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SECTORIAL COVERS FOR CURVES OF CONSTANT LENGTH

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1. In answer to a question raised by Leo Moser, A. Meir proved some years ago that every plane arc of unit length lies in some closed semidisk of radius $\frac{1}{2}$. His elegant, unpublished argument is reproduced here with his kind permission.

THEOREM 1 (A. Meir). *Every plane arc of length L lies in some closed semidisk of radius $L/2$.*

Proof. The assertion is clear for closed curves, for such a curve plainly lies in a semidisk of radius $L/2$ centered at a point of contact of any support line of the curve. Let Γ be an arc of length L having distinct endpoints P and Q, let l be a line of support parallel to the line PQ and touching Γ at a point R, and let P' and Q' be the points symmetric to P and Q in l (Figure 1). Let O be the point in which the lines PQ' and QP' meet l. Each point X on Γ lies between R and P or between R and Q along Γ, and we may suppose that X lies between R and P. Because the median of a triangle is shorter than the average of the lengths of the two adjacent sides,

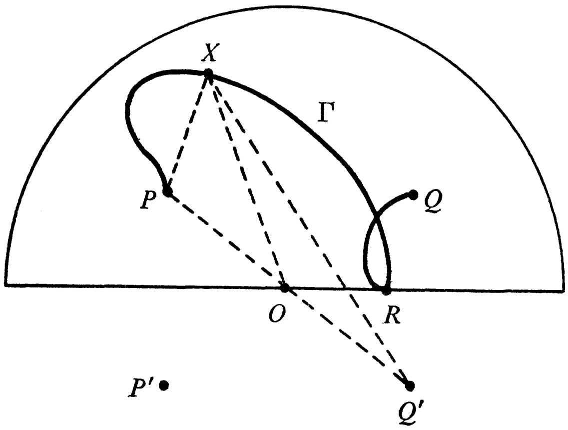

Figure 1

$$OX \leq \tfrac{1}{2}(XP+XQ') \leq \tfrac{1}{2}(PX+XR+RQ) \leq \tfrac{1}{2}L.$$

Thus Γ lies in the semidisk of radius $L/2$ and edge l centered at the point O.

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In §3 of this note we generalize this result to circular sectors and show that there is a sector of area less than $0.3451L^2$ that can accommodate every arc of length L. Meir's semidisk has area $\pi L^2/8 \approx 0.3927L^2$.

Section 2 is devoted to a characterization of circular sectors that contain a translate of every closed curve of length L.

In §4 we show that the least area of a convex set that contains a translate of every closed curve of length L lies between $0.15544L^2$ and $0.15900L^2$, and in §5 we show that the least area of a convex set that contains a displacement of every arc of length L lies between $0.21946L^2$ and $0.34423L^2$.

2. A circular sector is *circumscribed* about a curve if the curve lies in the sector and has a point on the circular boundary arc and a point on each of the boundary radii. We begin with a result about circular sectors that are circumscribed about a closed curve of length L.

Let $\text{Csc}\, x = \csc x$ when $0 < x < \pi/2$ and $\text{Csc}\, x = 1$ when $\pi/2 \leq x \leq \pi$. For $r > 0$ and $0 < \theta \leq \pi$, we denote the circular sector with radius r and vertex angle θ by $S(r, \theta)$.

LEMMA 2. *If a circular sector $S(r, \theta)$ is circumscribed about a closed curve of length L, then $r \leq (L/2)\,\text{Csc}\,\theta$.*

