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ON SOME INTEGRAL INEQUALITIES RELATED TO HARDY'S

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ABSTRACT. We obtain mainly by using Jensen's inequality for convex functions an integral inequality, which contains as a special case Shum's generalization of Hardy's inequality.

1. Introduction. The following inequality, which is of wide application, is due to Hardy ([3], Theorem 319):

$$(1.1)\quad \int_0^\infty dy\left[\int_0^\infty K(x,y)f(x)\,dx\right]^p \le \left[\int_0^\infty K(x,1)x^{-1/p}\,dx\right]^p \int_0^\infty f(x)^p\,dx, \quad p \ge 1,$$

where $K(x, y)$ is non-negative and homogeneous of degree -1. The inequality is reversed if $0<p\le 1$.

Two special cases of inequality (1.1) are (i) the Hilbert integral inequality ([3], Theorem 316) and (ii) the two useful inequalities due to Hardy ([3], Theorem 330):

$$(1.2)\qquad \int_0^\infty y^{-r}\left[\int_0^y f(x)\,dx\right]^p dy \le \left[\frac{p}{r-1}\right]^p \int_0^\infty x^{-r}[xf(x)]^p\,dx \quad (r>1)$$

and

$$(1.3)\qquad \int_0^\infty y^{-r}\left[\int_y^\infty f(x)\,dx\right]^p dy \le \left[\frac{p}{1-r}\right]^p \int_0^\infty x^{-r}[xf(x)]^p\,dx \quad (r<1)$$

provided $p\ge 1$.

The inequalities are reversed if $0<p\le 1$.

Various generalizations and applications of inequalities (1.2) and (1.3) and their series analogues in different areas of mathematics have appeared in the literature during the past decade (see for example [1], [3], [4] and [5]).

A recent trend in inequalities is to establish, mainly by the Jensen inequality and its generalization due to Steffensen, some very general inequalities that include as special cases, many that are of independent interest and that were originally proved by quite different methods. The object of this note, therefore, is to obtain some generalizations of inequalities (1.2) and (1.3) from the Jensen inequality for convex functions. Indeed, our main result is a generalization of Shum's result [6] obtained by replacing x by $g(x)$.

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The result is as follows:

THEOREM. *Let g be continuous and non-decreasing on* $[0, \infty]$ *with* $g(0)=0$, $g(x)>0$ *for* $x>0$ *and* $g(\infty)=\infty$. *Let* $p\geq 1$, $r\neq 1$ *and* $f(x)$ *be non-negative and Lebesgue-Stieltjes integrable with respect to* $g(x)$ *on* $[0, b]$ *or on* $[a, \infty]$ *according as* $r>1$ *or* $r<1$ *where* $a>0$ *and* $b<0$. *Suppose*

$$F(x)=\begin{cases}\int_0^x f(t)\,dg(t) & (r>1)\\ \int_x^\infty f(t)\,dg(t) & (r<1).\end{cases}$$

Then

$$\text{(1.4)}\quad \int_0^b g(x)^{-r}F^p(x)\,dg(x)+\frac{p}{r-1}\,g(b)^{1-r}F^p(b) \leq \left[\frac{p}{r-1}\right]^p\int_0^b g(x)^{-r}[g(x)f(x)]^p\,dg(x)\quad (r>1)$$

and

$$\text{(1.5)}\quad \int_a^\infty g(x)^{-r}F^p(x)\,dg(x)+\frac{p}{1-r}\,g(a)^{1-r}F^p(a) \leq \left[\frac{p}{1-r}\right]^p\int_a^\infty g(x)^{-r}[g(x)f(x)]^p\,dg(x)\quad (r<1),$$

with both inequalities reversed if $0<p\leq 1$.

Equality holds in either inequality when either $p=1$ *or* $f\equiv 0$. *The constant* $[p/(r-1)]^p$ *or* $[p/(1-r)]^p$ *is the best possible when the left side of* (1.4) *or* (1.5) *is unchanged.*

We note, however, that the left sides of inequalities (1.4) and (1.5) exist when the right sides do.

