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The Donaldson-Hitchin-Kobayashi Correspondence for Parabolic Bundles over Orbifold Surfaces

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Abstract. A theorem of Donaldson on the existence of Hermitian-Einstein metrics on stable holomorphic bundles over a compact Kähler surface is extended to bundles which are parabolic along an effective divisor with normal crossings. Orbifold methods, together with a suitable approximation theorem, are used following an approach successful for the case of Riemann surfaces.

In [6] it is shown that if $\mathcal{E}$ is a stable parabolic bundle over a Kähler surface parabolic along a smooth divisor D then $\mathcal{E}$ admits an irreducible Hermitian-Einstein metric. Here we give a different proof of this theorem, extending it to the case when D is an effective divisor with normal crossings. The strategy of the proof is that of [28], which was indeed undertaken as a test of the method. It rests on two pillars: the proof of S. K. Donaldson in the ordinary (non-parabolic) case, a proof which extends straightforwardly to orbifolds [18], [37], and an approximation argument [28] which shows that if one has the result for rational weights one can extend to irrational ones. (These are described in Section 2 and Sections 6 and 7 respectively.)

If $D \subset Y$ is an effective divisor there is a notion of bundle with parabolic structure along D [21], [17], extending that of curves [22], and a fairly simple correspondence between parabolic bundles with rational monodromy and V-bundles over certain associated orbifolds. It is described for curves in [12] and its extension to surfaces—when the divisor D is made up of smoothly embedded curves intersecting normally—is described briefly in [18], [37] and in some detail in Sections 3 and 4 here because we need the precise local forms of the metrics. As for curves, stability (with respect to an underlying Kähler metric) is preserved and one immediately establishes a Donaldson-Hitchin-Kobayashi correspondence when the parabolic weights are rational. However, to carry out the proof on the orbifold we need an *orbifold Kähler metric* and on the underlying smooth surface this degenerates along the divisor: it is what is called a *cone-like* metric in [17]. Moreover, the correspondence carries genuine metrics on the V-bundle to singular ones on the parabolic bundle with the singularity determined by the weight (that is, the connexion has specified holonomy about D). Such metrics we call *parabolic metrics*. This is acceptable, but on the complex manifold one's preference is for a standard Kähler metric. With some effort one can indeed pass from a (singular) Hermitian-Einstein connexion for a cone-like metric to one for a standard Kähler metric lying in the same cohomology class and so get the desired correspondence. It is much harder to carry out for surfaces than for

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curves and we are obliged first to reduce the case when the first Chern class is zero (or 'small') by tensoring with a line V-bundle and then to argue over $Y - D$ using the theory of [19] and [23]. This is done in Section 5 and constitutes the arch linking the 2 pillars. The final theorem is the following which is proved in Section 7.

Theorem 7.4 *If $\mathcal{E}$ is a stable parabolic over Y, parabolic along D where D is a divisor with normal crossings, then $\mathcal{E}$ admits a parabolic Hermitian-Einstein metric with curvature in L^p for some p a little larger than 2, where here L^p is taken with respect to an ordinary metric.*

Except in Section 5, and to a tiny extent in Section 7, the analysis takes place on a compact space as in [28]. Because many of the ideas and proofs are known in the case of Riemann surfaces and because there is a careful discussion of analytical ideas in [18], [28] we have not laboured proofs. (For the analysis in this section it is convenient to have parabolic weights in $[-1/2, 1/2)$ rather than $[0, 1)$ which, on the other hand, is better for the geometry, as Section 2 suggests.)

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1

Orbifolds occur naturally in certain situations and from some points of view they are little harder to handle than manifolds. As for a smooth manifold a distinction has to be made between the orbifold structure and the underlying space which may itself be a topological manifold—as it is in the case of complex dimension 1. An orbifold structure on a Riemann surface with chosen points enables one to consider fractional powers of line bundles in the same way that one would ordinary bundles. The same is true in higher dimensions and this section has a brief discussion, see [32], with the case of complex surfaces especially in mind.

The definition of analytic subvariety is a local one and extends to complex orbifolds. There is a theory of divisors, as is noted in [3] and to each divisor there is an associated V-line bundle defined as for manifolds [13, p. 133] using an atlas or more abstractly. The group $\mathrm{Pic}^V(X)$ of line V-bundles over a compact complex orbifold is studied in [38]. (Because the local ring is a unique factorization domain, there is no distinction between Weil and Cartier divisors [24].)

Later we shall use orbifolds as a tool—roughly as branched covers are needed—so we are interested in the case when the underlying space $|X|$ of an orbifold X is a manifold. Suppose that X is a compact complex orbifold or V-manifold [30], [33] of complex dimension n. Locally, then, any point P has a chart $U = \{\pi\colon \tilde{U} \to \tilde{U}/G_U \cong |U|\}$ where $\tilde{U}$ is an open subset of $\mathbf{C}^n$ and G_u a finite group of analytic maps acting faithfully. We can always suppose the chart is Euclidean in that G_u acts via a faithful unitary representation. Then there is a simple criterion for $\mathbf{C}^n/G_u$ to be $\mathbf{C}^n$ due essentially to Chevalley. It is in terms of complex reflexions: *i.e.*, linear maps of which all eigenvalues except one are 1.

Lemma 1.1 **[7], [20]** *If G is a finite group acting faithfully on $\mathbf{C}^n$ and G is generated by complex reflexions then $\mathbf{C}^n/G \cong \mathbf{C}^n$.*

When $n = 1$, G has to be cyclic, and this criterion is trivially satisfied. In higher dimensions, the situation is more complicated. Much of the time we shall restrict to the cases where every G_u is *abelian* and the action *satisfies the criterion* above: such an orbifold we will call an *abelian orbifold*. For an abelian orbifold, where the underlying space is a smooth complex surface, the singular set (*i.e.*, where the isotropy group is non-trivial) is the union of smooth hypersurfaces intersecting normally. There are no isolated points because then G would have to act freely away from P and intersections are generic since G is abelian; in particular, for surfaces there are no triple or higher intersections. For such orbifolds the relation between divisors and line bundles is exact if the orbifold is algebraic.

Proposition 1.2 **[32]** *If X is an abelian orbifold with $|X|$ a smooth algebraic surface then every line bundle is the line bundle of a divisor.*

Suppose now that Y is a smooth compact Kähler surface and that $D = \Sigma_1 \cup \cdots \cup \Sigma_d$ is an effective divisor where $\Sigma_1, \ldots, \Sigma_d$ are smooth complex curves embedded in Y and intersecting normally. Given integers $n_1, \ldots, n_d$ we can build, as for Riemann surfaces, a corresponding orbifold X by declaring that

(i) a chart at $P \in Y - D$ is an ordinary one;

(ii) a chart at $P \in \Sigma_i \setminus \bigcup_{i \neq j} \Sigma_j$ is of the form $\Delta \times \Delta \to \Delta \times \Delta/\mathbf{Z}_{n_i} \to Y$ where $\mathbf{Z}_{n_i}$ acts by $\zeta(z, w) = (\zeta z, w)$, $\Sigma_{i\cap}|\Delta \times \Delta/\mathbf{Z}_{n_i}| = \{(0, w) : w \in \Delta\}$ and $\Delta \times \Delta/\mathbf{Z}_{n_i} \to Y$ is a chart for Y;

(iii) a chart at $P \in \Sigma_{i\cap}\Sigma_j$ $(i \neq j)$ if of the form $\Delta \times \Delta \to \Delta \times \Delta/\mathbf{Z}_{n_i} \times \mathbf{Z}_{n_j} \to Y$ where $(\zeta, \eta)(z, w) = (\zeta z, \eta w)$, $\Sigma_{i\cap}|\Delta \times \Delta/\mathbf{Z}_{n_i} \times \mathbf{Z}_{n_j}| = \{(0, w) : w \in \Delta\}$, and $\Delta \times \Delta/\mathbf{Z}_{n_i} \times \mathbf{Z}_{n_j} \to Y$ is a chart for Y.

(Here $\Delta/\mathbf{Z}_k$ is identified with $\Delta \subset \mathbf{C}$ in the usual way and we suppose that at $P \in \Sigma_i - D \cdot D$ the normal direction is given by first coordinate.)

Then $\mathcal{L}_D^Y = \mathcal{L}_{\Sigma_1}^Y \otimes \cdots \otimes \mathcal{L}_{\Sigma_d}^Y$ and $\mathcal{L}_D^X = \mathcal{L}_{\Sigma_1}^X \otimes \cdots \otimes \mathcal{L}_{\Sigma_d}^X$, but $\left(\mathcal{L}_{\Sigma_i}^X\right)^{n_i} = \mathcal{L}_{\Sigma_i}^Y$ so that we have effectively extracted the roots. (Of course, $\mathcal{L}_{\Sigma_i}^X/\Sigma_i$ is probably not a genuine bundle: it will only be if n_i divides $c_1\left(\mathcal{L}_{\Sigma_i}^Y\right) \cdot \Sigma_i$.)

A similar construction could also be made if a curve Σ_i should have a simple double point.

As a rule we shall write (z, w) for coordinates on X and (u, v) for coordinates on Y so that $u = z^{n_i}$, $v = w^{n_j}$ in case (iii).

Definition 1.3 X will be called the associated orbifold to $(Y, D, \underline{n})$ and the notation will often be amplified to $(X, D, \underline{n})$.

Sections of bundles can be defined in the usual way and we have differential operators as for manifolds. The Riemann-Roch theorem for complex orbifolds is due to Kawasaki. We state it in the case of an abelian orbifold surface with singular set $D = \Sigma_1 \cup \cdots \cup \Sigma_d$, where each Σ_i is a smooth complex curve.

Theorem 1.4 **[16] (Kawasaki, Riemann-Roch)** *If X is an abelian orbifold surface, as above, and $\mathcal{E}$ a holomorphic V-bundle then*

$$\chi(\mathcal{E}) = \int_X \mathfrak{I}(X,\mathcal{E}) + \sum_1^d (1/n_i) \int_{\Sigma_i} \mathfrak{I}(X,\mathcal{E}) + \sum_{i,j} (1/n_i n_j) \Sigma_i \cdot \Sigma_j,$$

where $\mathfrak{I}(X,\mathcal{E})$ is the total Todd class of X coupled to $\mathcal{E}$.

Theorem 1.5 **[27] (Serre Duality)** $H^i(\mathcal{E}^* \otimes \mathcal{K}_X) \cong H^{n-i}(\mathcal{E})$ *for an abelian orbifold X with canonical bundle $\mathcal{K}_X$.*

An analogue of the adjunction formula also holds for orbifolds and, with the appropriate definition of sheaf cohomology [32], one has $\mathrm{Pic}^V(X) \cong H^1(X, \mathcal{O}^*)$ and $\mathrm{Div}(X) \cong H^0(X, \mathcal{M}^*/\mathcal{O}^*)$.

