

## A RECURRENCE FORMULA FOR $q$ -BERNOULLI NUMBERS ATTACHED TO FORMAL GROUP

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**Abstract.** Kaneko [2] proved a new recurrence formula for the Bernoulli numbers and gave two proofs. One of them was due to Don Zagier. We shall apply Zagier's idea to the  $q$ -Bernoulli numbers attached to formal group.

### 1. Generalization of Kaneko's recurrence formula

Let  $\mathfrak{B} = \mathfrak{B}(X)$  be the generating function of the Bernoulli numbers, i.e.,

$$\mathfrak{B} = \frac{X}{e^X - 1},$$

then it is anti-invariant under a map:  $\mathfrak{B} \mapsto \mathfrak{B}e^X$ , i.e.,

$$\mathfrak{B}(X)e^X = \mathfrak{B}(-X).$$

Zagier gave another proof of Kaneko's recurrence formula for the Bernoulli numbers by using this property [2]. On the other hand because of  $\mathfrak{B}(-X) = \mathfrak{B}(X) + X$ , we can see that  $\mathfrak{B}$  is transformed to the sum of a polynomial and itself under the above map. We use the second property in order to generalize Kaneko's recurrence formula and prove a formula for the  $q$ -Bernoulli numbers attached to formal group.

First we suppose a power series  $B$  in  $X$  which satisfies the following:

ASSUMPTION 1.

$$Be^X = B + C,$$

where  $C$  is a polynomial.

If  $C = X$ , then  $B$  is equal to  $\mathfrak{B}$ , and if  $C = X^2$ , then  $B$  is essentially equal to the generating function of  $\tilde{B}_n$  which was defined in [2], i.e.,

$$\sum_{n \geq 0} \tilde{B}_n \frac{X^n}{n!} = \left( \frac{X^2}{e^X - 1} \right)'.$$

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The starting point of our argument is the following trivial lemma:

LEMMA 1. *For any power series  $A$  and non-negative integer  $n$ , we have*

$$A^{(n)}e^X = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} (Ae^X)^{(i)},$$

where  $A^{(n)}$  means the  $n$ -th derivative of  $A$ .

*Proof.* Because of  $A = (Ae^X)e^{-X}$ , we can get what we want. □

Set  $A = B$  and compare the coefficient of  $\frac{X^m}{m!}$  for any non-negative integer  $m$ , then we have a generalization of Kaneko’s recurrence formula as follows:

PROPOSITION 1. *If  $B$  satisfies Assumption 1, then we have*

$$\sum_{i=0}^m \binom{m}{i} b_{n+i} = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} (b_{m+i} + c_{m+i}),$$

where  $B = \sum_{n \geq 0} b_n \frac{X^n}{n!}$  and  $C = \sum_{n \geq 0} c_n \frac{X^n}{n!}$ .

If  $m > \deg C$ , then we have

COROLLARY 1.

$$\sum_{i=0}^m \binom{m}{i} b_{n+i} = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} b_{m+i}.$$

Furthermore if  $m = n$ , then we have

COROLLARY 2.

$$\sum_{\substack{i=0 \\ i \neq n \pmod 2}}^n \binom{n}{i} b_{n+i} = 0 .$$

If  $C = X$ , then there is no new information about the Bernoulli numbers. But if  $C = X^2$ , then this is equivalent to Kaneko’s recurrence formula.

### 2. $q$ -recurrence formula

In this section we shall extend results in the previous section for the  $q$ -Bernoulli numbers attached to formal group. Let  $q$  be an indeterminate and let  $\mathfrak{o}$  be the formal power series ring in  $q - 1$  over some  $\mathbb{Q}$ -algebra. Furthermore let  $\mathfrak{F}(X, Y)$  be a 1-dimensional commutative formal group defined over  $\mathfrak{o}$  and let  $f(X)$  be an isomorphism from the additive formal group  $X + Y$  to  $\mathfrak{F}(X, Y)$ . We note that there exists a unique isomorphism  $f_{\mathfrak{F}}(X)$  from  $X + Y$  to  $\mathfrak{F}(X, Y)$  defined over  $\mathfrak{o}$  such that  $f'_{\mathfrak{F}}(0) = 1$ . And  $f(X)$  is equal to  $f_{\mathfrak{F}}(cX)$  for some invertible element  $c \in \mathfrak{o}^\times$ . Conversely for any  $c \in \mathfrak{o}^\times$ ,  $f_{\mathfrak{F}}(cX)$  is an isomorphism from  $X + Y$  to  $\mathfrak{F}(X, Y)$ . Throughout this paper we assume that

ASSUMPTION 2.

$$\text{ord}_{q-1} f_{\mathfrak{F}}^{(n)}(0) \geq n - 1 \quad \text{for all } n \geq 1.$$

We note that by this assumption  $\mathfrak{F}_n(A, B)$  (see Definition 1 below) and  $f(a)$  are convergent in  $\mathfrak{o}$  for any  $A, B \in \mathfrak{o}[[X]]$  and  $a \in \mathfrak{o}$  as formal power series (see [6, Remark 3]).