2. Preliminary Lemmas. We shall make use of the following:

LEMMA 2.1. *Let g be continuous and non-decreasing on* $[a, b]$, *where* $-\infty\leq a<b\leq\infty$. *Let φ be continuous and convex and let* $h(x, t)$ *be non-negative,* $x\geq 0$, $t\geq 0$ *and λ be non-decreasing. Let* $-\infty\leq\xi(x)\leq\eta(x)\leq\infty$, *and suppose φ has a continuous inverse (which is necessarily concave). Then*

$$\text{(2.1)}\quad \int_a^b g(x)^{-1}\left[\int_{\xi(x)}^{\eta(x)} h(x,t)\,d\lambda(t)\right]dg(x)\geq\int_a^b g(x)^{-1}\int_{\xi(x)}^{\eta(x)} d\lambda(t) \times\varphi\left\{\frac{\int_{\xi(x)}^{\eta(x)}\varphi^{-1}[h(x,t)]\,d\lambda(t)}{\int_{\xi(x)}^{\eta(x)} d\lambda(t)}\right\}dg(x)$$

with the inequality reversed if φ is concave.

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In particular, if $p\geq 1$,

$$\int_0^b g(x)^{-1}\left[\int_0^x h(x,t)\,d\lambda(t)\right]dg(x)\geq \int_0^b g(x)^{-1}\left[\int_0^x d\lambda(t)\right]^{1-p} \times\left[\int_0^x h(x,t)^{1/p}\,d\lambda(t)\right]^p dg(x) \tag{2.2}$$

and

$$\int_a^\infty g(x)^{-1}\left[\int_x^\infty h(x,t)\,d\lambda(t)\right]dg(x)\geq \int_a^\infty g(x)^{-1}\left[\int_x^\infty d\lambda(t)\right]^{1-p} \times\left[\int_a^\infty h(x,t)^{1/p}\,d\lambda(t)\right]^p dg(x) \tag{2.3}$$

with the inequalities reversed if $0<p\leq 1$.

Proof. Let φ be convex. Then Jensen's inequality says

$$\varphi^{-1}\left[\frac{\int_{\xi(x)}^{\eta(x)} h(x,t)\,d\lambda(t)}{\int_{\xi(x)}^{\eta(x)} d\lambda(t)}\right]\geq \frac{\int_{\xi(x)}^{\eta(x)} \varphi^{-1}[h(x,t)]\,d\lambda(t)}{\int_{\xi(x)}^{\eta(x)} d\lambda(t)}. \tag{2.4}$$

Consequently,

$$\int_{\xi(x)}^{\eta(x)} h(x,t)\,d\lambda(t)\geq \int_{\xi(x)}^{\eta(x)} d\lambda(t)\varphi\left\{\frac{\int_{\xi(x)}^{\eta(x)} \varphi^{-1}[h(x,t)]\,d\lambda(t)}{\int_{\zeta(x)}^{\eta(x)} d\lambda(t)}\right\}.$$

Inequality (2.1) follows by integrating the above inequality with respect to the measure $g(x)^{-1}\,dg(x)$ on the set $[a,b]$.

LEMMA 2.2. *Let g be as in the theorem and let $\delta=(1-r)/p$ where $r\neq 1$ and $p\geq 1$. Suppose*

$$\theta(x)=\begin{cases}\int_0^x g(t)^{-(1-p)(1+\delta)}f(t)^p\,dg(t) & (r>1)\\ \int_x^\infty g(t)^{-(1-p)(1+\delta)}f(t)^p\,dg(t) & (r<1),\end{cases}$$

where f is non-negative and Lebesgue-Stieltjes integrable with respect to g on $[0,b]$ or on $[a,\infty]$ according as $r>1$ or $r<1$, $a>0$ and $b>0$. Then

$$g(b)^\delta\theta(b)\geq(-\delta^{-1})^{1-p}g(b)^{\delta p}F^p(b)\qquad (r>1) \tag{2.5}$$

and

$$g(a)^\delta\theta(a)\geq\delta^{p-1}g(a)^{\delta p}F^p(a)\qquad (r<1), \tag{2.6}$$

with the inequalities reversed if $0<p\leq 1$.