2

Suppose that X is a compact Kähler orbifold-surface: this means that for each chart $U = \{\pi_U : \tilde{U} \to \tilde{U}/G_u \cong |U|\}$ there is an invariant hermitian metric h_u and an associated 2-form $\tilde{w}_u$ on $\tilde{U}$ such that $d\tilde{w}_u = 0$ and all fit together under the transition functions. As one knows from the 1-dimensional case such a hermitian metric will be singular on $|U|$ if $G_u \neq 1$, even if $|U|$ is a polydisc. Using the two form $\tilde{w}$, stability can be defined in the usual way.

Definition 2.1 A holomorphic bundle $\mathcal{E}$ of rank l is *stable* if, for every holomorphic bundle $\mathcal{F}$ of rank $r < l$ and map $i : \mathcal{F} \to \mathcal{E}$ injective at some point, $\mu\mathcal{F} < \mu\mathcal{E}$ where $\mu\mathcal{F} = c_1\mathcal{F} \cdot w/r = \deg \mathcal{F}/\operatorname{rank} \mathcal{F}$ is the slope of $\mathcal{F}$.

A connexion on a V-bundle E is, locally, an invariant operator on $\tilde{U}$ and so is determined there by an invariant 1-form. The curvature F_A of a connexion A is, as for manifolds, a 2-form with values in End E. If E is holomorphic we insist that $\bar{\partial}_A = \bar{\partial}$, where $\bar{\partial}_A$ is the anti-holomorphic part of d_A. If E is hermitian there is a bijection, described in [10], between holomorphic structures on E and orbits of the space A of integrable- or $(1,1)$-connexions under the group G of complex automorphisms of E. The holomorphic structure on E associated to A may be written $\mathcal{E}_A$.

Definition 2.2 A connexion on E is Hermitian-Einstein if $\Lambda F_A = \lambda I$, where $\Lambda : \Omega^2 \to \Omega^0$ is the adjoint of the operator $\alpha \to \alpha \wedge w$ and $\lambda = -4\pi i \mu E$ (if we normalize and suppose $\mathrm{vol}\, X = 1$).

Theorem 2.3 **[38], [17]** *A holomorphic bundle $\mathcal{E}$ is stable if and only if $\mathcal{E}$ admits an irreducible Hermitian-Einstein connexion.*

The proof of this involves no more than checking that each stage of S. K. Donaldson's proof ([8], [9], or [11]) extends to orbifolds.

The Weitzenböck formulae are local and the orbifold operators obey the same rules as for those on manifolds. So one has (correcting a sign and writing $\hat{F}_A = \Lambda F_A$

as in [11]) that, as operators on $\Omega^0(E)$,

$$\bar\partial_A^*\bar\partial = \frac{1}{2}(\nabla_A^*\nabla_A - i\hat F_A) \quad \text{and} \quad \partial_A^*\partial_A = \frac{1}{2}(\nabla_A^*\nabla_A + i\hat F_A). \tag{2.4}$$

There are the two following consequences, where one uses integration by parts in the first and considers the eigenspaces of $i\hat F_A$ in the second.

Proposition 2.5 *If A is a unitary $(1,1)$-connexion on E and $iF_A \le 0$ as a self-adjoint endomorphism of E then any holomorphic section of E_A is a covariant constant, zero unless $\hat F_A \equiv 0$.*

Proposition 2.6 *If $d_A^*F_A = 0$ (i.e., A is a critical point of Yang-Mills functional) then A is a direct sum of Hermitian-Einstein connexions.*

Definition 2.7 $\mathcal{E}$ is *polystable* if $\mathcal{E}$ is a direct sum of stable bundles of the same slope.

Corollary *If Theorem 2.3 is true for rank $< r$ then any rank r bundle $\mathcal{E}$ which admits a Hermitian-Einstein connexion is polystable and stable if the connexion is irreducible.*

The essential part of the proof, the existence of such a connexion on a stable bundle, uses the evolution equation

$$\frac{\partial}{\partial t}A_t = d_{A_t}^*F_{A_t} \tag{$*$}$$

on the orbifold surface X.

Proposition 2.8 **[8], [9], [37]** *If A is a $(1,1)$-connexion, there is a unique solution A_t to the equation $(*)$ valid for all t and with $A_0 = A$. Moreover, there is a 1-parameter family of complex gauge transformations g_t such that $A_t = g_t(A)$.*

The long-time existence follows exactly as in [8], [9] once one has short-time existence for the initial value problem. As the linearization of $(*)$ is *not* parabolic a modified problem has to be considered. We choose the approach of working directly with the complex gauge transformations. The equation

$$\frac{\partial g_t}{\partial_t} = i\hat F_{g_tA}g_t; \quad g_0 = I$$

is still not parabolic. However, it is the unitary transformations which are redundant and we can remove this problem either by taking g_t to be self-adjoint or by considering $h_t = g_t^*g_t$ as in [8], [9]. The new equation for h_t, namely

$$\frac{\partial h_t}{\partial t} = -2ih_t\{\hat F_{A_0} + \Lambda\bar\partial_{A_0}(h_t^{-1}\partial_{A_0}h_t)\}, \quad h_0 = I$$

is *parabolic*. Hence, as its linearization is linear parabolic, we have the short-term existence and uniqueness from the inverse function theorem for Banach spaces [14, Part 4, Section 11], given the existence and uniqueness for linear ones. The basis of the argument for this in [14, Part 3] is a local result which holds equivariantly. Hence we have the local result for orbifolds and consequently the global result as in [8], [9].

But the solution was for h_t. We now lift as in Section 6.3.1. of [11].

The next step is to show that a limit connexion A_∞ exists. The patching argument of [11, Lemmas 4.4.4 and 4.4.5] extends directly to orbifolds and yields a connexion smooth over $X - \{x_1, \dots, x_k\}$, where $\{x_1, \dots, x_k\}$ is a finite set of points, and such that $d^*_{A_\infty} A_\infty = 0$. So, as in 4.4.5, either

(a) $\hat{F}_{A_\infty} = \lambda I$ or
(b) A_∞ is the direct sum of Hermitian-Einstein connexions.

Now the removal of singularities theorem [36] gives a smooth connexion on a bundle E' and a sequence of unitary maps

$$\rho_\alpha \colon E|X - \{x_1, \dots, x_k\} \to E'|X - \{x_1, \dots, x_k\}$$

such that $\rho^*_\alpha(A_\alpha) \to A_\infty$ in L^2_2 on every compact subset. From this, one deduces from Hartog's theorem that there is a holomorphic map

$$f \colon \mathcal{E} \to \mathcal{E}'$$

and one shows it is non-zero. As $\operatorname{tr}(F_\alpha)$ converges, we know that $\deg \mathcal{E}' = \deg \mathcal{E}$. They have the same rank as well. Finally one shows that if $\mathcal{E}' = \mathcal{F}^+ \oplus \mathcal{F}^-$ where $\mu(\mathcal{F}^-) < \mu\mathcal{E}$ and $\mu(\mathcal{F}^+) \geq \mu\mathcal{E}$ then $f^+ \colon \mathcal{E} \to \mathcal{F}^+$ is non-zero, contradicting stability.

For the proof of Theorem 2.3 above, a hermitian metric is chosen on $\mathcal{E}$. We take the initial connexion for the equation $(*)$ to be that unitary connexion A associated to the holomorphic structure $\mathcal{E}$ on E and the chosen hermitian metric. As the final connexion is in the same orbit as A, it determines the same holomorphic structure $\mathcal{E}$.

(If we carry this connection down onto the associated parabolic as in the next section we get a Hermitian-Einstein one with respect to the orbifold metric, but at the moment we cannot say much more than that it is in L^p.

Proposition 2.9 *If $\mathcal{E}$ is a stable parabolic bundle on Y with the denominators of the weights along Σ_i dividing n_i then there is an irreducible continuous L^p_1 parabolic metric on $\mathcal{E}$, Hermitian-Einstein with respect to an orbifold metric on $X = (X, D, \underline{n})$, and in the same class as that of Y.)*

3

Suppose that Y is a smooth Kähler surface with an effective divisor D. A general definition of what it means for a bundle to be parabolic along D is given in [21]. We take a simpler case where D is made up of smooth irreducible curves each embedded in Y and with only normal crossings (of precisely two at any crossing) but generalize a little and consider bundles on orbifolds parabolic along such divisors. (This is helpful in an inductive proof, for then one can handle a single embedded divisor at a time.) There are two possible conventions: the *positive* one, where weights are in the range $[0, 1)$ and representations of $\Delta/\mathbf{Z}_{n_i}$ are classified by an integer in $\{0, 1, \dots, k-1\}$, and the *balanced* one with weights in $[-1/2, 1/2)$ and representations classified by an integer in the set $\{-[k/2], \dots, [(k-1)/2]\}$.

Definition 3.1

(a) A bundle with a *quasi-parabolic structure* along $D = \Sigma_1 \cup \dots \cup \Sigma_d$ is a bundle E together with a filtration $E = F_1^{(i)} \supset F_2^{(i)} \supset \dots \supset F_{l_{i+1}}^{(i)} = 0$ along Σ_i such that on $\Sigma_i \cap \Sigma_j$ these have common refinement.

(b) A (positive) *weighted structure* along D consists of a quasi-parabolic structure and weights $\{\lambda_j^{(i)}\}$, $1 \leq j \leq l_i$, for each i such that

$$0 \leq \lambda_1^{(i)} < \lambda_2^{(i)} < \dots < \lambda_{l_i}^{(i)} < 1.$$

The weights satisfy $-\frac{1}{2} \leq \lambda_1^{(i)} < \lambda_2^{(i)} < \dots < \lambda_{l_i}^{(i)} < \frac{1}{2}$ in the case of a balanced weighted structure.

(c) If E is a holomorphic bundle and the filtration along each Σ_i is holomorphic, then the bundle will be called *parabolic* (along D). Often the bundle will simply be smooth, when it will be called *weighted*.

For notational reasons it is sometimes convenient to have a full set of weights and write $0 \leq \lambda_1^{(i)} \leq \lambda_2^{(i)} \leq \dots \leq \lambda_l^{(i)} < 1$ with each repeated according to the rank of the corresponding factor of the associated graded bundle—in the balanced case similarly.

Definition 3.2 If each weight λ_i^s is rational and $\mathcal{E}$ is holomorphic we say the bundle $\mathcal{E}$ is a *rational* parabolic one.

In this case, let n_i be the lowest common multiple of the denominators along the embedded surface Σ_i. Then we can associate an orbifold X to $(Y; D; \underline{n})$ as in Definition 1.3.

We can also carry back a bundle, rational parabolic along D and with weights whose denominators on Σ_i divide n_i, to obtain a V-bundle over X. The simplest case is that of a line bundle; as explained in [12] and [18] it is a generalization of the construction of the line bundle of a divisor—indeed, if we took $\lambda = 1$ then the line bundle we would get from $\mathcal{L}$ would be $\mathcal{L}(-D)$.