DEFINITION 1. For each non-negative integer  $n$ , we denote the expansion of  $\mathfrak{F}(X, Y)^n$  by

$$\mathfrak{F}(X, Y)^n = \sum_{i, j \geq 0} \binom{n}{i, j}_{\mathfrak{F}} X^i Y^j,$$

and we set

$$\mathfrak{F}_n(A, B) = \sum_{i, j \geq 0} \binom{n}{i, j}_{\mathfrak{F}} a_i b_j$$

for any power series  $A = \sum_{n \geq 0} a_n \frac{X^n}{n!}$  and  $B = \sum_{n \geq 0} b_n \frac{X^n}{n!}$  in  $\mathfrak{o}[[X]]$ . Then we define the  $*_{\mathfrak{F}}$ -product by

$$A *_{\mathfrak{F}} B = \sum_{n \geq 0} \mathfrak{F}_n(A, B) \frac{X^n}{n!}.$$

We can prove  $(\mathfrak{o}[[X]], +, *_{\mathfrak{F}})$  is an  $\mathfrak{o}$ -algebra (see [6, Proposition 1]). Next we extend the following map:

$$X^n \mapsto c^n \underbrace{X *_{\mathfrak{F}} \cdots *_{\mathfrak{F}} X}_{n \text{ times}}$$

$\mathfrak{o}$ -linearly. Hence we can get a natural homomorphism from  $(\mathfrak{o}[[X]], +, \cdot)$  to  $(\mathfrak{o}[[X]], +, *_{\mathfrak{F}})$ . We call this map  $q$ -operator and denote the image of  $A \in \mathfrak{o}[[X]]$  under the  $q$ -operator by  $A_{\mathfrak{F},c}$ . We define a  $q$ -analogue of power series  $A$  attached to  $\mathfrak{F}$  and  $c$  by  $A_{\mathfrak{F},c}$ . The following proposition is essential for our theory of  $q$ -analogue (see [6, Theorem 1 and Proposition 2]).

PROPOSITION 2. For any  $a, b \in \mathfrak{o}$ , we have

- (i)  $(e^{aX})_{\mathfrak{F},c} = e^{f(a)X}$ ,
- (ii)  $e^{f(a)X} *_{\mathfrak{F}} e^{f(b)X} = e^{f(a+b)X}$ .

We define the  $q$ -Bernoulli numbers  $\beta_n(\mathfrak{F}, c)$  attached to  $\mathfrak{F}$  and  $c$  as follows:

DEFINITION 2. For each non-negative integer  $n$ , we define the  $n$ -th  $q$ -Bernoulli number  $\beta_n(\mathfrak{F}, c)$  attached to  $\mathfrak{F}(X, Y) \in \mathfrak{o}[[X, Y]]$  and  $c \in \mathfrak{o}^\times$  by the coefficient of  $\frac{X^n}{n!}$  in  $\mathfrak{B}_{\mathfrak{F},c} = \left(\frac{X}{e^X - 1}\right)_{\mathfrak{F},c}$ .

We note that if  $\mathfrak{F} = X + Y + (q - 1)XY$  and  $c = \frac{\log q}{q - 1}$ , then  $f(X) = \frac{q^X - 1}{q - 1}$  and  $\beta_n(\mathfrak{F}, c)$  satisfies the following recurrence formula:

$$\beta_0(\mathfrak{F}, c) = 1, \quad (q\beta(\mathfrak{F}, c) + 1)^n - \beta_n(\mathfrak{F}, c) = \begin{cases} \frac{\log q}{q - 1} & \text{for } n = 1, \\ 0 & \text{for } n > 1, \end{cases}$$

where we use the usual convention about replacing  $\beta(\mathfrak{F}, c)^i$  by  $\beta_i(\mathfrak{F}, c)$  for each non-negative integer  $i$ .