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Proof. First, we show that the integrals defining θ exist under the hypothesis that the integrals on the right sides of inequalities (1.4) and (1.5) exist.

Since

$$g(t)^{-(1-p)(1+\delta)}f(t)^p = g(t)^{(r-1)/p}[g(t)^{p-r}f(t)^p],$$

we have, on using the non-decreasing property of g, when $r>1$,

$$\text{(2.7)}\quad 0<\theta(x)=\int_0^x g(t)^{(r-1)/p}[g^{p-r}(t)f^p(t)]\,dg(t)\le g(x)^{(r-1)/p}\int_0^x g^{p-r}(t)f^p(t)\,dg(t).$$

Similarly, if $r<1$, then

$$\text{(2.8)}\quad 0<\theta(x)=\int_x^\infty g(t)^{(r-1)/p}[g^{p-r}(t)f^p(t)]\,dg(t)\le g(x)^{(r-1)/p}\int_x^\infty g^{p-r}(t)^p\,dg(t).$$

Hence, the existence of the integrals on the right sides of inequalities (1.4) and (1.5) implies the existence of the integrals defining $\theta(x)$.

Now let $\varphi(u)=u^p$, $p\ge 1$,

$$h(x,t)=g(x)^\delta g(t)^{p(1+\delta)}f(t)^p$$

and

$$d\lambda(t)=g(t)^{-(1+\delta)}\,dg(t)$$

in the Jensen inequality (2.4). Then, if $r>1$,

$$\begin{aligned} g(b)^\delta\theta(b) &= g(b)^\delta\int_0^b g(t)^{-(1-p)(1+\delta)}f(t)^p\,dg(t)\\ &= \int_0^b h(b,t)\,d\lambda(t)\\ &\ge \left[\int_0^b d\lambda(t)\right]^{1-p}\left[\int_0^b f(t)\,dg(t)\right]^p g(b)^\delta\\ &= (-\delta^{-1})^{1-p}g(b)^{-\delta+\delta p}g(b)^\delta F(b)^p.\end{aligned}$$

Consequently,

$$g(b)^\delta\theta(b)\ge(-\delta^{-1})^{1-p}g(b)^{\delta p}F(b)^p.$$

A similar argument shows that

$$g(a)^\delta\theta(a)\ge\delta^{p-1}g(a)^{\delta p}F(a)^p.$$

This proves inequalities (2.5) and (2.6) when $p\ge 1$; when $0<p\le 1$, Jensen's inequality is reversed; hence so are inequalities (2.5) and (2.6). The lemma is proved.

We remark on passing that inequalities (2.5) and (2.6) are strict unless $p=1$ or $h(x,t)$ is independent of t, and the latter is the case when $f\equiv 0$.

LEMMA 2.3. *With notation as in Lemma 2.2, we have for* $p \geq 1$,

$$\int_0^b g(x)^{\delta-1}\theta(x)\,dg(x) \geq (-\delta^{-1})^{1-p}\int_0^b g(x)^{\delta p-1}F(x)^p\,dg(x) \quad (r>1) \tag{2.9}$$

and

$$\int_a^\infty g(x)^{\delta-1}\theta(x)\,dg(x) \geq \delta^{p-1}\int_a^\infty g(x)^{\delta p-1}F(x)^p\,dg(x) \quad (r<1) \tag{2.10}$$

with the inequalities reversed if $0<p\leq 1$.

Proof. With the definitions of the functions θ, h and λ in the proof of Lemma 2.2, we obtain from inequalities (2.2) and (2.3) the assertions of the lemma.

This completes the proof of the lemma.