Suppose $D = \Sigma_1 \cup \dots \cup \Sigma_d$ is such a divisor and $n_1, \dots, n_d$ are associated positive integers. We can think of D as a divisor on X and take the corresponding line bundle $\mathcal{L}_D^X = \mathcal{L}_{\Sigma_1}^X \otimes \dots \otimes \mathcal{L}_{\Sigma_d}^X$ defined as usual, so that $(\mathcal{L}_{\Sigma_i}^X)^{n_i} = \mathcal{L}_{\Sigma_i}^Y$, a genuine line bundle on Y. If (U_α), $\alpha \in A$, is an orbifold atlas, take $g_{\alpha\beta} = 1$ if $|U_\alpha| \cap D = \varnothing = |U_\beta| \cap D$. If $|U_\alpha| \cap D \neq \varnothing$ and of type (ii) but $|U_\beta| \cap D = \varnothing$ then we have $U_\alpha = \{\pi_\alpha \colon \Delta \times \Delta \xrightarrow{p} \Delta \times \Delta/\mathbf{Z}_{n_i} \to Y\}$, $U_\beta = \{\pi_\beta \colon \Delta \times \Delta \to Y\}$, and $D \cap |U_\alpha|$ given by $z = 0$. Suppose that $\theta \colon \pi_\beta^{-1}(|U_\alpha| \cap |U_\beta|) \to \tilde{U}_\alpha$ is an inclusion of charts; then take $g_\theta \colon \pi_\beta^{-1}(|U_\alpha| \cap |U_\beta|) \to \mathbf{C}^*$ to be the function $z \circ \theta$, because z is a local defining function. (Similarly for the other charts; for example, if U_γ is another chart like U_α such that $\big| |U_\gamma| \cap \Sigma_i \big| \neq \varnothing$ and $\phi \colon \pi_\gamma^{-1}(|U_\alpha| \cap |U_\gamma|) \to \tilde{U}_\alpha$ is a morphism of charts then $g_\phi \colon \pi_\gamma^{-1}(|U_\alpha| \cap |U_\gamma|) \to \mathbf{C}^*$ is the function $(z \circ \phi) \cdot w^{-1}$, where $w \colon \tilde{U}_\gamma \to \mathbf{C}$

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defines D. Because both only vanish to first order the quotient is in $\mathbf{C}^*$.) In this way we manufacture a line bundle $\mathcal{L}_D^X$ over X; it is a V-bundle because on U_α above we have

$$\begin{array}{ccc} \Delta \times \Delta \times \mathbf{C} & \longrightarrow & \Delta \times \Delta \\ \downarrow & & \downarrow \\ \frac{\Delta\times\Delta\times\mathbf{C}}{\mathbf{Z}/n} & \longrightarrow & \frac{\Delta\times\Delta}{\mathbf{Z}/n} \end{array}$$

where $\mathbf{Z}/n$ acts on $\mathbf{C}$ with the generator multiplying by $\zeta = e^{2\pi i/n}$ in order that the transition function $g_\theta\colon \pi_\beta^{-1}(|U_\alpha| \cap |U_\beta|) \to \mathbf{C}^*$ be $z \circ \theta$.

Suppose that $\mathcal{L}$ is a parabolic line bundle with weight $\lambda_i = x_i/n_i$ along Σ_i.

Definition 3.3 $\varphi\mathcal{L} = \mathcal{L} \otimes \left(\mathcal{L}_{\Sigma_1}^X\right)^{x_1} \otimes \cdots \otimes \left(\mathcal{L}_{\Sigma_d}^X\right)^{x_d}$.

Proposition 3.4 $\varphi\colon \operatorname{Pic}_{\mathbf{n}}^{\mathrm{Par}}(Y, D) \cong \operatorname{Pic}^V(X)$.

The proof is as for Riemann surfaces [12] and as in [18]. The definition and proposition hold in each convention, but starting from the same underlying bundle one may well get a different V-bundle, and conversely. When we need to distinguish we shall put *dashes* on the functors for the *balanced* convention.

To extend to rank l bundles we shall work with cocycle representatives. Recall that D is a divisor with normal crossings such that each Σ_i is smoothly embedded. Suppose that we have a covering $(|U_\alpha|)_{\alpha\in A}$ of Y and suppose $\mathcal{E}$ is given by 'transition' functions $g_{\alpha\beta}$, where $g_{\alpha\beta}$ lies in the parabolic subgroup defined by the (partial) flags along D at points on D. (So we are insisting that the local trivializations respect the flags.) On X we take the orbifold covering $(U_\alpha)_{\alpha\in A}$ and declare that $g_{\alpha\beta}^X = g_{\alpha\beta}$ if $|U_\alpha| \cap D = \varnothing = |U_\beta| \cap D$. If, however, $|U_\alpha| \cap D \subset \Sigma_i - D \cdot D$ and $|U_\beta| \cap D = \varnothing$ then for $\theta\colon \pi_\beta^{-1}(|U_\alpha| \cap |U_\beta|) \to \tilde{U}_\alpha$ we take (where z is a coordinate on $\tilde{U}_\alpha$ defining Σ_i)

$$g_\theta^X = \begin{pmatrix} (z\theta)^{x_1} & & \\ & \ddots & \\ & & (z\theta)^{x_l} \end{pmatrix} \cdot g_\theta = c_\theta g_\theta.$$

In the other case if $|U_\alpha| \cap D \subset \Sigma_i - D \cdot D$, $|U_\gamma| \cap D \neq \varnothing$ too and $\phi\colon \pi_\gamma^{-1}(|U_\alpha| \cap |U_\gamma|) \to \tilde{U}_\alpha$ then we set

$$g_\phi^X = \begin{pmatrix} (z\phi)^{x_1} & & \\ & \ddots & \\ & & (z\phi)^{x_l} \end{pmatrix} \cdot g_\phi \cdot \begin{pmatrix} f^{-x_1} & & \\ & \ddots & \\ & & f^{-x_l} \end{pmatrix},$$

where f is a coordinate function defining Σ_i in U_γ. If we are in the situation where $|U_\alpha| \cap D \cdot D \neq \varnothing$ and $|U_\beta| \cap D \subset \Sigma_i - D \cdot D$ with $\theta\colon \pi_\beta^{-1}(|U_\alpha| \cap |U_\beta|) \to \tilde{U}_\alpha$ then we set

$$g_\theta^X = \begin{pmatrix} (z\theta)^{x_1}(w\theta)^{y_1} & & \\ & \ddots & \\ & & (z\theta)^{x_l}(w\theta)^{y_l} \end{pmatrix} \cdot g_\theta \cdot \begin{pmatrix} f^{-x_1} & & \\ & \ddots & \\ & & f^{-x_l} \end{pmatrix},$$

where f defines Σ_i in U_β. Notice that the 'cocycle' condition is still satisfied and that, in fact, we get a V-bundle $\tilde{\mathcal{E}} = \varphi(\mathcal{E})$ over X. It is tedious to check that if we have two altases and cocycles defining the same $\mathcal{E}$ then the corresponding new cocycles define the same $\varphi(\mathcal{E})$.

The correspondence works backwards, so that—just as for line-bundles—we have a correspondence between V-bundles over X and parabolic bundles over Y with rational weights whose denominator divide $\underline{n}$. The topological type of the V-bundles might be quite distinct—as the case of elliptic surfaces shows.

The above maps c_θ for the morphisms θ are referred to as *clutching* maps. Suppose the curve Σ_i is defined by a holomorphic section s_i of the associated line bundle $\mathcal{L}_i$ on Y and $\mathcal{L}_i^X$, $\tilde{s}_i$ are those on X, so that $\left(\mathcal{L}_i^X\right)^n = \mathcal{L}_i$ and $(\tilde{s}_i)^{n_i} = s_i$, where n_i is the isotropy order of Σ_i. Choose a hermitian metric on $\mathcal{E}$ and so decompose E over each Σ_i, and then over $\nu(D)$, into $E|\nu(D) = p^*(E_1 \oplus \cdots \oplus E_l)$, where $E_1 \oplus \cdots \oplus E_l = G(E)$ is the associated graded bundle to the filtered bundle $E|_D$ and $p\colon \nu(D) - D$ is projection. Form

$$E|(X-D) \cup_\psi \tilde{p}^*(E_1) \otimes (L_x)^{x_1} \oplus \cdots \oplus \tilde{p}^*(E_l) \otimes (L_X)^{x_l}$$

by taking $\psi(x) = \operatorname{diag}(\tilde{s}^{x_1}, \ldots, \tilde{s}^{x_l}) \circ \varphi(x)$, where

$$\phi\colon E|\big(\nu(D) - D\big) \to p^*E_1 \oplus \cdots \oplus p^*E_l|\big(\nu(D) - D\big)$$

is the clutching map for E, $\tilde{p}\colon \tilde{\nu}(D) \to D$ is the projection of the tubular neighbourhood in X, and $(L_X)^{x_i} = \prod_1^d \left(L_j^X\right)^{x_i(j)}$ with each factor raised to the i-th index corresponding to the j-th smooth component. This construction, *modified by replacing $\tilde{s}$ by $\tilde{s}/|\tilde{s}|$*, i induces a map $\bar{\varphi}$, used in Sections 5 and 7, from the parabolic unitary connexions on Y to standard ones on X. It will be called *unitary clutching* and is defined by $\bar{\varphi}(\bar{\partial}_E)|_{U_\alpha} = b_\alpha \bar{\partial}_E b_\alpha^{-1}$ on U_α, where

$$b_\alpha(x) = \operatorname{diag}\left(\frac{\tilde{s}_a^{x_1}(x)}{|\tilde{s}_a^{x_1}(x)|}, \ldots, \frac{\tilde{s}_a^{x_l}(x)}{|\tilde{s}_a^{x_l}(x)|}\right) \tag{3.5}$$

and $\tilde{s}_\alpha = \tilde{s}|U_\alpha$ is a representative function for $\tilde{s}$ over U_α in some local trivialization of L.

If $\mathcal{E}$ and $\mathcal{E}'$ are two parabolic bundles a morphism $\mathcal{E} \xrightarrow{f} \mathcal{E}'$ is required to be filtration preserving and such that the weights along each curve coincide: if $\mathcal{E} = \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots \supset \mathcal{F}_{r+1} = 0$ along Σ with weights $0 \le \lambda_1 < \cdots < \lambda_r < 1$ and $\mathcal{E}' = \mathcal{F}_1' \supset \mathcal{F}_2' \supset \cdots \supset \mathcal{F}_{s+1}' = 0$ along Σ with weights $0 \le \mu_1 < \cdots < \mu_s < 1$ then require that

(1) $f(\mathcal{F}_i) \subset \mathcal{F}_j'$ if $\lambda_i \ge \mu_j$;

(2) $f(\mathcal{F}_i) \subset \mathcal{F}'_j = 0$ unless λ_s weight of $\mathcal{F}'_j$ for some $s \leq i$.

