A'_\alpha \subseteq C$, we certainly have that $C' - C$ is of first category. Since we are assuming that A has the Baire property, $C - A$ is of first category. Suppose $A - C'$ were *not* of first category; then $A - C'$ is a second-category set having the Baire property, and hence $A - C'$ has a Borel subset of cardinality $2^{\aleph_0}$; a contradiction. Thus, $A - C'$ is of first category. But, obviously, $C - C' \subseteq (C-A) \cup (A-C')$; hence, $C - C'$ is of first category and therefore $A'_\alpha \approx_B A$.

(b) Letting J be a Borel set such that $A'_\alpha \subseteq J \subseteq M$ and $\mu^*(J) = \mu(J) = \mu^*\big(A'_\alpha\big)$, we see (since, by hypothesis, $\mu(A)$ is defined) that the set $A - J$ is μ-measurable. Lemma 3.2 gives $\operatorname{card}(A-J) < 2^{\aleph_0}$; so, since we are dealing with a non-atomic, countably additive, inner-regular Borel measure in a separable complete metric space, we have $\mu(A-J) = 0$. It now follows, via routine observations, that $\mu(A) = \mu^*(A) = \mu^*(J) = \mu^*\big(A'_\alpha\big)$. Finally, suppose $\mu_*\big(A'_\alpha\big) > 0$. Then there is a Borel set $K \subseteq A'_\alpha$ such that $\operatorname{card}(K) = 2^{\aleph_0}$. But we have already seen, in the proof of *(a)*, that no such K exists. So $\mu_*\big(A'_\alpha\big) = 0$, and

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the proof is complete.

Let us now examine our proof of Theorem 4.1 in light of the Martin's Axiom results, Corollaries 1 and 2, stated in §1. As far as *(a)* is concerned, we used the assumption that A *has the Baire property* at just one place in the proof; namely, we used it in order to argue that $C - C'$ is of first category (in our verification that $A'_\alpha \approx_B A$). Suppose we assume merely that A is of second category. Then, if we adopt Martin's Axiom and hence Corollary 1, we can proceed in the following alternative manner with regard to $C - C'$: since $A'_\alpha \subseteq C' \in F$, we have $\mathrm{card}(A - C') < 2^{\aleph_0}$; hence (since points are of first category in M), $A - C'$ is of first category. So, there exists an F_σ set K of first category such that $A - C' \subseteq K$. But then $C' \cup K$ is a Borel set having A as a subset; whence $C - (C' \cup K)$ is of first category. Hence, $(C - (C' \cup K)) \cup K$ is a first-category set having $C - C'$ as a subset; so, once again, we conclude that $C - C'$ is of first category and that $A'_\alpha \approx_B A$.

As for *(b)*, we needed the μ-measurability of A only to obtain the equation $\mu(A - J) = 0$. But that equation follows at once from Corollary 2, since $\mathrm{card}(A - J) < 2^{\aleph_0}$ and we are working with a non-atomic, complete, bi-regular Borel measure in a second-countable space.

In view of these remarks, we see that the following is true:

THEOREM 4.2. *Assume Martin's Axiom, and let* M, A, *and* κ *be as in Theorem* 4.1. *Let* μ *be a complete, non-atomic, bi-regular Borel measure defined in* M. *Then there exists a partition of* A *into* κ *pieces* A_τ, $\tau < \kappa$, *such that the following conditions are satisfied:*

(a) $(\forall \tau < \kappa)\ [A_\tau \approx_B A]$ *and*

$(\forall B)\ [(B \subseteq A_\tau\ \&\ B$ *has the Baire property*$) \Rightarrow$

B *is of first category*$]$;

and

(b) $(\forall \tau < \kappa)\ [\mu^*(A_\tau) = \mu^*(A)\ \&\ \mu_*(A_\tau) = 0]$.

5. Sets which are strongly Lusin or strongly (μ^*, μ)-Sierpiński

THEOREM 5.1. *Let T be a second-countable space, of second category on itself, with no isolated points, such that $\mathrm{card}(T) = 2^{\aleph_0}$. Let μ be a complete, non-atomic, bi-regular Borel measure defined in T, and let μ^*, μ_* be the outer and inner measures which induce μ. Assume Martin's Axiom. Then T has both a strongly Lusin subset and a strongly (μ^*, μ)-Sierpiński subset.*

Proof. First, we construct a strongly Lusin subset of T. Since T is a T_1-space (recall our standing hypothesis that all spaces are T_1), and since T is second-countable with $2^{\aleph_0}$ points, we see that the Borel subsets of T are $2^{\aleph_0}$ in number; in particular, there are exactly $2^{\aleph_0}$ F_σ sets of first category and, also, exactly $2^{\aleph_0}$ second-category sets D such that D is representable in the form $G - F$ where G is a G_δ set and F is an F_σ set. Let $\langle K_\alpha \rangle_{\alpha < 2^{\aleph_0}}$ be an enumeration of all first-category F_σ sets, and let $\langle D_\alpha \rangle_{\alpha < 2^{\aleph_0}}$ be an enumeration of all sets D of the kind just described, *such that* for each such D there are $2^{\aleph_0}$ distinct ordinals $\alpha < 2^{\aleph_0}$ for which $D = D_\alpha$. We observe that a *strongly* Lusin set L is obtained simply by constructing a Lusin set L for which we have

$$(\forall B)\ \left[(B \text{ a second-category set having the Baire property}) \Rightarrow \mathrm{card}(L \cap B) = 2^{\aleph_0}\right].$$

In order to produce such a set we proceed, very straightforwardly, as follows:

STEP 0. Let $L_0 = \emptyset$.