3. Proof of Theorem. From the inequalities (2.7) and (2.8), we have

$$\lim_{x\to 0} g(x)^\delta\theta(x)=0, \quad (r>1),$$

and

$$\lim_{x\to\infty} g(x)^\delta\theta(x)=0, \quad (r<1).$$

Consequently, we obtain on using integration by parts,

$$\begin{aligned}\int_0^b g(x)^{\delta-1}\theta(x)\,dg(x) \\ = \delta^{-1}g(b)^\delta\theta(b)-\delta^{-1}\int_0^b g(x)^{\delta p-1}[g(x)f(x)]^p\,dg(x) \quad (r>1)\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\int_a^\infty g(x)^{\delta-1}\theta(x)\,dg(x) \\ =(-\delta^{-1})g(a)^\delta\theta(a)+\delta^{-1}\int_a^\infty g(x)^{\delta p-1}[g(x)f(x)]^p\,dg(x) \quad (r<1).\end{aligned} \tag{3.2}$$

Combining inequalities (2.5), (2.9) and (3.1) on one hand and inequalities (2.6), (2.10) and (3.2) on the other hand gives the desired assertion of the theorem.

Since the conditions for equality in inequalities (1.4) and (1.5) follow from those for inequalities (2.5) and (2.6), it follows from our earlier observation that equality holds in inequalities (1.4 and (1.5) if either $p=1$ *or* $f\equiv 0$.

Suppose $K(p, r, b)$ is the best possible constant in inequality (1.4). It is easily shown by taking $f(x)=g(x)^{-1+\varepsilon+(r-1)/p}$ where $\varepsilon>0$, in inequality (1.4), and then letting $\varepsilon\to 0^+$, that $K(p, r, b)\geq[p/(r-1)]^p$. For inequality (1.5), the corresponding result follows by taking $f(x)=g(x)^{-1-\varepsilon(r-1)/p}$.

Consequently, the constants $[p/(r-1)]^p$ and $[p/(1-r)]^p$ are the best possible in inequalities (1.4) and (1.5) respectively when the left sides of these inequalities are unchanged. This completes the proof of the theorem.

REMARK 1. Taking $g(x)=x$ in the theorem produces Shum's results, namely

$$\int_0^b x^{-r}F(x)^p\,dx+\frac{p}{r-1}\,b^{1-r}F(b)^p\le\left(\frac{p}{r-1}\right)^p\int_0^b x^{-r}[xf(x)]^p\,dx\quad(r>1). \tag{3.3}$$

and

$$\int_a^\infty x^{-r}F(x)^p\,dx+\frac{p}{1-r}\,a^{1-r}F(a)^p\le\left(\frac{p}{1-r}\right)^p\int_a^\infty x^{-r}[xf(x)]^p\,dx,\quad(r<1), \tag{3.4}$$

where

$$F(x)=\begin{cases}\int_0^x f(t)\,dt & (r>1)\\ \int_x^\infty f(t)\,dt & (r<1).\end{cases}$$

REMARK 2. It is readily seen that if g is absolutely continuous, then we obtain from the theorem inequalities involving integrals of functions and their derivatives.

REFERENCES

1. Boas, R. P. Jr.; *Integrability Theorems for Trigonometric Transforms.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. **38,** (1967).

2. Boas, R. P. Jr.; *Some Integral Inequalities Related to Hardy's Inequality.* Journal d'Analyse Math. **23** (1970), 53–63.

3. Hardy, G. H., Littlewood, J. E. and Pólya, G: *Inequalities.* Second edition, Cambridge (1952).

4. Izumi, M, Izumi, S. and Peterson, G: *On Hardy's inequality and its Generalizations.* Tohoku Math. J. **21** (1969) 601–613.

5. Németh, J: *Generalizations of Hardy–Littlewood Inequality* 11. Acta Sci. Math. **32** (1971), 295–299.

6. Shum, D. T: *On Integral Inequalities Related to Hardy's.* Can. Math. Bull. **14** (2) (1971), 225–230.

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