Given a map $f\colon \mathcal{E} \to \mathcal{E}'$ we can think of it as a section s of $\mathrm{Hom}(\mathcal{E}, \mathcal{E}')$, so we need only discuss the case when $\mathcal{E}$ is trivial line bundle with weight 0. Then, taking a cover (U_α), we have $s_\alpha \in \Gamma(U_\alpha, \mathbf{C}^r)$ such that $g_{\alpha\beta} s_\beta = s_\alpha$ where $g_{\alpha\beta}$ are transition functions for $\mathcal{E}'$. Then we get $t \in H_0(\varphi E')$ by procedure above: namely

$$t_\alpha = s_\alpha \quad \text{if } |U_\alpha| \cap D = \varnothing;$$

$$\begin{pmatrix} z^{x_1} & & \\ & \ddots & \\ & & z_{x_l} \end{pmatrix} s_\alpha \quad \text{if } |U_\alpha| \cap D \subset \Sigma_i; \quad \text{and so on.}$$

Thus we get an identification of morphisms too. If, then, we can define degree so that $\deg_V$ and par deg coincide we will have a correspondance between stable bundles as well as we already have a Kähler class. For V-bundles we already have Chern classes defined in various ways—using connexions and differential forms, for example. For parabolic bundles a definition is required. Essentially this is chosen so as to make $c_i(\mathcal{E}) = c_i(\varphi\mathcal{E})$ as classes in $H^n(Y, \mathbf{R})$. It is given in the next definition, where we denote Poincaré duals by stars.

Definition 3.6 If the bundle $\mathcal{E}$ is parabolic along D with weights $\lambda_1, \dots, \lambda_d$ then

$$\begin{cases} \operatorname{par} c_1(\mathcal{E}) = c_1(\mathcal{E}) + \sum_{i=1}^d \sum_{j=1}^l \lambda_j^{(i)} [\Sigma_i]^* \\ \operatorname{par} c_2(\mathcal{E}) = c_2(\mathcal{E}) + c_1(\mathcal{E}) \cdot \{\sum_{i=1}^d \sum_{j=1}^l \lambda_j^{(i)} [\Sigma_i]^*\} \\ \qquad\qquad - \sum \lambda_j^{(i)} d_j^* + \sum_{i \neq s}^d \sum_{j \neq k}^l \lambda_k^{(s)} \lambda_j^{(l)} (\Sigma_s \cdot \Sigma_i)^*. \end{cases}$$

Proposition 3.7 par $c_i(\mathcal{E}) = c_i(\varphi\mathcal{E})$*, and this holds in both conventions.*

Proof For line bundles this follows exactly as in [12]. Now one checks for 2-plane bundles and the general case follows by using the splitting principle or arguments of [12, p. 400].

Theorem 3.8 *The correspondence $\mathcal{E} \to \varphi\mathcal{E}$ and $s \to \varphi s$ gives a bijective correspondence between $V_{\mathrm{par}}(Y, D, \underline{n})$ and $V(X, D, \underline{n})$. Moreover degree coincides and a parabolic bundle $\mathcal{E}$ is stable if and only if $\varphi\mathcal{E}$ is. (Here, $V_{\mathrm{par}}(Y, D, \underline{n})$ denotes the bundles rational parabolic along D with denominators dividing $\underline{n}$. In fact, the functors φ and its inverse, ϑ, define an equivalence of categories [38].)*

In [18], and also [38], there is a more sophisticated description of this in terms of sheaves.

It is much simpler to handle just one smooth divisor at a time by extending the above definitions to orbifolds. Suppose that Y is an abelian orbifold surface (Section 1) with underlying smooth surface $|Y|$ and with the singularity set $\Sigma_Y = S$ made up of smoothly embedded curves with normal crossings as described (1.3). If Σ is a curve in Y meeting the divisor S normally and missing $S \cdot S$, think of Σ as an orbifold curve with multiplicity n_i at any point P where Σ intersects $\Sigma_i \subset S$. We can talk about V-bundles with a parabolic structure along Σ and the argument above generalizes to such orbifolds. (The sole difference is that the weights along Σ_i have to lie in the interval $[0, \frac{1}{n_i}]$.)

Theorem 3.9 *If Y is an abelian orbifold surface with Σ_Y the union of smooth divisors with normal crossings and if $D \subset Y$ is a similar divisor meeting Σ_Y normally then there is a bijective correspondance between $V(X)$ and $V_{\text{par}}(Y, D, \underline{n})$.*

The functor $\mathcal{E} \to \varphi\mathcal{E}$ and its inverse ϑ have good properties with respect to cotangent bundles, though the analogous functors defined by the *balanced convention*—denoted here by *dashes*—do not. (The relation between the two is explained in [18]: if $\mathcal{E}$ is a V-bundle then $\vartheta'(\mathcal{E}) = \vartheta(\mathcal{E}) \otimes \mathcal{L}_{\Sigma_1}^{-[\frac{n_1}{2}]} \otimes \cdots \otimes \mathcal{L}_{\Sigma_d}^{-[\frac{n_d}{2}]}$ as bundles, with translated weights. Similiarly, if $\mathcal{F}'$ is a balanced convention parabolic bundle and $\mathcal{F}$ the same quasi-parabolic bundle but with weights shifted by $[\frac{n_i}{2}]$, then $\varphi'(\mathcal{F}') = \varphi(\mathcal{F}) \otimes \mathcal{L}_{\Sigma_1}^{-[\frac{n_1}{2}]} \otimes \cdots \otimes \mathcal{L}_{\Sigma_d}^{-[\frac{n_d}{2}]}$.)

Proposition 3.10 *$\vartheta\Lambda_X^p\mathcal{T}_X^* = \Lambda^p\mathcal{T}_Y^*$ with quasi-parabolic structure $\mathcal{T}|\Sigma_i \supset \nu^* \supset 0$ along Σ_i and weights $(0, \frac{n_i-1}{n_i})$ if $p = 1$, and weights $(\frac{n_i-1}{n_i})$ if $p = 2$.*

The proof is straightforward.

Corollary

(1) $H^i(X, \mathcal{E}) \cong H^i(Y, \vartheta\mathcal{E})$;
(2) $H^i(X, \mathcal{L}^*) \cong H^{2-i}(X, \mathcal{K} \otimes \mathcal{L})^*$ for a line bundle $\mathcal{L}$.

The result shows that, as expected, the dimension of moduli spaces is unchanged by changes in the weights—provided that the pattern of inequalities is preserved.

4

The correspondence of Section 3 extends to connexions and metrics as it does for Riemann surfaces [28, p. 141], provided that one allows certain singularities (or degeneracies); in the case of a connexion it is a logarithmic singularity. The connexions and metrics obtained upon carrying across the standard orbifold definitions are called *parabolic* connexions and metrics. (In order to carry out the correspondence in the smooth category the bundle has to be hermitian. The bundles obtained from the two constructions—holomorphic and hermitian—are topologically isomorphic but the local trivializations are not related by smooth transformations so that, in particular, the $\bar{\partial}$-operator is smooth for one but singular for the other, a situation seen in the b-calculus of Melrose [23].) Recall that we write (z, w) for coordinates on the orbifold and (u, v) for the corresponding ones on the smooth surface.

Definition 4.1

(a) A *parabolic hermitian metric* is a hermitian one over $Y - D$ but which has the form

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(i) $\begin{pmatrix} |u|^{2\lambda_1^i} & & \\ & \ddots & \\ & & |u|^{2\lambda_l^i} \end{pmatrix}$ near $P \in \Sigma_i - D \cdot D$, and

(ii) $\begin{pmatrix} |u|^{2\lambda_1^i}|v|^{2\lambda_1^j} & & \\ & \ddots & \\ & & |u|^{2\lambda_l^i}|v|^{2\lambda_l^j} \end{pmatrix}$ near $P \in \Sigma_i \cdot \Sigma_j$

in some smooth, but probably not holomorphic, local trivialization respecting the flags; where Σ_i is defined by $u = 0$ and $\Sigma_i \cup \Sigma_j$ by $uv = 0$.

(b) A *parabolic gauge transformation* is an automorphism which preserves the parabolic metric.

(c) A *parabolic unitary connexion* is one which respects the flag along each Σ_i, has a smooth $\bar{\partial}$-operator of square 0 and respects the parabolic metric.

(d) A *weighted hermitian metric* on a weighted bundle E (Definition 3.1 (c)) is defined as in (a) and a bundle with such a structure is called *weighted hermitian*. Weighted unitary maps are similarly defined, as are weighted unitary connexions.

(e) A frame near $P \in \Sigma_i - D \cdot D$ is *(parabolic or weighted) unitary* if there the metric has the standard form

$$\begin{pmatrix} |u|^{2\lambda_1^i} & & \\ & \ddots & \\ & & |u|^{2\lambda_l^i} \end{pmatrix}$$

in some holomorphic coordinates (u, v), where Σ_i is defined by $u = 0$. (So this depends upon the coordinates.)

(f) A *weighted (or parabolic) unitary connexion* is one which respects the flag along each Σ_i, has a smooth $\bar{\partial}$-operator of square 0 and respects the *weighted* metric.

(g) Given a *genuine* hermitian metric on E, a bundle automorphism is in L^p_k if it is in the usual sense. An L^p_k-parabolic hermitian metric is one obtained from type (a) by an L^p_k-automorphism. (Usually this will be applied when $k - \frac{4}{p} > 0$ so that the automorphisms form a group.)

(h) An L^p_k-weighted unitary connexion is one where the $\bar{\partial}$-operator is L^p_k, again with respect to a genuine hermitian metric. (A better definition is probably to say it is L^p_k if it differs from a smooth weighted connexion by something in L^p_k.)

A parabolic hermitian metric can be constructed using the defining sections $s_i \in O(\mathcal{L}_{\Sigma_i})$, together with a hermitian metric on $\mathcal{L}_{\Sigma_i}$ and compatible decompositions of E along each Σ_i; decompositions which are extended to $\nu(D)$, a small tubular neighbourhood of D. Then, if h is a genuine metric on E such that the decompositions are perpendicular, define a parabolic one by taking over $\nu(D)$ the metric

$$h(z) \cdot \begin{pmatrix} \prod_i |s_i|^{2\lambda_1^i} & & \\ & \ddots & \\ & & \prod |s_i|^{2\lambda_l^i} \end{pmatrix}.$$

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Thus we have the first assertion of the next proposition, which is set in the holomorphic category. (*Unitary* clutching (3.5) is appropriate in the hermitian one and produces a *genuine* hermitian bundle (with weights) over Y: the metric is not weighted; it is only so when seen through holomorphic eyes.)

Proposition 4.2

- *(i) If $\mathcal{E}$ is a parabolic bundle over Y, with parabolic structure along D, then $\mathcal{E}$ admits a parabolic hermitian metric. Moreover,*
- *(ii) if h is a parabolic hermitian metric on $\mathcal{E}$ then the corresponding metric on $\vartheta(\mathcal{E})$ is a genuine hermitian metric. However,*
- *(iii) if $\tilde{h}$ is a (genuine) hermitian metric on $\tilde{\mathcal{E}} = (\mathcal{E})$ then $\hat{h}$, the induced metric on the associated parabolic bundle $\mathcal{E} = \varphi(\tilde{\mathcal{E}})$, is not necessarily a parabolic one as it may only be in L_1^p, for $2 \le p \le \frac{2}{(1-2/n)}$. But,*
- *(iv) if $\tilde{h}$ is Hermitian-Einstein with respect to a Kähler metric w on Y then $\hat{h}$ is in L_2^p for the same p.*

Proof As the first claim is proved we turn to the second.