STEP α, $0 < \alpha < 2^{\aleph_0}$. Assume, inductively, that $\mathrm{card}\left(\bigcup_{\tau<\alpha} L_\tau\right) \le \mathrm{card}(\alpha)$. Pick an element d_α of the set $D_\alpha - \bigcup_{\tau<\alpha} (L_\tau \cup K_\tau)$; the latter set is non-empty in view of Corollary 1 and our inductive hypothesis, since D_α is of second category while (points being of first category in T) both $\bigcup_{\tau<\alpha} L_\tau$ and $\bigcup_{\tau<\alpha} K_\tau$ are of first category. Observe that the inductive hypothesis will remain true at Step $\alpha + 1$. Let $L_\alpha = \bigcup_{\tau<\alpha} L_\tau \cup \{d_\alpha\}$.

Now set $L = \bigcup_{\alpha<2^{\aleph_0}} L_\alpha$. Since a new point enters L at every step $\alpha < 2^{\aleph_0}$, we certainly have $\mathrm{card}(L) = 2^{\aleph_0}$. To see that L is Lusin, we simply note: if K is any first-category subset of T, then $K \subseteq K_\tau$ for some $\tau < \alpha$; so, by construction, we have $\mathrm{card}(K \cap L) \leq \mathrm{card}(\tau) < 2^{\aleph_0}$. Now suppose B is a second-category subset of T such that B has the Baire property. Then B is of the form $(G - K') \cup K''$ where G is a G_δ set, K' is an F_σ set of first category, and K'' is of first category. Thus $(\exists\alpha < 2^{\aleph_0})\ [D_\alpha \subseteq B]$. Hence, by the construction of L and the fact that $(\forall\alpha < 2^{\aleph_0})\ (\exists\beta < 2^{\aleph_0})\ [\alpha < \beta\ \&\ D_\alpha = D_\beta]$, we have $\mathrm{card}(L \cap B) = 2^{\aleph_0}$. Therefore L is a strongly Lusin subset of T.

To obtain a strongly (μ^*, μ)-Sierpiński set, we proceed in an analogous fashion, using Corollary 2 in place of Corollary 1: let $\langle Z_\alpha \rangle_{\alpha<2^{\aleph_0}}$ be an enumeration of the Borel sets of μ-measure 0, and let $\langle Y_\alpha \rangle_{\alpha<2^{\aleph_0}}$ be an enumeration of the Borel sets of positive μ-measure, such that for each such Borel set Y there are $2^{\aleph_0}$ distinct values of $\alpha < 2^{\aleph_0}$ for which $Y = Y_\alpha$. Again, it is only necessary to construct a (μ^*, μ)-Sierpiński set S such that $\mathrm{card}(S \cap P) = 2^{\aleph_0}$ whenever P is a subset of T for which $\mu(P)$ is defined and is greater than 0. The construction is exactly as before, with $\langle Z_\alpha \rangle_{\alpha<2^{\aleph_0}}$ in place of $\langle K_\alpha \rangle_{\alpha<2^{\aleph_0}}$, $\langle Y_\alpha \rangle_{\alpha<2^{\aleph_0}}$ in place of $\langle D_\alpha \rangle_{\alpha<2^{\aleph_0}}$, and S_α as the partial value of S after step α (rather than L_α as the partial value of L). Using Corollary 2 in place of Corollary 1, we then readily verify that

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$S = \bigcup_{\alpha<2^{\aleph_0}} S_\alpha$ is strongly (μ^*, μ)-Sierpiński.

REMARK 5.2. Let the space T be the usual n-dimensional euclidean space R^n, and let μ, μ^* be, respectively, the usual (that is, *Lebesgue*) measure and outer measure in R^n. As is pointed out in [3, Chapter 20], the existence of *both* a classical Lusin set and a classical Sierpiński set follows from the Duality Principle ([3, Chapter 19]) *plus* the existence of *either one* of them. Since (as is quite easy to show) Martin's Axiom implies the Duality Principle, we have (for instance) that Martin's Axiom plus a Lusin set (in our present sense) yields a (μ^*, μ)-Sierpiński set (in our present sense). When we consider *strongly* Lusin and *strongly* (μ^*, μ)-Sierpiński sets, however, it is no longer the case that the existence of one of them yields the existence of the other via measure-category duality; for, the kind of duality required for such a transition is (as noted in [3, Chapter 21]) known to be false.

References

[1] K. Kuratowski, *Topology*. Vol. I (new edition, revised and augmented. Translated by J. Jaworowski. Academic Press, New York and London; PWN - Polish Scientific Publishers, Warszawa, 1966).

[2] D.A. Martin and R.M. Solovay, "Internal Cohen extensions", *Ann. Math. Logic* 2 (1970), 143-178.

[3] John C. Oxtoby, *Measure and category. A survey of the analogies between topological and measure spaces* (Graduate Texts in Mathematics, 2. Springer-Verlag, Berlin, Heidelberg, New York, 1971).

[4] Wacław Sierpiński, *Hypothèse du continu*, 2nd ed. (Chelsea, New York, 1956).

[5] Wacław Sierpiński, *Cardinal and ordinal numbers* (Państwowe Wydawnictwo Naukowe, Warsawa, 1958).

[6] W. Sierpiński et N. Lusin, "Sur une décomposition d'un intervalle en une infinité non dénombrable d'ensembles non mesurables", *C.R. Acad. Sci. Paris* 165 (1917), 422-424.

https://doi.org/10.1017/S0004972700023996 Published online by Cambridge University Press

[7] R.M. Solovay and S. Tennenbaum, "Iterated Cohen extensions and Souslin's problem", *Ann. of Math.* (2) 94 (1971), 201-245.

[8] Franklin D. Tall, "An alternative to the continuum hypothesis and its uses in general topology", submitted.

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