Proof. Let the sector $S(r, \theta) = \langle \text{BAC} \rangle$ be circumscribed about a closed curve Γ of length L, and let X, Y, and Z be points of Γ on the circular arc BC and radial segments AB and AC, respectively. The perimeter p of ΔXYZ is at most L, and p equals L precisely when the curve Γ coincides with ΔXYZ. Let X' and X'' be the points symmetric to X in the lines AB and AC respectively. If $\theta < \pi/2$, then

$$p = X'Y + YZ + ZX'' \geq X'X'' = 2r \sin \theta.$$

If $\pi/2 \leq \theta \leq \pi$, then

$$p = X'Y + YZ + ZX'' \geq X'Z + ZX'' \geq X'A + AX'' = 2r.$$

In either case,

$$r \leq \tfrac{1}{2}p\,\text{Csc}\,\theta \leq \tfrac{1}{2}L\,\text{Csc}\,\theta.$$

When θ is acute, the equality occurs precisely when Γ coincides with ΔXYZ and the points X', Z, Y, and X'' are collinear. When θ is not acute, the equality occurs precisely when Γ is a radial segment (traversed twice).

A compact, convex set in the plane is a *translation cover* for a family of plane arcs if for each arc in the family there is a translation that carries the arc into the set. We can use Lemma 2 to characterize sectorial translation covers for the family $\mathscr{C}_L$ of all closed curves of length L.

THEOREM 3. *A sector $S(r, \theta)$ is a translation cover for $\mathscr{C}_L$ if and only if $r \geq (L/2)$ $\text{Csc}\,\theta$.*

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Proof. Suppose that $S(r, \theta)=\langle BAC\rangle$ is a sectors atisfying $r\geq(L/2)$ Csc θ, and suppose that Γ is a given closed curve of length L. By a translation we can suppose that Γ lies in $\angle BAC$ and touches each of the rays AB and AC. Let r_0 be the maximum distance from the vertex A to any point on the (translated) curve. Then the sector with vertex A and radius r_0 is circumscribed about the curve, and according to Lemma 2, $r_0\leq(L/2)$ Csc $\theta\leq r$. It follows that the (translated) curve lies in $S(r, \theta)$. Conversely, it is plain that no sector with angle θ and smaller radius can be a translation cover for all closed curves of length L, because the width $r \sin \theta$ of any covering sector $S(r, \theta)$ in the direction perpendicular to a boundary ray must be at least $L/2$.

COROLLARY 4. *The circular sector of least area that is a translation cover for* $\mathscr{C}_L$ *has angle* θ_0 *and radius* $(L/2) \csc \theta_0$, *where* $\theta_0\approx 1.16556$ *is the least positive root of the equation* $\tan \theta=2\theta$. *The area of this sector is approximately* $0.1725L^2$.

Proof. For each θ, the smallest sector $S(r, \theta)$ that is a translation cover for $\mathscr{C}_L$ has radius $r=(L/2)$ Csc θ and area $f(\theta)=\frac{1}{8}L^2\theta$ Csc^2 θ. This function has a unique minimum on the interval $(0, \pi]$ at the least positive root θ_0 of the equation $\tan \theta=2\theta$.

3. A compact, convex (plane) set is a *displacement cover* for a family of (plane) arcs if for each arc in the family there is a displacement (i.e., a map of the plane that can be factored as a product of a translation and a rotation) that carries the arc into the set. By combining Lemma 2 with a reflection of J. Ralph Alexander's, we obtain a result on sectorial displacement covers for the family $\mathscr{A}_L$ of arbitrary arcs of length L that generalizes Meir's semidisk result.

THEOREM 5. *If* $r\geq(L/2) \csc \theta$, *then the circular sector* $S(r, 2\theta)$ *is a displacement cover for* $\mathscr{A}_L$; *and conversely when* $\theta\geq\pi/6$.

Proof. Suppose that $r\geq(L/2) \csc \theta$, and let Γ be an arc of length L. The assertion follows from Theorem 3 if Γ is closed, so suppose that Γ has distinct endpoints P and Q. Let m be the perpendicular bisector of the segment PQ, and let Γ' be the closed curve of length L that results from reflecting the points of Γ lying on one side of m through m (Figure 2). Let l be a support line of Γ' that makes an angle θ with m. Then the circular sector with radius r, sides on l and m, and center at the point A in which l and m intersect surrounds Γ'; and it is evident that the sector $\langle BAC\rangle$ with radius r and vertex angle 2θ covers Γ. Every displacement cover must have diameter at least L, so when $\theta\geq\pi/6$ no smaller sector is a cover.