(ii) Let $\{e_i'\}$, $1 \le i \le l$, be a local frame respecting the extension and $\{e_i\}$ the corresponding one on $\tilde{\mathcal{E}} = \vartheta(\mathcal{E})$. Let g be be the map carrying the frame $\{e_i'\}$ to a parabolic unitary frame $\{f_i\}$. So g is smooth in the coordinates (u, v) on Y and diagonal on the intersection of the domain of the coordinates with D. A short calculation shows that, when the intersection lies in a single embedded curve Σ defined by $u = 0$, the form $\langle e_i, e_j\rangle = \langle e_i', e_j'\rangle u^{-\lambda_i}\bar{u}^{-\lambda_j}$ is non-degenerate, smooth in $z = u^{\frac{1}{n}}$ and so defines a hermitian metric. (The case of a crossing is no different.)

(iii) Take again a local holomorphic frame respecting the decomposition along Σ. We may suppose that $\langle e_i, e_j\rangle$ vanishes on Σ if $i \ne j$. Then if $\{e_i'\}$ is a corresponding frame for $\mathcal{E} = \varphi(\tilde{\mathcal{E}})$ we have again $\langle e_i, e_j\rangle = \langle e_i', e_j'\rangle u^{-\lambda_i}\bar{u}^{-\lambda_j}$. Let g be an upper triangular matrix carrying $\{e_i\}$ to a unitary frame $\{f_i\}$. Then $\langle e_i | e_j\rangle = (g^*g)_{i,j}$ and so $\langle e_i' | e_j'\rangle = \hat{g}^* q \hat{g}$, where $\hat{g} = cgc^{-1}$, $q = (cc^*)^{-1}$ and $c = \begin{pmatrix} u^{-\lambda_1} & & \\ & \ddots & \\ & & u^{-\lambda_l} \end{pmatrix}$. (Here interpret $u^{-\lambda_i}$ as z^{-x_i}, where $\lambda_i \cdot n = x_i$.)

The map $\hat{g}$ is upper triangular and smooth in the coordinates (z, w) on X but probably not in coordinates (u, v) on Y. It could perfectly well have a term like $|u|^{\frac{2}{n}}$ in its expression so that the most that we could hope in general is that, if $2 \le p \le \frac{2}{(1-2/n)}$, $\hat{g}$ should be in L_1^p (with respect to a *standard* hermitian metric) near Σ. It is, however, continuous. But if we know that all the derivatives, of order up to $n-1$, of g in directions perpendicular to Σ vanish on Σ then we may deduce that $\hat{g} \in L_2^p(Y)$.

For the final clause (iv) we lift the bundle to $X' = X^{(2)} = (Y, D, 2\underline{n})$. (This is for convenience because it means that on D equivariant transformations will differ from diagonal ones by something of order $|z|^2$.) We give the result as a separate lemma.

Lemma 4.3 *If a metric $\tilde{h}$ is Hermitian-Einstein on $\tilde{\mathcal{E}}$ with respect to a* Kähler metric *w on Y and smooth (or indeed just in $L_3^p(X)$) with respect to a Kähler metric on X, then the associated metric on $\mathcal{E}$ over Y can be taken in $L_2^p(Y)$.*

Proof We shall argue on X', and the bundle $\tilde{\mathcal{E}}$ will be thought of over X'. As usual, we work locally with cocycles $\Phi = \{\varphi_U\}_{U\in \mathrm{U}}$ for a covering U and take local coordinates (z, w) on X'. As above, (part (iii)), let g be an upper triangular matrix carrying a local holomorphic frame for $\tilde{\mathcal{E}}$, say $\{e_i\}$, to one $\{f_i\}$ unitary with respect to the metric $\tilde{h}$. Suppose to start with that g is smooth. Equivariance implies that

$$g(z, w) = C + Bz\bar{z} + Az^2 + A'\bar{z}^2 + D(z, \bar{z}),$$

where D has order greater than 2 in z, C is diagonal and A, A', C are triangular matrices which are functions of w and $\bar{w}$ alone. The connexion on X' has, locally, the form

$$d + (gg^*)^{-1}\partial(g^*g) = d + Gdz + Hdw,$$

and the curvature is then

$$F = \bar{\partial}\{(g^*g)^{-1}\partial(g^*g)\} = \frac{\partial G}{\partial\bar{z}}d\bar{z}\wedge dz + \frac{\partial G}{\partial\bar{w}}d\bar{w}\wedge dz + \frac{\partial H}{\partial\bar{z}}d\bar{z}\wedge dw + \frac{\partial H}{\partial\bar{w}}d\bar{w}\wedge dw,$$

since the connection is integrable.

Locally the Kähler form can be written

$$w = \gamma|z|^{4n-2}dz\wedge d\bar{z} + \beta z^{2n-1}dz\wedge d\bar{w} + \beta'\bar{z}^{2n-1}d\bar{z}\wedge dw + \delta dw\wedge d\bar{w},$$

where γ, δ and $\gamma\delta + \beta\beta'$ do not vanish on Σ, and the volume form is

$$(\gamma\delta + \beta\beta')|z|^{4n-2}dz\wedge d\bar{z}\wedge dw\wedge d\bar{w} = \mathrm{vol}\,.$$

Because the connexion is Hermitian-Einstein $F\wedge w = \lambda\,\mathrm{vol}$. Consequently $F\wedge w = O(|z|^{4n-2})$:

$$\delta\frac{\partial G}{\partial\bar{z}} + \bar{z}^{2n-1}\beta'\frac{\partial G}{\partial\bar{w}} - z^{2n-1}\beta\frac{\partial H}{\partial\bar{z}} + |z|^{4n-2}\gamma\frac{\partial H}{\partial\bar{w}} = \lambda|z|^{4n-2}.$$

This tells us that $\frac{\partial G}{\partial\bar{z}}$ vanishes to order $2n-1$ on Σ. If, for fixed w, the partial Taylor expansion of the connexion with respect to the variable z is written

$$d + \Big\{\sum_{p+q<2n} a^{p,q}z^p\bar{z}^q + o(|z|^{2n-1})\Big\}dz + Hdw,$$

then $(a^{p,q})_{i,j} = 0$ unless $q = 0$ and $p \equiv 2(i-j) - 1 \mod 2n$. Hence the connexion has the form

$$d + \sum_{p=1}^{2n-1} a_p z^p dz + o(|z|^{2n-1})dz + Hdw,$$

where the terms $a_p(w)$ are given by an integral formula

$$a_p(w) = \frac{1}{2\pi i}\lim\oint_{|\zeta|=r} G(\zeta, w)\zeta^{-p-1}d\zeta, \quad \text{as } r\to 0,$$

and are thus as smooth as G is. So the connexion would vanish to order $2n-1$ if the polynomial in z were not there. Now we use the following lemma, of which proof later.

Lemma 4.4 *There is a change of frame, given by a matrix which is upper triangular in terms of the old frame (as in Proposition 4.2 (iii)) and of which the entries are polynomial in z with coefficients as differentiable as G (so in $L_k^p(w)$ if G is in L_k^p), such that the connexion has locally the form*

$$d + O(|z|^{2n-1})dz + bdw + cd\bar{w}.$$

From the above discussion and the lemma one can see that in the new frame the curvature is locally

$$F = O(|z|^{2n-1})dz \wedge d\bar{z} + \text{terms involving } dw \text{ or } d\bar{w},$$

and further, upon taking a partial Taylor expansion in terms of z and $\bar{z}$ for the transformation carrying this frame to a unitary one for h, the lemma tells us that there is a change of frame such that the resulting transformation, a composition $\tilde{g}$ say, has the form $\tilde{g} = C + O(|z|^{2n-1})$ in terms of z. Since g is already triangular we deduce from this and the remark at the end of the first paragraph of Proposition 4.2 (iii) that $\hat{\tilde{g}} \in L_2^p$ for $p < 2 + \frac{1}{2n-1}$. Taking each Σ_i into account, we have $\hat{\tilde{g}} \in L_2^p(Y)$ for $p < 2 + \frac{1}{2n-1}$, where $N = \max(n_1, \dots, n_d)$. The curvature F of the Hermitian-Einstein metric on the holomorphic V-bundle $\tilde{\mathcal{E}}$ is smooth or in L_{2n-2}^p, so because $\hat{\tilde{g}} \in L_2^p$ the corresponding metric on the parabolic bundle $\mathcal{E} = \varphi(\tilde{\mathcal{E}})$ is indeed an L_2^p-parabolic one (with respect to a standard metric) and the curvature is in L^p. So we have Lemma 4.3 in the smooth case. The argument goes through (over X') in much greater generality, because once we have that $\frac{\partial G}{\partial \bar{z}}$ exists and is continous we know that it is of order $2n$ and so has derivatives, vanishing on Σ, of orders up to $2n-1$. Stokes' theorem now gives an expansion of the same order in z as above, so that if the tranformation is only in L_3^p then we may still conclude the existence of transformation in L_2^p on Y.

Proof of 4.4 Look for a holomorphic (in z) automorphism $h(z) = A_0 + B_0 z + \cdots$, such that

$$h^{-1}\partial_z h = -\sum_{1}^{2n-1} a_p z^p dz + \text{higher terms}.$$

Solve the equation successively, starting off with $h = I - (\frac{1}{2})a_1 z^2$. Then

$$\begin{aligned} h^{-1}\partial_z h &= \left(I + \left(\frac{1}{2}\right) a_1 z^2 + \left(\frac{1}{4}\right) a_1^2 + \cdots\right)(-a_1 z)dz \\ &= \left(-a_1 z - \left(\frac{1}{2}\right) a_1^2 z^3 + \cdots\right) dz. \end{aligned}$$

Now solve for $h = I - (\frac{1}{2})a_1 z - (\frac{1}{4})b_3 z^3$. Equating coefficients we find that $b_3 = a_3 - (\frac{1}{2})a_1^3$. Similarly, adding a term $-(\frac{1}{6})b_5 z^6$ to h we can calculate the value of b_5

in terms of the a's and the lower b's and so on up to order $2n-2$. From the formula above for the coefficients $a_p(w)$, we see that each is as differentiable as G is and in the same L^p_k-space, $k > 2n-2$ in this case.

Though at the moment one can only see, on a stable parabolic bundle, a continous L^p_1-parabolic metric, Hermitian-Einstein with respect to the *orbifold* metric, in Section 5 we prove the following result using Lemma 4.3.

Theorem 4.5 *Let $\mathcal{E}$ be a stable parabolic bundle on Y, parabolic along the divisor D with normal crossings, and with rational weights. Suppose that the denominators divide $\underline{n}$. Then there is, for some p with $2 < p < 2 + \frac{1}{(2N-1)}$, $N = \max(n_1, \dots, n_d)$, an L^p_2-parabolic Hermitian-Einstein metric, so one with curvature in L^p. (Here, L^p is with respect to a genuine metric on $\mathcal{E}$ and the Kähler metric on Y.)*

Notes 1) The full connexion form in Theorem 4.5 is in $\Omega^1(\log D)$ completed with respect to L^p_1 for $2 < p < 2 + \frac{1}{(2N-1)}$. If there are two Hermitian-Einstein L^p_2-metrics on E they will differ by an L^p_2-automorphism η and one can set $\sigma(\eta) = \operatorname{tr}\eta + \operatorname{tr}\eta^{-1} - 2r$ as in [9]. As there, $\Delta(\operatorname{tr}\eta) \le 0$ and $\Delta(\operatorname{tr}\eta^{-1}) \le 0$; hence $\Delta(\sigma)$ too. But η and so σ are in L^p_2 for some $p > 2$. Consequently one may use the formula

$$\int_Y |\nabla\sigma|^2 = \int_Y \sigma\nabla\sigma \le 0,$$

and deduce that $\nabla\sigma = 0$ and hence $\sigma = 0$.