*Proof.* Apply the  $q$ -operator to  $\mathfrak{B}e^X - \mathfrak{B} = X$ , then we have  $\mathfrak{B}_{\mathfrak{F},c} *_{\mathfrak{F}} e^X - \mathfrak{B}_{\mathfrak{F},c} = cX$  and  $\mathfrak{F}_n(\mathfrak{B}_{\mathfrak{F},c}, e^X) = (q\beta(\mathfrak{F}, c) + 1)^n$ . Hence the above recurrence formula holds. □

Now we may get a  $q$ -analogue of Proposition 1 by applying the  $q$ -operator to Lemma 1, but it is unknown the commutativity of the  $q$ -operator and the derivative on  $\mathfrak{o}[[X]]$ . So we need to take another approach to get a  $q$ -analogue of Lemma 1.

LEMMA 2. For any power series  $A, B$  and non-negative integer  $n$ , we have

$$(1) \quad (A *_{\mathfrak{F}} B)^{(n)} = \sum_{i,j \geq 0} \binom{n}{i,j}_{\mathfrak{F}} A^{(i)} *_{\mathfrak{F}} B^{(j)}.$$

*Proof.* For any non-negative integer  $m$ , the coefficient of  $\frac{X^m}{m!}$  in the left hand side of (1) is equal to

$$\mathfrak{F}_{m+n}(A, B) = \sum_{i,j \geq 0} \binom{m+n}{i,j}_{\mathfrak{F}} a_i b_j.$$

On the other hand that in the right hand of (1) is equal to

$$\sum_{i,j \geq 0} \binom{n}{i,j}_{\mathfrak{F}} \mathfrak{F}(A^{(i)}, B^{(j)}) = \sum_{i,j \geq 0} \binom{n}{i,j}_{\mathfrak{F}} \sum_{k,l \geq 0} \binom{m}{k,l}_{\mathfrak{F}} a_{i+k} b_{j+l}.$$

Hence it is sufficient to show that

$$\binom{m+n}{i,j}_{\mathfrak{F}} = \sum_{\substack{0 \leq k \leq i \\ 0 \leq l \leq j}} \binom{m}{k,l}_{\mathfrak{F}} \binom{n}{i-k, j-l}_{\mathfrak{F}}$$

for all  $i \geq 0$  and  $j \geq 0$ . Because of  $\mathfrak{F}(X, Y)^{m+n} = \mathfrak{F}(X, Y)^m \mathfrak{F}(X, Y)^n$ , we can get what we want. □

Apply this lemma to  $A *_{\mathfrak{F}} e^{f(1)X}$  and  $e^{f(-1)X}$ , then we have

LEMMA 3.

$$A^{(n)} *_{\mathfrak{F}} e^{f(1)X} = \sum_{i,j \geq 0} \binom{n}{i,j}_{\mathfrak{F}} f(-1)^j (A *_{\mathfrak{F}} e^{f(1)X})^{(i)}.$$

This is a  $q$ -analogue of Lemma 1. If  $B$  satisfies Assumption 1, by applying the  $q$ -operator, we have

$$B_{\mathfrak{F},c} *_{\mathfrak{F}} e^{f(1)X} = B_{\mathfrak{F},c} + C_{\mathfrak{F},c}.$$

If  $C$  is a polynomial, then  $C_{\mathfrak{F},c}$  is also a polynomial and  $\deg C = \deg C_{\mathfrak{F},c}$  (see [6, Lemma 2]). Hence we have the following:

PROPOSITION 3. *If  $B$  satisfies Assumption 1, then we have*

$$\sum_{i,j \geq 0} \binom{m}{i,j}_{\mathfrak{F}} f(1)^j \beta_{n+i} = \sum_{i,j \geq 0} \binom{n}{i,j}_{\mathfrak{F}} f(-1)^j (\beta_{m+i} + \gamma_{m+i})$$

for any non-negative integer  $m$  and  $n$ , where  $B_{\mathfrak{F},c} = \sum_{n \geq 0} \beta_n \frac{X^n}{n!}$  and  $C_{\mathfrak{F},c} = \sum_{n \geq 0} \gamma_n \frac{X^n}{n!}$ .