Whether a sector $S(r, 2\theta)$ with radius smaller than $(L/2) \csc \theta$ can accommodate each arc of length L when $\theta<\pi/6$ is not known.

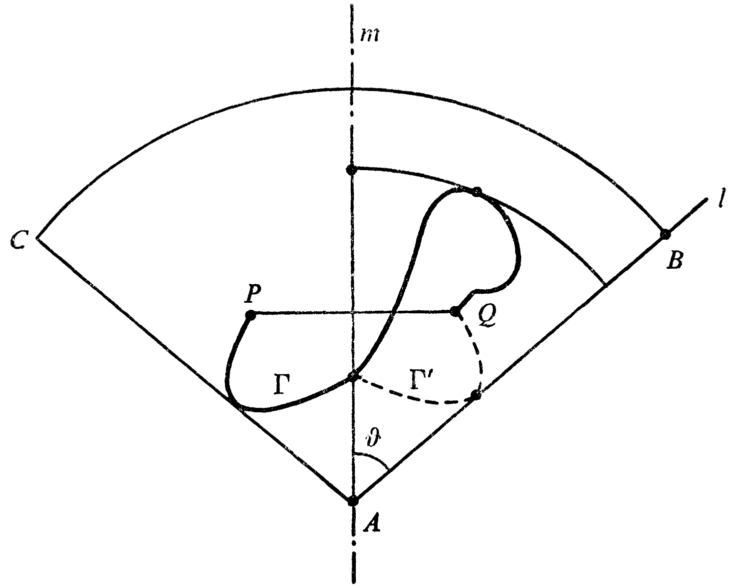

Figure 2

COROLLARY 6. *There is a circular sector with area less than* $0.3451L^2$ *that is a displacement cover for* $\mathscr{A}_L$.

Proof. For each θ in $(0, \pi/2]$, the sector $S(r, 2\theta)$ with radius $r=(L/2)\csc\theta$ is a displacement cover for $\mathscr{A}_L$ and has area $f(\theta)=\frac{1}{4}L^2\theta\csc^2\theta$. This function has a unique minimum value $f(\theta_0)\approx 0.34501L^2$ at the least positive root $\theta_0\approx 1.16556$ of the equation $\tan\theta=2\theta$.

4. Problems of finding sets of certain kinds that can accommodate in a specified way each arc from a specified family are called "worm" problems, and a great variety of such problems, mostly unsolved, can be found in the literature. (For examples and further references, see [4] and the lists of research problems compiled by Croft [2], [3] and Moser [6].)

The smallest triangular translation cover for the family $\mathscr{C}_L$ of all closed curves of length L is the equilateral triangle of side $2L/3$ (see [9]). The smallest sectorial translation cover, determined in Corollary 4, is a little smaller than this smallest triangle.

But we can do a bit better. Once a given closed curve has been translated into a covering sector, a further translation will produce a point of contact with the circular boundary arc without taking the curve outside the sector. Then the curve surely cannot enter the small sector $\langle DAE\rangle$ having the same vertex and boundary rays and radius $(L/2)(\text{Csc}\,\theta-1)$, or its length would be greater than L (Figure 3). It follows that the truncated sector obtained by clipping the small isosceles triangle ΔDAE with $AD=AE=(L/2)(\text{Csc}\,\theta-1)$ from a covering sector $S((L/2)\,\text{Csc}\,\theta, \theta)$ is