2) Even if the $\bar\partial$-operator is smooth, the ∂-part of the connexion will, in general, be much less regular. It is determined solely by the condition that the connexion be 'unitary' and so the (i, j)-th entry has terms of the order of $|u|^{2(\lambda_i-\lambda_j)}$.

3) In the special case when the divisor is ample we might hope to argue completely over X, using Theorem 8.7.1 on principally normal operators in [15].

4) For the case of ordinary bundles one has, once a hermitian metric has been fixed, an exact correspondence between holomorphic structures and orbits of the space A of integrable unitary connexions under the complexified gauge group G^c [8]. This correspondence extends in a fairly obvious way to the situation of parabolic bundles [6]. Suppose now that a decomposition of E along D, together with an extension to $\nu(D)$—by parallel translation, say—has been chosen, and a corresponding parabolic metric, as in Proposition 4.2. Then one may apply the construction $\bar\vartheta$, and its inverse $\bar\varphi$, of Section 3. Let A^w be the space of weighted integrable $(1, 1)$ connexions and $G^{c,w}$ the group of smooth automorphisms preserving the flags. The Newlander-Nirenberg theorem says that $A^w/G^{c,w}$ is just the set of holomorphic structures. (As in the usual case, if we take $\bar\partial$-operators which are simply in L^p_1 and complex transformations in L^p_2 ($p > 2$) then the correspondence extends to this.) The weighted hermitian metric gives a choice of decomposition along D. Suppose now that we are in the situation above and that the weights are all rational with denominators dividing $\underline{n}$. We can construct the orbifold X associated to $(Y, D, \underline{n})$, as in Section 1, and, extending the decomposition to $\nu(D)$ and noting Proposition 4.2, find

$$\bar\varphi\colon A^w \to A_X, \quad \bar\varphi\colon G^{c,w} \to G^c_X,$$

inducing $\bar{\varphi}\colon A^w/G^{c,w} \to A_X/G^c_X$.

Proposition 4.6 *$\bar{\varphi}\colon A^w/G^{c,w} \cong A_X/G^c_X$ is a bijection.*

Proof Here we use L^p_k on the orbifold X for some $p > 2$ and consider the composition $A^w/G^{c,w} \to A_X/G^c_X \to (A_X)^p_k/(G^c_X)^p_k$. Lemmas 14.6–14.8 of [2] apply to this situation and we may conclude the following:

(i) $dF\colon (G^c_X)^p_k \to (A_X)^p_k$ is a Fredholm map near $g = I$, where $Fg = gA$;
(ii) every $G^{c,w}$-orbit contains a smooth connexion;
(iii) if A and B are in A^w and $gA = B$ then $g \in G^{c,w}$.

The points to check are (ii) and (iii), where here we are thinking of A^w as embedded in A_X by $\bar{\varphi}$. The latter holds because $\bar{\partial}_A$ is smooth, and the essential question for (ii) is whether $\bar{\varphi}(A^w) \subset (A_X)^p_k$ is dense. It is if $3 > p > 2$ and we can conclude that $\bar{\varphi}\colon A^w/G^{c,w} \cong A_x/G^c_X$.

5

Here we prove Theorem 4.5 using results of [17], [19] and [23]. The section corresponds to Section 5b and part of Section 6 of [28] and the argument takes place over Y using weighted spaces since we no longer have equivariance to control growth (or decay). We work on a hermitian bundle with singular operators so the hermitian structure will appear as parabolic with respect to any underlying complex structure. When we say L^p_k we shall mean with respect to *this hermitian metric* so that L^p_k results will not have the meaning as in Section 4: Theorem 5.4 is different from Theorem 4.5.

Let $X = (Y, D, \underline{n})$ be an abelian Kähler orbifold surface with an underlying smooth complex surface $Y = |X|$, as in Section 1. Let $\mathcal{E} = \vartheta(\tilde{\mathcal{E}})$ be a stable rational parabolic bundle of rank l over Y with denominators dividing $\underline{n}$, where $\tilde{\mathcal{E}}$ is a stable holomorphic V-bundle with Hermitian-Einstein metric k. Suppose that $w = w_{|X|}$ is a Kähler metric on the underlying smooth complex surface $Y = |X|$ in the same class as $w_X = \tilde{w}$. Because $|w_{|X|}| = |w_X|$, the definition of stability is the same with respect to w_X as with respect to $w_{|X|}$. We show that there is a bundle automorphism carrying the metric k on $\tilde{\mathcal{E}}$ Hermitian-Einstein with respect to $\tilde{w}$ (given by Theorem 2.3) to one which is Hermitian-Einstein with respect to $w_{|X|}$: a metric which, thought of on X, degenerates along the singularities. At some point we also need $X^{(q)} = (Y, D, q\underline{n})$ with a Kähler metric $\tilde{w}^{(q)}$.

We continue to write $\tilde{w}$ for w_X, $\tilde{w}^{(q)}$ for $w_{X^{(q)}}$ and w for $w_{|X|}$ and suppose that $\tilde{w} = w + \sum_{j=1}^d (\frac{\varepsilon}{2i})\partial\bar{\partial}\beta_j\varphi_j$, with a similar relation between w and $\tilde{w}^{(q)}$, where ε is suitably small, each $\|\partial\bar{\partial}\varphi_j\|_\infty \leq 1$ and each β_j is zero outside a small neighbourhood of Σ_j. This can be achieved as follows: see [18]. Let $D = \Sigma_1 \cup \cdots \cup \Sigma_d$ where each Σ_j is embedded, any two intersect normally in at most one point, and there are no triple points. Let $\mathcal{L}^{|X|}_D = \mathcal{L}_{\Sigma_1} \cdots \mathcal{L}_{\Sigma_d}$ be the corresponding line bundle on $|X|$, and let $s_j \in O(\mathcal{L}_{\Sigma_j})$ be a section vanishing solely on Σ_j and to order 1. Let $\| \bullet \|_j$ be the norm coming from a hermitian metric on $\mathcal{L}_{\Sigma_j}$ and set $\varphi = \sum_{j=1}^d \|s_j\|_j^{\frac{2}{n_j}} = \sum_{j=1}^d \varphi_j$.

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Then φ is continuous on $|X|$ and smooth on X. Moreover each $(\frac{1}{2i})\bar{\partial}\partial\varphi_j$ is bounded below with respect to $w_{|X|} = w$: that is, $(\frac{1}{2i})\bar{\partial}\partial\varphi_j \geq -K_j w$ for some constant K_j. As $(\frac{1}{2i})\bar{\partial}\partial\varphi_j = (\frac{\varphi_j}{2i})\{F_{L_{\Sigma_j}} + \frac{\bar{\partial}\varphi_j \wedge \partial\varphi_j}{2i\varphi_j^2}\}$, this is because the first is bounded and the second is non-negative. Now set $\beta(\varphi_j)$ to be a cut-off function, equal to 1 near Σ_j and zero outside a neighbourhood of D. Then $(\frac{1}{2i})\bar{\partial}\partial(\beta_j\varphi_j)$ is still closed and for small enough ε the form $\tilde{w} = w + \sum_{j=1}^{d}(\frac{\varepsilon}{2i})\overline{\partial\partial}(\beta_j\varphi_j)$ is positive on X and non-degenerate. So it defines an orbifold Kähler metric which is in the same class. (In one approach we would write

$$
\begin{aligned}
w &= \tilde{w} - \sum_{j=1}^{d}\Big(\frac{\varepsilon}{2i}\Big)\overline{\partial\partial}\Big(\beta_j\prod_{s\neq j}(1-\beta_s)\varphi_j\Big) \\
&\qquad - \sum_{j=1}^{d}\Big(\frac{\varepsilon}{2i}\Big)\overline{\partial\partial}\Big(\beta_j\Big(1-\prod_{s\neq j}(1-\beta_s)\Big)\varphi_j\Big) \\
&= w_0 - \sum_{j=1}^{d}\Big(\frac{\varepsilon}{2i}\Big)\overline{\partial\partial}\Big(\beta_j\prod_{s\neq j}(1-\beta_s)\varphi_j\Big)
\end{aligned}
$$

and first find a metric Hermitian-Einstein with respect to w_0, and then with respect to w.)

The basic idea is to consider the limit of the Hermitian-Einstein connexions with respect to the metrics $\tilde{w}_\varepsilon = w + \sum_{j=1}^{d}(\frac{\varepsilon}{2i})\bar{\partial}\partial\beta_j\varphi_j$ as $\varepsilon \to 0$. Unfortunately, if we choose an orbifold reference metric $\tilde{w}_{\varepsilon_0}$ the curvatures are not obviously L^2-bounded with respect to it nor, on the other hand, do they lie in $L^2(Y)$.

Suppose, then, that the holomorphic V-bundle $\tilde{\mathcal{E}}$ is stable and $\tilde{A}$ is a connexion associated to the metric k on $\tilde{\mathcal{E}}$ and Hermitian-Einstein with respect to the Kähler class $\tilde{w}$ on X. We look for a metric h such that the unitary connexion A associated to h is Hermitian-Einstein with respect to w. If, following [11], we take a local holomorphic frame $(s_1, \ldots, s_r)$ and write $H_{ij} = h(s_i, s_j)$, $K_{ij} = k(s_i, s_j)$ and $\eta = K^{-1}H$, $H = K\eta$ then

$$A = H^{-1}\partial H = \eta^{-1}K^{-1}\partial(K\eta) = \eta^{-1}\partial_K\eta + K^{-1}\partial K.$$

So $\partial_H = \partial_K + \eta^{-1}\partial_K\eta$ and consequently $F_H = F_K + \bar{\partial}(\eta^{-1}\partial_K\eta)$. Let $\tilde{\Lambda}$ denote the adjoint of $\wedge\tilde{w}$ and $\tilde{*}$ the star operator for $\tilde{w}$—similarly for w and $\tilde{w}^{(d)}$. *We now suppose* $c_1\tilde{\mathcal{E}} = 0$. (This can be achieved by tensoring with a line-bundle.) In this case we need to solve the equation

$$\Lambda F_H = 0.$$

This becomes

$$\Lambda(F_K) + \bar{\partial}(\eta^{-1}\partial_K\eta) = 0.$$

Now $*(\Lambda\alpha) = \alpha \wedge w$ and $[\Lambda, \bar\partial] = -i\partial_H^*$. Therefore

$$\begin{aligned}
*\Lambda\{F_K + \bar\partial(\eta^{-1}\partial_K\eta)\} &= \{F_K + \bar\partial(\eta^{-1}\partial_K\eta)\} \wedge w \\
&= \{F_K + \bar\partial(\eta^{-1}\partial_K\eta)\} \wedge \Big\{\tilde w - \Big(\frac{\varepsilon}{2i}\Big)\sum \beta_j\partial\bar\partial\varphi_j\Big\} \\
&= -F_K \wedge \mu + \bar\partial(\eta^{-1}\partial_K\eta) \wedge \tilde w - \bar\partial\{(\eta^{-1}\partial_K\eta)\} \wedge \mu
\end{aligned}$$

writing

$$\mu = \Big(\frac{\varepsilon}{2i}\Big)\sum \beta_j\partial\bar\partial\varphi_j.$$

