Adv. Appl. Prob. **25**, 472–476 (1993)
Printed in N. Ireland

LETTERS TO THE EDITOR

FLUCTUATION THEORY FOR THE EHRENFEST URN VIA ELECTRIC NETWORKS

JOSÉ LUIS PALACIOS,* *New Jersey Institute of Technology*

Abstract

Using the electric network approach, we give very simple derivations for the expected first passage from the origin to the opposite vertex in the d-cube (i.e. the Ehrenfest urn model) and the Platonic graphs.

FIRST PASSAGE; EFFECTIVE RESISTANCE

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY: 60C05
SECONDARY: 60J15

1. Introduction

Based on the standard theory of skip-free random walks, Bingham (1991) gave a thorough account of that aspect of the fluctuation theory having to do with the first passage from the origin to the opposite vertex in the d-cube (i.e. the Ehrenfest urn) and the Platonic graphs. Two other earlier treatments of the material in Bingham's paper are due to Baróti (1988) and Matthews (1989), who give asymptotics of the generating functions—thus providing limit distributions—of the first passages. Here we give very simple derivations for the expected values of those first passages (or hitting times) based on the electric approach to random walks on graphs. Similar examples of hitting-time calculations using this approach were given by Palacios (1992).

The Ehrenfest model consists of d balls distributed between two urns, I and II; at each time $n = 0, 1, \cdots$ a ball is chosen—each with probability $1/d$—and changed to the other urn. A classical textbook account of this model is given by Kac (1959). If we use a 'full description' with 2^d states representing the possible configurations by d-tuples $i = (i_1, \cdots, i_d)$, where $i_k = 1, 0$ if ball k is in urn, I, II, one may identify the states with the vertices of the d-cube, and the evolution of the Ehrenfest model is that of a simple random walk on the cube.

In general, a simple walk on a finite connected undirected graph, $G = (V, E)$, is the Markov chain X_n, $n \geqq 0$, that from its current vertex v jumps to one of the $d(v)$ neighboring vertices with uniform probability. The hitting time (or first passage) T_v of the vertex v is the minimum number of steps the random walk takes to reach that vertex: $T_v = \inf\{n \geqq 0 : X_n = v\}$. The expected value of T_v when the walk starts at the vertex w is denoted by $E_w T_v$. The commute time between vertices i and j is $E_i T_j + E_j T_i$.

2. The electric approach

There is a strong connection between random walks on graphs and electric networks, spelled out beautifully in the monograph of Doyle and Snell (1984). We will use the following fact of this electric analog involving the commute time between a and b and the effective resistance R_{ab} between those two vertices when every edge of the graph is considered to be a

Received 21 July 1992; revision received 13 October 1992.

* Postal address: Department of Mathematics, New Jersey Institute of Technology, Newark, NJ 07102, USA.

unit resistor:

$$E_aT_b + E_bT_a = 2\,|E|\,R_{ab}. \tag{1}$$

If we can ensure further that $E_aT_b = E_bT_a$, for instance under some symmetry assumption, then (1) simplifies to

$$E_aT_b = |E|\,R_{ab}. \tag{2}$$

Formulas (1) and (2) enable one to compute the commute time—and hitting time in the presence of symmetry—not only of a single vertex but also of *a set* of vertices because one can short together those vertices (they have 'zero voltage', see Doyle (1984), p. 53) into a single vertex. Moreover one can short together all vertices sharing the same potential, a fact that simplifies things considerably. Formula (1) can be proved in a number of ways (see Chandra et al. (1989)) of which we include here the following, for the sake of completeness.

Proof of (1). It is a well-known fact (Proposition 9-58 in Kemeny et al. (1966)) that the commute time between two states a and b of an irreducible recurrent Markov chain can be expressed as

$$E_aT_b + E_bT_a = \frac{1}{\pi_a}\,E_aN_a^b, \tag{3}$$

where π_a is the value of the stationary distribution at a and N_a^b is the number of visits to a prior to time T_b.

It is easy to verify that for random walks on graphs the stationary distribution is given by

$$\pi_a = \frac{d(a)}{2\,|E|}. \tag{4}$$

Also, on p. 50 in Doyle and Snell (1984) it is proved that: (i) $E_aN_x^b/d(a)$ equals the voltage at x when we put a battery from a to b that establishes a voltage $E_aN_a^b/d(a)$ at a and a voltage 0 at b, and (ii) the total current flowing into the network at a under these conditions is 1. Since the effective resistance R_{ab} is the quotient of the voltage at a over the total current at a, we can conclude that

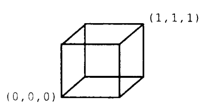
$$\frac{E_aN_a^b}{d(a)} = R_{ab}. \tag{5}$$

Now (3), (4) and (5) together imply (1).

In what follows, we use the above ideas and the fact that the effective resistance of a set of resistors is: (i) the sum of the individual resistances in case the resistors are in series and (ii) the inverse of the sum of the inverse individual resistances in case the resistors are in parallel.