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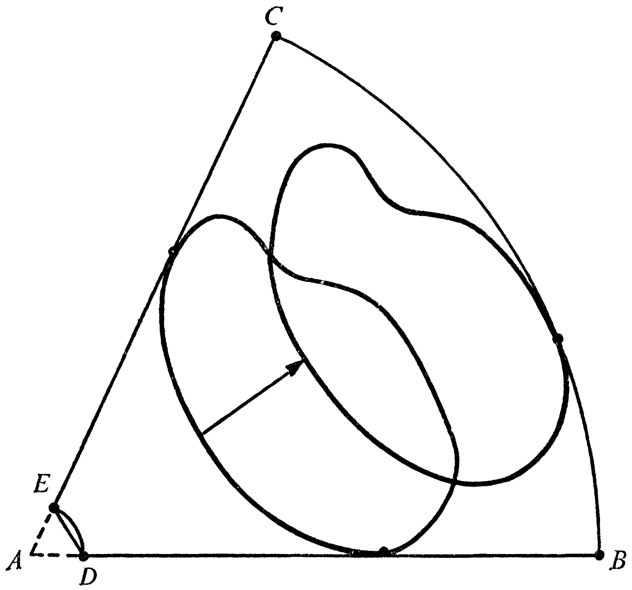

Figure 3

again a translation cover. This truncated sector has area

$$f(\theta) = \frac{L^2}{2}\,[\theta \operatorname{Csc}^2 \theta - \sin \theta\,(\operatorname{Csc} \theta - 1)^2],$$

and minimizing this function on $(0, \pi]$ shows that there is such a truncated sector having area less than $0.15900L^2$; the minimum truncated sector has angle equal to the least positive root $\theta_1 \approx 1.12120$ of the equation

$$\tan \theta = 2\theta - \sin \theta \cos^2 \theta.$$

This minimal truncated sector is the one pictured in Figure 3.

In 1921, Pál [7] proved that every convex set having minimal width t has area at least $t^2/\sqrt{3}$ (see also [10, pp. 60, 221–222]). It follows that every translation cover for $\mathscr{C}_L$ has area at least $L^2/(4\sqrt{3}) \approx 0.144L^2$, because every such set obviously has minimal width $t \geq L/2$. By modifying Pál's argument, we can strengthen this lower bound to approximately $0.15544L^2$.

For the following discussion, let T be a translation cover for $\mathscr{C}_L$. We say that a triangle ΔABC is embedded in T if A, B, and C lie on the boundary of T and if there are support lines at A, B, and C that form a triangle enclosing T.

LEMMA 7. *If ΔABC is embedded in T and if B' and C' are points of T across the line BC from A so that the segments $B'C'$ and BC are parallel, then $B'C' < BC$.*

Proof. If $B'C' \geq BC$, then (by the parallel postulate) every support line at B meets

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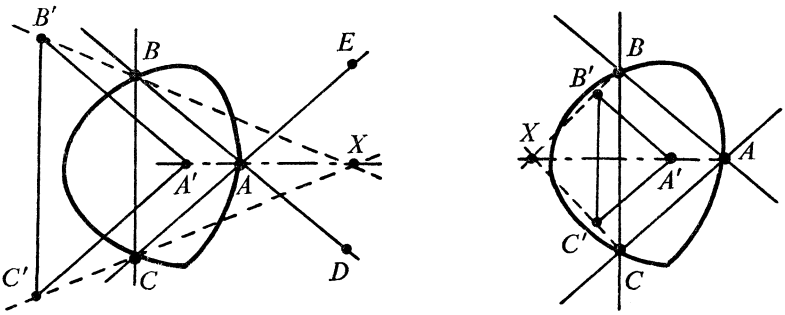
each (non-parallel) support line at C on the same side of the line BC as A, contrary to the assumption that there be support lines at A, B, and C that surround T.