So we must solve the equation

$$F_k \wedge \mu = \bar\partial(\eta^{-1}\partial_K\eta) \wedge w, \tag{5.1}$$

where both μ and F_K are smooth on X (so that, off the diagonal, $F_K \wedge \mu \in L^p_{k,\delta-2}(Y)$ provided that $\delta < \frac{3}{n}$ and all entries if $\delta < \frac{2}{n}$). Applying the star operator $\tilde *$ on X does not give an elliptic operator; applying $*$ on Y does, but because the functions are multivalued we use *unitary* clutching to solve what is formally the same equation, but with F_K replaced by $F_K' = bF_Kb^{-1}$ and η by $\xi = b\eta b^{-1}$, on the associated (genuine) *hermitian* bundle $E = \bar\vartheta(\tilde{\mathcal{E}})$ on Y. (So we are at the moment using a metric on $\mathrm{End}(E)$ which is *not smooth with respect to the holomorphic structure*: in terms of a holomorphic chart it is weighted.) The $\bar\partial$-operator in (5.1)′—the transform of equation (5.1)—is now *singular*, having locally near a typical Σ (the subscript j is suppressed) the form $\bar\partial_A = \bar\partial + A + B$, where B is a matrix of 1-forms, smooth when lifted to X, and A is a diagonal matrix with entries $\frac{\alpha_i}{2}d\bar z = \frac{x_i}{2n}d\bar z$, where x_i is isotropy about the component Σ of the divisor; so x_i is the difference of 2 isotropies of $\tilde E$. More generally, for use in Section 7, we need to consider $\frac{\alpha_i}{2} = \frac{\tau_i}{2} + \frac{x_i}{2n}$, where the τ_i are weights, x_i the isotropy again and the τ_i are small.

We would like that the singular operator $\bar\partial_A + \bar\partial_A^*$, $\bar\partial_A^*\partial_A$, or $\partial_A + \partial_A^*$, for the Hermitian bundle $E = \bar\vartheta(\tilde E)$ be Fredholm on certain weighted L^p_k-spaces. Difficulties arise because applying $*$ may introduce singularities so that we can only suppose that $*F_K' \wedge \mu \in L^p_{k,\frac{2}{n}-2-\sigma}\big(Y, \mathrm{End}(E)\big)$ for $\sigma > 0$ and the spaces we are forced to use are the following. (Here, and always, X has the metric $\tilde w$, Y has w and the Hermitian bundle is E.)

Definition 5.2

$$L^p_{k,\underline\delta}(Y) = \Big\{ f \in L^p_{k,\mathrm{loc}}(Y - D) : \prod_{j=1}^d (\|s_j\|^{l_j-\delta_j-\frac{2}{p}})\nabla^l f \in L^p_k(Y);$$

$$0 \le l_j \le k, \sum_{j=1}^d l_j = l \le k\Big\}.$$

Observe that,

$$f \in L^p_{k,\underline\delta}(Y) \Longleftrightarrow f\pi \in L^p_{k,\underline{n\delta}}(X),$$

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where $\pi\colon X \to Y$ and from the Sobolev embedding we have, [5], [17], that if $f \in L^p_{k,\sigma}(Y)$ and $\underline{\delta} = \frac{2}{\underline{n}}(1 - 1/qp)$, $p > 2$, as $f\pi \in L^p_{k,2q-2/p}(X^{(q)})$ then $f\pi$ is $r = \min(k-2, 2q-2)$-times differentiable and these derivatives vanish on D.

We prove that if ε is small enough we can solve the equation (5.1) for $\xi \in L^p_{k,\delta}(Y)$, where $\underline{\delta} = \frac{2}{\underline{n}} - \underline{\sigma}$ for some $\underline{\sigma} = (\sigma_1, \dots, \sigma_d) > 0$. The value of ε is determined by the operators ∂_A^* and ∂_A which do not depend upon it. If the left-hand side is small enough this follows from the Fredholm alternative for the linearisation $*F'_K \wedge \mu = -i\partial_A^*\partial_A\xi$ if we know that the operator is Fredholm between the appropriate spaces.

Lemma 5.3 *Let F_ε denote the curvature of the Hermitian-Einstein connection on $\tilde{E}$ corresponding to the metric $\tilde{w}_\varepsilon = w - (\frac{\varepsilon}{2i})\sum_{j=1}^d \overline{\partial\overline{\partial}}\beta_j\varphi_j = w - \mu_\varepsilon$. Then in $L^p_{0,\delta-2}(Y)$ the curvatures are bounded and $\| *F'_\varepsilon \wedge \mu_\varepsilon\| \to 0$ as $\varepsilon \to 0$. (Here $\underline{\delta} = \frac{2}{\underline{n}} - \underline{\sigma}$.)*

Let Λ^- and Λ^+ denote the positive and negative spaces with respect to $w = w_{|X|}$ and let $m_\varepsilon\colon \Lambda^- \to \Lambda^+$ be the map which gives the relative position of the positive space Λ^+_ε for the metric $\tilde{w}_\varepsilon$ [11]. A straightforward computation using this map gives the lemma, because $m_\varepsilon \to 0$ pointwise as D is approached.

Theorem 5.4 *Let X be a compact V-surface with underlying space a compact Kähler manifold ($|X|, w_{|X|}$) and $\mathcal{E} = \vartheta(\tilde{\mathcal{E}})$ a rank l rational parabolic bundle over $|X|$, endowed with the parabolic metric associated to a Hermitian-Einstein one on $\theta(\tilde{\mathcal{E}})$ and stable with respect to $w_{|X|}$. Suppose that the denominators of the weights divide the orders of isotropy. Then there is a $p \geq 2$ such that $\mathcal{E}$ admits a parabolic metric which is Hermitian-Einstein with respect to $w_{|X|}$ and in $L^p_{2,\frac{2}{n}-\frac{2}{pnq}}$, measured with respect to the original parabolic metric.*

Moreover, the stable V-bundle $\tilde{\mathcal{E}}$ has a Hermitian-Einstein metric in $L^p_2(|X|)$.

Proof For line bundles the result holds by Hodge theory—as in [28]—so we may indeed reduce the general case to that when $c_1(\tilde{\mathcal{E}}) = 0$ by tensoring with a line V-bundle at the expense, perhaps, of working on an orbifold where some n_i have to be replaced by multiples.

As described above, we use unitary clutching and look for a solution to the transformed equation

$$F'_K \wedge \mu = \bar{\partial}_A(\xi^{-1}\partial_A\xi) \wedge w. \tag{5.1$'$}$$

We can proceed in several ways: directly using [23], but preferably for a p a little bigger than 2; by proving, as in [16] but using a patching argument with model the product of two copies of $\mathbf{C}P^1$ with orbifold points of isotropy n and m at ∞, that the linearized operator is Fredholm; or inductively on the number d of smooth components of D, after having first proved that there is a metric Hermitian-Einstein with respect to w_0. (Here in the last method, again an argument using the product of complex projective lines with orbifold points and separation of variables is needed.)

It is simplest to use [23] with p a little bigger than 2, but using Sobolev embedding we can in fact deduce the result from the case of $p = 2$ because once we have a solution in $L^p_{2,2q-\frac{2}{p}}(X^{(q)} - D)$ and smooth on $X^{(q)}$ we have that it is in $L^p_{2q-2}(X^{(q)})$.

The general case needs the results of [23] for manifolds with corners. However, to begin with, consider the more familiar case (A) of a *single* smooth divisor Σ, where there are no corners.

(A) Using Lemma 5.3 we first choose the parameter ε in the definition of $\tilde{w}$ sufficiently small so that then we can solve the nonlinear equation if the linear equation is Fredholm: this depends only on w and a choice of q. There is a metric k, Hermitian-Einstein with respect to $\tilde{w}$, on the bundle $\tilde{E}$ and we argue over $Y = |X|$ with $E = \bar{\vartheta}(\tilde{E})$, the associated hermitian bundle. The end of $Y - \Sigma$ is a circle bundle over the curve Σ. Let Λ be the adjoint of $\wedge w$ and $*$ the star operator. As the relation $[\Lambda, \bar{\partial}_A] = -i\partial_A^*$ is still valid for a singular connexion the linearization of $(5.1)'$ is

$$*F' \wedge \mu = -i\partial_A^* \partial_A \xi,$$

where $F' = bFb^{-1}$, the symbol F denotes the curvature of the metric k and b is the unitary clutching map of Section 2. So, as a matrix near Σ, the (r,s)-entry of ξ is $\xi_{r,s} = (\frac{z}{|z|})^{\frac{x_r}{n_i}} \eta_{r,s} (\frac{z}{|z|})^{\frac{-x_s}{n_i}}$.

We know [10], [19], [23] that for fixed A the operators $\bar{\partial}_A + \bar{\partial}_A^* \colon L^p_{k-1,\delta-1}(\Omega^{1,0}, E) \to L^p_{k-2,\delta-2}(\Omega^{1,1}E)$, $\bar{\partial}_A + \bar{\partial}_A^* \colon L^p_{k,\delta}(E) \to L^p_{k-1,\delta-1}(\Omega^{1,0}, E)$ and both $\partial_A^*\partial_A$ and $\bar{\partial}_A^*\bar{\partial}_A$: $L^p_{k,\delta}\big(\Omega^0 \operatorname{End}(E)\big) \to L^p_{k-2,\delta-2}\big(\Omega^{1,1} \operatorname{End}(E)\big)$ will be Fredholm except for an *isolated* set of weights δ. There is thus a q—as large as we wish—for which they are Fredholm when $\delta = \frac{2}{n} - \frac{2}{pqn}$. From the Fredholm alternative for $\partial_A^* + \partial_A$ we have that

$$L^p_{k-1,\delta-1}(\Omega^{1,0}E) = \operatorname{Im}(\partial_A + \partial_A^*) \oplus \operatorname{Ker} \partial_A \cap \operatorname{Ker} \partial_A^*,$$

and we can solve the equation $\gamma = *F' \wedge \mu = \partial_A^* \partial_A \xi$ for ξ in the standard way using this because the cohomology class of μ and so of $*F' \wedge \mu$ is zero, but it is simpler to use the operator $\partial_A^*\partial_A$ directly. The inverse function theorem for Banach spaces now tells us that, for $k = 2$, because we chose ε small we may solve the non-linear equation. Consequently, we have a transformation ξ solving the equation. The solution ξ is not very regular because all we know on $|X|$ is that $*F' \wedge \mu \in L^p_{2,\delta-2}(|X|)$, where $\delta = \frac{2}{n} - \frac{2}{pqn}$. However, lifted to X, $\xi - I$ will lie in $L^p_{2,2-\frac{2}{pq}}\big(\Omega^0 \operatorname{End}(E)\big)$, and lifted to $X^{(q)}$ it will lie in $L^p_{2,2p-\frac{2}{p}}$. As it is also smooth on $X^{(q)} - D$ by elliptic regularity we may suppose $\xi \in L^p_{2q}$, and vanishes, together with its first derivative, on Σ. The transformation η is still, over $X^{(q)}$, in $L^p_{2,2q-\frac{2}{p}}\big(\Omega^0 \operatorname{End}(\tilde{E})\big)$ since the clutching b is unitary, so we have Theorem 5.4.