If  $m > \deg C$ , then we have

COROLLARY 3.

$$\sum_{i,j \geq 0} \binom{m}{i,j}_{\mathfrak{F}} f(1)^j \beta_{n+i} = \sum_{i,j \geq 0} \binom{n}{i,j}_{\mathfrak{F}} f(-1)^j \beta_{m+i}.$$

Furthermore if  $m = n$ , then we have

COROLLARY 4.

$$\sum_{i,j \geq 0} \binom{n}{i,j}_{\mathfrak{F}} \{f(1)^j - f(-1)^j\} \beta_{n+i} = 0.$$

Hence if  $C = X$ , then we get a recurrence formula for  $\beta_n = \beta_n(\mathfrak{F}, c)$ . On the other hand if  $C = X^2$ , then  $\beta_n$  is the coefficient of  $\frac{X^n}{n!}$  in

$$\left( \frac{X^2}{e^X - 1} \right)_{\mathfrak{F},c} = cX *_{\mathfrak{F}} \mathfrak{B}_{\mathfrak{F},c} = cX d_{\mathfrak{F}}(\mathfrak{B}_{\mathfrak{F},c}),$$

where  $d_{\mathfrak{F}} = \sum_{i \geq 0} \binom{1}{i,1}_{\mathfrak{F}} \frac{d^i}{dX^i}$  (see [6, Lemma 1]). Furthermore if  $\mathfrak{F}(X, Y) = X + Y + (q - 1)XY$  and  $c = \frac{\log q}{q - 1}$ , then  $\frac{1}{c} d_{\mathfrak{F}}(\mathfrak{B}_{\mathfrak{F},c})$  is equal to the generating function of Carlitz’s  $q$ -Bernoulli numbers (see the next section). This means that we get a Kaneko’s type of recurrence formula for Carlitz’s  $q$ -Bernoulli numbers.

**3.  $X + Y + (q - 1)XY$**

If  $\mathfrak{F}(X, Y) = X + Y + (q - 1)XY$  and  $c = \frac{\log q}{q - 1}$ , then we can state results in the previous section as follows:

COROLLARY 5. *If  $B$  satisfies Assumption 1, then we have*

- (i)  $\sum_{i=0}^m \binom{m}{i} q^{n+i} \beta_{n+i} = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} (\beta_{m+i} + \gamma_{m+i}),$
- (ii)  $\sum_{i=0}^m \binom{m}{i} q^{n+i} \beta_{n+i} = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} \beta_{m+i}$  if  $m > \deg C,$
- (iii)  $\sum_{i=0}^n \binom{n}{i} \{q^{n+i} - (-1)^{n-i}\} \beta_{n+i} = 0$  if  $m = n > \deg C,$

where  $B_{\mathfrak{F},c} = \sum_{n \geq 0} \beta_n \frac{X^n}{n!}$  and  $C_{\mathfrak{F},c} = \sum_{n \geq 0} \gamma_n \frac{X^n}{n!}.$

*Proof.* If  $\mathfrak{F}(X, Y) = X + Y + (q - 1)XY,$  then, by the definition of  $\binom{m}{i, j}_{\mathfrak{F}},$  we have

$$(2) \quad \binom{m}{i, j}_{\mathfrak{F}} = \binom{m}{i} \binom{i}{i+j-m} (q-1)^{i+j-m}.$$

Hence for any  $a \in \mathfrak{o},$  we have

$$\begin{aligned} & \sum_{i, j \geq 0} \binom{m}{i, j}_{\mathfrak{F}} f(a)^j \beta_{n+i} \\ &= \sum_{i, j \geq 0} \binom{m}{i} \binom{i}{i+j-m} (q-1)^{i+j-m} f(a)^j \beta_{n+i} \\ &= \sum_{i=0}^m \binom{m}{i} f(a)^{n-i} \beta_{n+i} \sum_{j=m-i}^m \binom{i}{i+j-m} \{(q-1)f(a)\}^{i+j-m} \\ &= \sum_{i=0}^m \binom{m}{i} f(a)^{n-i} \beta_{n+i} q^{ai}. \end{aligned}$$

Hence we can get what we want. □

Let  $\bar{\beta}_n(\mathfrak{F}, c)$  be the coefficient of  $\frac{X^n}{n!}$  in  $\frac{1}{c} d_{\mathfrak{F}}(\mathfrak{B}_{\mathfrak{F},c}),$  i.e., that of  $c^2 \frac{X^{n+1}}{n!}$  in  $\left(\frac{X^2}{e^X - 1}\right)_{\mathfrak{F},c},$  then  $\bar{\beta}_n(\mathfrak{F}, c)$  satisfies the following Carlitz’s recurrence formula ([1]):

$$\bar{\beta}_0(\mathfrak{F}, c) = 1, \quad q(q\bar{\beta}(\mathfrak{F}, c) + 1)^n - \bar{\beta}_n(\mathfrak{F}, c) = \begin{cases} 1 & \text{for } n = 1, \\ 0 & \text{for } n > 1. \end{cases}$$