LEMMA 8. *Every triangle embedded in T has perimeter at least L.*

Proof. Suppose ΔABC, with perimeter p, is embedded in T. By hypothesis there are points A', B', and C' in T so that $\Delta A'B'C'$ is (positively) homothetic to ΔABC and has perimeter $p'=L$. If $\Delta A'B'C'$ is a translate of ΔABC, then $p=p'=L$. Otherwise, let X be the center of the homothety.

We show first that X cannot lie in one of the (open) angular regions formed by the sides of ΔABC off the vertices. Suppose X lies in such a region, say in the interior of $\angle DAE$ (Figure 4a). Since this set is disjoint from T (otherwise A would



(a) (b)

Figure 4

be an interior point of T), A must lie on the segment $A'X$. Suppose $A'\neq A$. Then since the segments $B'C'$ and BC are parallel and ΔABC is embedded in T, $B'C'<BC$, an obvious contradiction.

Consequently X lies in one of the closed angular regions bounded by an angle of ΔABC, say in the closed region bounded by $\angle BAC$ (Figure 4*b*). Then A' lies on the segment AX, and it follows at once that $p\geq p'=L$.

It was proved by Blaschke [1, pp. 370–371] that if S is an incircle of a compact, convex set T with boundary ∂T, then either $S\cap\partial T$ contains two points that are the ends of a diameter of S, or $S\cap\partial T$ contains three points that are the vertices of an acute triangle (see also [10, pp. 59, 215–216]).

COROLLARY 9. *The inradius of T is at least $L\sqrt{3}/9$.*

Proof. Let r be the inradius of T, and let S be an incircle. If $S\cap\partial T$ contains two points P and Q that are the ends of a diameter of S, then $PQ=2r\geq L/2$, and so $r\geq L/4>L\sqrt{3}/9$. If on the other hand $S\cap\partial T$ contains three points A, B, and C

that form an acute triangle, then ΔABC is embedded in T (because the unique support lines to T at A, B, and C are perpendicular to the radii of S to these points, and the center of S lies inside ΔABC); and it follows from the lemma and Jensen's inequality [5, pp. 23–25, 28] that

$$\begin{aligned} r &= \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma} \\ &= \frac{p}{2(\sin\alpha+\sin\beta+\sin\gamma)} \\ &\geq \frac{L}{2(\sin\alpha+\sin\beta+\sin\gamma)} \\ &\geq \frac{L}{6\sin\frac{\pi}{3}} = \frac{L\sqrt{3}}{9}, \end{aligned}$$

proving the assertion.

For the sake of completeness, we include a sketch of the relevant portions of Pál's argument (from [7, pp. 313–314]). For each r in $[L/6, L/4]$, let Φ_r be the convex hull of a circle of radius r and three points X, Y, and Z at distance $(L/2)-r$ from the center of the circle, arranged so that the "caps" that are added to the circle do not overlap (see Figure 5). The area $f(r)$ of a figure Φ_r is given by

$$f(r) = \pi r^2+\frac{3r}{2}(L^2-4rL)^{1/2}-3r^2 \arccos\frac{2r}{L-2r}; \tag{1}$$

and since $f'(r)>0$ for $L/6\leq r<L/4$, the area $f(r)$ is an increasing function of r. We claim that T contains a figure Φ_r for some $r\geq L\sqrt{3}/9$ and that consequently the area of T is at least $f(L\sqrt{3}/9)$.

Let r be the inradius of T, and let S be an incircle with center 0. If $S\cap\partial T$ contains two points that are the ends of a diameter of S, then $r\geq L/4$ (as observed before), and the circle $\Phi_{L/4}$ is a subset of T. Otherwise $S\cap\partial T$ contains three points A, B, and C that form an acute triangle. The support lines l_A, l_B, and l_C to T at A, B, and C are tangent to S. Let D, E, and F be points on ∂T at which the support lines l_D, l_E, and l_F are parallel to l_A, l_B, and l_C respectively (Figure 5).