3. The d-cube

If we start at any vertex that we relabel as the 'origin' $(0, \cdots, 0)$, the vertex it takes longest to hit is $(1, \cdots, 1)$. Now if we apply a unit voltage between these two vertices so that the voltage at $(0, \cdots, 0)$ is 1 and the voltage at $(1, \cdots, 1)$ is 0, then all vertices having the same number of 1's share the same voltage and can be shorted. Figure 1 shows the effect of doing this on the 3-cube.



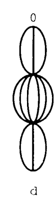


Figure 1.

https://doi.org/10.2307/1427663 Published online by Cambridge University Press

In general what we obtain is a new graph with $d+1$ vertices, where the kth new vertex consists of the shorting of all vertices in the unit cube with k 1's. Since every vertex in the unit cube with k 1's is connected to $d-k$ vertices with $k-1$ 1's, there are

$$(d-k)(d/k) = d\binom{d-1}{k}$$

resistors between vertex k and $k+1$ in the new graph, $0 \leqq k \leqq d-1$. Then (2) yields

$$\begin{aligned} E_{(0,\cdots,0)}T_{(1,\cdots,1)} = E_0 T_d = |E|\, R_{0d} &= d2^{d-1} \sum_{k=0}^{d-1} \frac{1}{d\binom{d-1}{k}} \\ &= 2^{d-1} \sum_{k=0}^{d-1} \frac{1}{\binom{d-1}{k}}. \end{aligned} \tag{6}$$

Bingham (1991) found that

$$E_0 T_d = d \sum_{1 \leqq j \leqq d,\, j \text{ odd}} \frac{\binom{d}{j}}{j}. \tag{7}$$

It is not entirely obvious at first glance that these two combinatorial expressions for $E_0 T_d$ are indeed equal! Perhaps in our derivation it is easier to see that $E_0 T_d = O(2^d)$, the order of the graph. In fact, whenever $2^k \leqq n < 2^{k+1}$ and $1 \leqq j \leqq \lfloor \frac{1}{2}(n-1) \rfloor$ one has that

$$\frac{1}{\binom{n}{j}} = \frac{1}{n} \frac{j}{n-1} \cdots \frac{2}{n-j+1} \leqq \frac{1}{2^k} \frac{1}{2} \cdots \frac{1}{2} = \frac{1}{2^{k+j-1}},$$

so that

$$\sum_{j=1}^{\lfloor \frac{1}{2}(n-1) \rfloor} \frac{1}{\binom{n}{j}} \leqq \sum_{j=1}^{\lfloor \frac{1}{2}(n-1) \rfloor} \frac{1}{2^{k+j-1}},$$

whenever $2^k \leqq n < 2^{k+1}$ and $1 \leqq j \leqq \lfloor \frac{1}{2}(n-1) \rfloor$. This implies that if we exclude the first and last terms (both equal to 1) in the summation in formula (6), then half the sum of all the remaining terms—excluding the central term—is bounded by the tail of a convergent series and thus, converges to 0 as n goes to ∞, so that $E_0 T_d = 2^{d-1}(2 + o(1))$.

Comments. Besides (6) and (7), there are other closed-form formulas for the hitting time $E_0 T_d$ in the d-cube. For instance, Karlin and McGregor (1965) study a slightly different version of the Ehrenfest urn presented here. In this model (we shall call it model B), every time we pick a ball at random we also toss a coin, and if it lands heads (with probability p) then we place the ball in urn I; otherwise we place the ball in urn II. The discrete model is embedded in a continuous-time model by picking the balls at exponential times, and for this continuous model it is proved that the Laplace transform of the (continuous) hitting time $\tilde{T}_d$ is given by

$$E_0(\exp(-s\tilde{T}_d)) = \frac{1}{K_d(-s)}, \tag{8}$$

where K_d is the Krawtchouk polynomial

$$K_d(x) = \sum_{\nu=0}^{d} (-1)^\nu \binom{x}{\nu} \frac{1}{p^\nu}. \tag{9}$$

If we choose $p = \frac{1}{2}$, model B describes a random walk on the d-cube such that at each vertex the walk either stays at that same vertex with probability $\frac{1}{2}$, or moves to one of the adjacent vertices with probability $1/2d$. With this choice of p, (9) becomes

$$K_d(x) = 1 + \sum_{\nu=1}^{d} (-1)^\nu \binom{x}{\nu} 2^\nu,$$

and

$$E_0 \tilde{T}_d = -K_d'(0) = \sum_{\nu=1}^{d} (-1)^{\nu+1} 2^\nu \left[\binom{x}{\nu}\right]'_{x=0} = \sum_{\nu=1}^{d} \frac{2^\nu}{\nu}. \tag{10}$$

In (10), the first equality uses the fact that $K_d(0) = 1$. The last equality follows because

$$\left[\binom{x}{\nu}\right]'_{x=0} = \frac{1}{\nu!}\,\text{coeff. of } x = (-1)^{n-1} \frac{(\nu-1)!}{\nu!}.$$