Hence  $\bar{\beta}_n(\mathfrak{F}, c)$  is equal to the  $n$ -th Carlitz's  $q$ -Bernoulli number. To prove this we need the following:

LEMMA 4. *If  $\mathfrak{F}(X, Y) = X + Y + (q - 1)XY$ , then  $d_{\mathfrak{F}}$  is a homomorphism on  $(\mathfrak{o}[[X]], +, *_{\mathfrak{F}})$ .*

*Proof.* In this case we can write

$$d_{\mathfrak{F}}(A) = A + (q - 1)A'$$

for any power series  $A \in \mathfrak{o}[[X]]$ . Hence by Lemma 2 we have

$$\begin{aligned} d_{\mathfrak{F}}(A *_{\mathfrak{F}} B) &= A *_{\mathfrak{F}} B + (q - 1)(A *_{\mathfrak{F}} B)' \\ &= A *_{\mathfrak{F}} B + (q - 1)\{A' *_{\mathfrak{F}} B + A *_{\mathfrak{F}} B' + (q - 1)A' *_{\mathfrak{F}} B'\} \\ &= (A + (q - 1)A') *_{\mathfrak{F}} (B + (q - 1)B') \\ &= d_{\mathfrak{F}}(A) *_{\mathfrak{F}} d_{\mathfrak{F}}(B) \end{aligned}$$

for any  $A$  and  $B$  in  $\mathfrak{o}[[X]]$ . Hence  $d_{\mathfrak{F}}$  is a homomorphism on  $(\mathfrak{o}[[X]], +, *_{\mathfrak{F}})$ . □

*Proof* (Carlitz's recurrence formula). Apply  $d_{\mathfrak{F}}$  to  $\mathfrak{B}_{\mathfrak{F},c} *_{\mathfrak{F}} e^X - \mathfrak{B}_{\mathfrak{F},c} = cX$ , then we have

$$d_{\mathfrak{F}}(\mathfrak{B}_{\mathfrak{F},c}) *_{\mathfrak{F}} qe^X - d_{\mathfrak{F}}(\mathfrak{B}_{\mathfrak{F},c}) = cX - \log q .$$

Hence  $\bar{\beta}_n(\mathfrak{F}, c)$  satisfies Carlitz's recurrence formula. □

Finally we give another proof of Corollary 5.

LEMMA 5. *If  $\mathfrak{F}(X, Y) = X + Y + (q - 1)XY$ , then the  $*_{\mathfrak{F}}$ -product is written by*

$$A *_{\mathfrak{F}} X^n = d_{\mathfrak{F}}^n(A)X^n$$

for any non-negative integer  $n$ .

*Proof.* It is sufficient to prove for  $\frac{X^i}{i!} *_{\mathfrak{F}} \frac{X^j}{j!}$  ( $i \geq 0$  and  $j \geq 0$ ). By the definition of  $*_{\mathfrak{F}}$  and (2), we have

$$\begin{aligned} \frac{X^i}{i!} *_{\mathfrak{F}} \frac{X^j}{j!} &= \sum_{m \geq 0} \binom{m}{i, j}_{\mathfrak{F}} \frac{X^m}{m!} \\ &= \sum_{m=j}^{i+j} \binom{m}{i} \binom{i}{i+j-m} (q - 1)^{i+j-m} \frac{X^m}{m!}. \end{aligned}$$



On the other hand

$$\begin{aligned} d_{\mathfrak{F}}^i \left( \frac{X^j}{j!} \right) \frac{X^i}{i!} &= \sum_{m=0}^i \binom{i}{m} (q-1)^m \frac{d^m}{dX^m} \left( \frac{X^j}{j!} \right) \frac{X^i}{i!} \\ &= \sum_{m=0}^i \binom{i}{m} \binom{i+j-m}{i} (q-1)^m \frac{X^{i+j-m}}{(i+j-m)!}. \end{aligned}$$

Hence we have what we want. □

*Remark 1.* If  $\mathfrak{F} = X + Y + (q - 1)XY$  and  $c = \frac{\log q}{q-1}$ , then, by Lemma 5, we can get

$$A(X) *_{\mathfrak{F}} e^{f(a)} = A(q^a X) e^{f(a)}$$

for any power series  $A \in \mathfrak{o}[[X]]$  and  $a \in \mathfrak{o}$ . By this we can get Corollary 5 from Lemma 1 not using Lemma 2.

*Remark 2.* Lemma 4 and Lemma 5 hold only for  $\mathfrak{F}(X, Y) = X + Y + (q - 1)XY$  (see [6, Lemma 1 and Proposition 4]).

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