The distance between l_A and l_D is at least $L/2$, so $AD\geq L/2$; and $DO\geq(L/2)-r$ since $AO=r$. Let X be the point on the segment OD so that $OX=(L/2)-r$. The tangent lines from X to S form a cap that lies entirely in T and on the opposite side of the line BC from A.

Similarly we find points Y on the segment OE and Z on the segment OF; and the circle S and points X, Y, and Z determine a figure $\Phi_r\subseteq T$, where r is the inradius of T. It follows that the area of T must be at least the area $f(r)$ of Φ_r, which, since $f(r)$ is increasing, must be at least $f(L\sqrt{3}/9)$. In summary, we have the following theorem.

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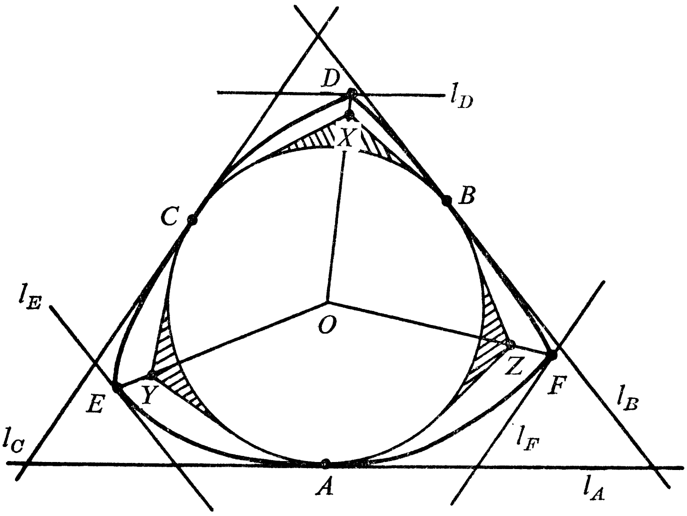

Figure 5

THEOREM 10. *Every translation cover for the family of closed curves of length L has area at least* $f(L\sqrt{3}/9) \approx 0.15544L^2$, *where* $f(r)$ *is given by* (1); *and there exists a translation cover for this family having area less than* $0.15900L^2$.

The circle $\Phi_{L/4}$ is the only figure Φ_r that is a translation cover for $\mathscr{C}_L$, because the minimal width of each Φ_r for $r<L/4$ is less than $L/2$.

5. By truncating a sectorial displacement cover, we can produce a smaller displacement cover for the family $\mathscr{A}_L$ of all arcs of length L. Indeed, the region produced by clipping the small isosceles triangle with vertex angle 2θ and sides of length $(L/2)(\csc\theta-1)$ from the vertex of a covering sector $S((L/2)\csc\theta, 2\theta)$ is again a displacement cover for $\mathscr{A}_L$, as can easily be seen by applying the reflection argument employed in the proof of Theorem 5. Its area,

$$f(\theta) = \frac{L^2}{8}[2\theta\csc^2\theta - \sin 2\theta(\csc\theta-1)^2],$$

has a unique minimum value $f(\theta_2) \approx 0.34423L^2$ at the least positive root $\theta_2 \approx 1.14687$ of the equation

$$\tan\theta = 2\theta - \tan\theta(\cos^2\theta - 2\sin^3\theta + 2\sin^4\theta).$$

The best lower bound we know for the area of such covers is $0.21946L^2$, which arises as follows. Schaer [8] showed that the arc of length 1 that has maximum thickness, i.e., whose minimum width is as large as possible, has thickness $b_0 \approx 0.43893$. Every displacement cover for $\mathscr{A}_L$ must have diameter d at least L and width w in the direction perpendicular to a diameter at least b_0L. Consequently its area must be at least $wd/2 \geq b_0L^2/2 \approx 0.21946L^2$. In summary, we have the following theorem.

THEOREM 11. *Every displacement cover for the family of all arcs of length L has area at least* $0.21946L^2$*; and there exists a displacement cover for this family with area less than* $0.34423L^2$.

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