Now the expected hitting time for the discrete B model is equal to the above $E_0 \tilde{T}_d$ multiplied by a factor of d, to account for the exponential holding time at each state. Finally, we argue that in the discrete model B, the walk behaves as in our model except that in every vertex there is a geometric holding time with mean 2, and thus the expected (discrete) hitting time in model B is twice that of our model, so that we finally get the alternative version of (6) and (7):

$$E_0 T_d = \frac{d}{2} \sum_{\nu=1}^{d} \frac{2^\nu}{\nu}.$$

Other expressions for the Krawtchouk polynomials (see Karlin and McGregor, p. 356, and references therein) will yield other closed-form formulas for $E_0 T_d$.

A final comment: formula (8) and (9) allow us to write

$$E_0\left(\exp\left(-\frac{s\tilde{T}_d}{2^d}\right)\right) = (1 + s + o(2^d))^{-1},$$

justifying an exponential limit distribution for the normalized hitting times, in an alternative way to the one proposed by Bingham (1991).

4. The Platonic graphs

Proceeding similarly to the case of the d-cube, the effective resistance of the Platonic graphs can be easily found once the vertices with the same potential have been shorted. Figure 2 shows the effect of doing that. In all cases 0 represents the initial vertex and d the opposite vertex.

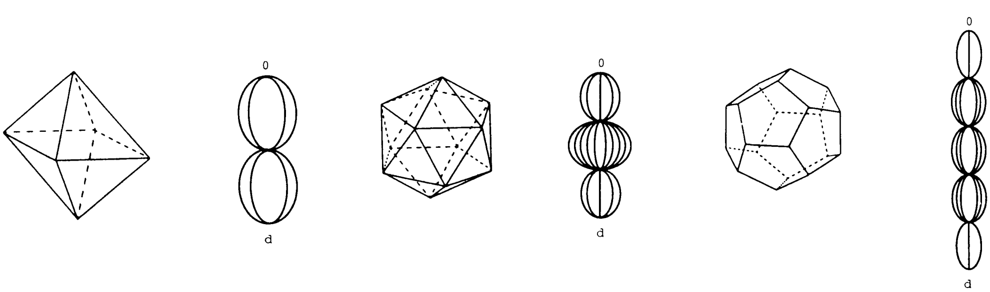


Figure 2.

It is immediate from the figure that the effective resistance R_{0d} equals

$\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$ for the octahedron

$\frac{1}{5}+\frac{1}{10}+\frac{1}{5}=\frac{1}{2}$ for the icosahedron

$\frac{1}{3}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{3}=\frac{7}{6}$ for the dodecahedron.

And therefore, by (2), the expected first passage is:

$12\times\frac{1}{2}=6$ for the octahedron

$30\times\frac{1}{2}=15$ for the icosahedron

$30\times\frac{7}{6}=35$ for the dodecahedron.

Bingham (1991) reports an incorrect figure of 33 for the dodecahedron, although the generating function from which is it derived is correct. As he points out, the tetrahedron is degenerate for this problem (no 'opposite vertex'); let us remark for completeness that for any two vertices a and b in the tetrahedron the effective resistance if $\frac{3}{5}$ and therefore $E_aT_b=3$.

References

BARÓTI, G. (1988) First passage problems for Ehrenfest model. In *Prob. Theo. Math. Stat. Appl.: Proc. 5th Pannonian Symp. Math. Stat.*, Visegrád, Hungary 1985, pp. 3–6, Reidel, Boston, MA.

BINGHAM, N. H. (1991) Fluctuation theory for the Ehrenfest urn. *Adv. Appl. Prob.* **23,** 598–611.

CHANDRA, A. K., RAGHAVAN, P., RUZZO, W. L., SMOLENSKY, R. AND TIWARI, P. (1989) The electrical resistance of a graph captures its commute and cover times. In *Proc.* 21*st Annual ACM Symp. on Theory of Computing*, Seattle, Washington, pp. 574–586.

DOYLE, P. G. AND SNELL, J. L. (1984) *Random Walks and Electrical Networks.* Mathematical Association of America, Washington, DC.

KAC, M. (1959) *Probability and Related Topics in Physical Sciences.* Interscience, London.

KARLIN, S. AND MCGREGOR, J. (1965) Ehrenfest urn models. *J. Appl. Prob.* **2,** 352–376.

KEMENY, J. G., SNELL, J. L. AND KNAPP, A. W. (1966) *Denumerable Markov Chains.* Van Nostrand, Princeton, NJ.

MATTHEWS, P. (1989) Some sample-path properties of a random walk on the cube. *J. Theoret. Prob.* **2,** 129–157.

PALACIOS, J. L. (1992) Expected hitting and cover times of random walks on some special graphs. *Proc. 'Random Graphs '91'*, Poznań, Poland.