(B) The General Case is exactly the same as case A, but using now the full force of Theorem 51 of [23, p. 224].

On the underlying space $|X|$ there will be a continuous L^p_2 parabolic transformation of $\mathcal{E}$ carrying the metric, k, to h by the additional observations on Lemma 4.3. So in the end we find a metric on $\mathcal{E}$ Hermitian-Einstein with respect to the Kähler metric on the underlying manifold Y and with curvature in $L^p(Y)$, as Theorem 4.5 claims.

6

In order to handle irrational weights we need to consider singular connexions on orbifolds—as in [28]. We use L^p_k-spaces over an orbifold defined using a *genuine* hermitian metric on the bundle and an underlying *orbifold* Kähler metric. (It might be more natural to use weighted spaces as in [6], [17], [25], [29] and as forced in the last section, but this choice gives us the advantage of a compact base space.) We follow [28], with a slight modification, and fix an integer $k_0 > 3$; though if we are working with $p = 2$ we shall take $k_0 > 4$ so as to be in L^2_3 at least. We exploit the fact that, for weighted bundles with a fixed underlying bundle filtered over D, proximity can be measured by the difference in weights, but (when the weights are rational) work on orbifolds where the corresponding bundles will be topologically different.

Let E be a weighted hermitian bundle of rank l with weights $0 \le \lambda^{(i)}_1 \le \cdots \le \lambda^{(i)}_l < 1$, as in 3.1(e), if we are using the positive convention. Extend the decomposition of E over D to $\nu(D)$. Take a coordinate z—appropriate to the weighted hermitian metric—such Σ_i is given locally by $z = 0$. Now take a representative cocycle for E, choose rational weights $\frac{x^{(i)}_1}{n_i}, \frac{x^{(i)}_2}{n_i}, \dots, \frac{x^{(i)}_l}{n_i}$ along each Σ_i and form the approximation $\tilde{E}$ over $(X, D, \underline{n})$ by 'clutching' with

$$(6.1)\qquad t(z,w) = \begin{pmatrix} \left(\frac{z}{|z|}\right)^{x^{(i)}_1} |z|^{-\lambda^{(i)}_1 n_i} & & \\ & \ddots & \\ & & \left(\frac{z}{|z|}\right)^{x^{(i)}_l} |z|^{-\lambda^{(i)}_l n_l} \end{pmatrix}.$$

The point of this choice—which is distinct from the clutchings c and b already used—is that it carries the weighted hermitian metric on E into a genuine one on $\tilde{E}$. If E should have a holomorphic structure, then the pullback of the $\bar{\partial}$-operator is no longer smooth: in the given frame it will have the form

$$\bar{\partial} + \begin{pmatrix} x^{(i)}_1 - \lambda^{(i)}_1 n_i & & \\ & \ddots & \\ & & x^{(i)}_l - \lambda^{(i)}_l n_i \end{pmatrix} \frac{d\bar{z}}{2\bar{z}} + B$$

so that the corresponding Chern (unitary) connexion will have the form

$$d + \begin{pmatrix} x^{(i)}_1 - \lambda^{(i)}_1 n_i & & \\ & \ddots & \\ & & x^{(i)}_l - \lambda^{(i)}_l n_i \end{pmatrix} \eta + B',$$

where $\eta = id\theta$ locally. (In terms of a *holomorphic* frame the pull-back is $\bar{\partial} + (\frac{1}{2})\Lambda d\bar{s}_i/\bar{s}_i$ near $\Sigma_i - D \cdot D$, where s_i is a defining section and Λ is a diagonal matrix with entries $x^p_i - \lambda^p_i/n_i$. The formulae we have given are local ones, corresponding global ones would be given by replacing z by $\tilde{s}$.) The next definition is the same for each convention: the crucial matter is the pattern of ordering.

Definition 6.2 **[28, Section 4b]** Such a V-bundle $\tilde{E}$ over $(X, D, \underline{n})$ constructed using 6.1 and with isotropy $x_1^{(i)}, \dots, x_l^{(i)}$ along Σ_i, where $\underline{n} = (n_1, \dots, n_d)$, is said to be a k_0-approximation to E if

(i) $\frac{k_0-1}{n_i} < |\lambda_s^{(i)} - \lambda_t^{(i)}| < 1 - \frac{k_0-1}{n_i}$ for each i,
(ii) if any $\lambda_s^{(i)}$ is rational then the denominator divides n_i,
(iii) $x_1^{(i)} \le x_2^{(i)} \le \cdots \le x_l^{(i)}$ with equality if and only if the corresponding λ's are equal,
(iv) $|\lambda_s^{(i)} - \frac{x_s^{(i)}}{n_i}| < \frac{1}{2n_i}$ for all s.

By taking the n_i's big enough and choosing the x_i's appropriately this can be achieved. A consequence of the inequalities is the following:

(v) $k_0 - 2 < |x_s^{(i)} - x_t^{(i)}| < n_i - k_0 + 2$ for all i, whenever $x_x^{(i)} \neq x_t^{(i)}$.

To obtain a k_0-approximation, just choose the n's and x's appropriately in 6.1.

Suppose henceforth that such a k_0-approximation $\tilde{E}$ has definitely been chosen for some $k_0 > 3$.

Let $\Lambda_{\underline{k}} = \mathrm{diag}(k_1, \dots, k_l)$ and let B_0 be a unitary connexion on $\tilde{E}$ with $\bar{\partial}$-operator of the form $\bar{\partial} - \Lambda_{\underline{k}^{(i)}}(d\bar{z}/2\bar{z})$ plus a smooth term near each Σ_j, and appropriately near each point of $D \cdot D$. As in [17], [28] call it the *model* connexion. We write $\Lambda = \Lambda^{0,1} + \Lambda^{1,0}$ for the form $\Lambda_{\underline{k}^{(i)}}(d\bar{z}/2\bar{z} + dz/2z)$.

Definition 6.3 **[28, p. 147]** A_Λ = connexions differing from the model connexion B_0 by a smooth 1-form.

Note that G, the gauge group of $\tilde{E}$, is independent of Λ as is its complexification G^c; it just depends upon the V-bundle $\tilde{E}$ and so on the x_i's.

Lemma 6.4 **[28, 4.1]** *If g is a weighted automorphism of E then $\tilde{g} = t^{-1}gt$ is an $L^p_{k_0}$ automorphism of $\tilde{E}$, for any $p > 1$.*

The proof is simply a matter of observing that $g_{s,t}$ has order at least $\alpha(\lambda_s^i - \lambda_t^i)$ in $|z|$; and similarly with respect to both z and w at a point $P \in \Sigma_i \cap \Sigma_j$.

Definition 6.5

(i) $A^p_{\Lambda,k} = \{$connexions A such that $A - B_0 \in L^p_k\}$, where B_0 is the model connexion of 6.3.
(ii) $G^p_k = L^p_k$ gauge transformations of the V-bundle $\tilde{E}$.

Proposition 6.6 **[28, 4.2 and 4.3]** *Let $p \ge 2$, $k \ge 1$ and let $(f_n)_{n=1}^\infty$ be an L^p_k-Cauchy sequence of smooth functions converging to f and with support in $\Delta \times \Delta$, where Δ is the unit disc in $\mathbf{C}$, and equivariant with respect to $\mathbf{Z}/n$ acting via $\zeta \cdot (z, w) = (\zeta^x \cdot z, w)$, where both $x \le k$ and $n - x \le k$.*

(i) There is a sequence, each function supported away from $0 \times \Delta$, converging to the same limit.

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(ii) *If $p \geq 2$ then there is a positive constant $c(p,i,j)$ such that*

$$\left\| f/|z|^i \right\|_{L^p_j} \leq c(p,i,j) \|f\|_{L^p_{i+j}} \quad \text{if } i+j \leq k.$$

(iii) $c(2,1,0) \leq 1/\min(x, n-x)$.

(iv) *The map $f \to f/r^i$ from $L^p_{i+j} \to L^p_j$ is compact on equivariant functions—provided always that $i+j \leq k$. (Here r denotes $|z|$.)*

(v) *For any y, with $-k \leq y < 1$ and $[y] \leq j \leq k+[y]$, $\left\| |z|^y f \right\|_{L^p_j} \leq c'(p,y,j)\|f\|_{L^p_{j-[y]}}$.*

Moreover, if each function f_n is equivariant with respect to $\mathbf{Z}/n \times \mathbf{Z}/m$ acting by $(\zeta,\eta)(z,w) = (\zeta^x z, \eta^y w)$ with $k \leq \min(x, y, n-x, m-y)$, then similar results hold with $|z|^i$ replaced by $|w|^s\,|z|^{i-s}$ upon repeating the argument on the second factor.

Corollary 1

(i) *If g is an L^p_k endomorphism of $\tilde{E}$ and a an L^p_k 1-form with values in $\mathrm{End}(\tilde{E})$, both supported in $\nu(\Sigma_i)$, then*

$$g \to [g,\Lambda] \quad \textit{and} \quad a \to [a,\Lambda]$$

are bounded linear maps from L^p_k to L^p_{k-1}, provided that $k \leq k_0 - 1$ in the first case and $k \leq k_0 - 2$ in the second. (Here, Λ denotes $\Lambda_{\underline{k}^{(i)}}(d\bar{z}/2\bar{z} + dz/2z)$ as noted before 6.3.)

(ii) *The operator norm of the map $L^2_1 \to L^2_0$ is no greater than $1/(k_0 - 2)$.*

Corollary 2

(i) *For any $p > 2$ the group $(G^c)^p_2$ of automorphisms of $\tilde{E}$ acts on $A^p_{\Lambda,1}$ and these connexions have curvature in L^p.*

(ii) *For $p = 2$ and $k > 2$ a similar statement holds for connexions in $A^2_{\Lambda,k-1}$ and group $(G^c)^2_k$ with k not necessarily integral.*

The proof of these corollaries is as that of the corresponding results (4.4 and 4.5) in [28]. For part (ii) of the second we use L^p_k-spaces defined as in [14] for possibly non-integral k.

Corollary 3 **[28, 4.6]** *If A_0 is the initial connexion on $\tilde{E}$ determined by a holomorphic structure on $\tilde{E}$ then, for any $p \geq 2$ and $k \leq k_0 - 2$, $A_0 \in A^p_{\Lambda,k}$. Moreover, given $P \in \Sigma_i$ there exists $\tilde{g}_0 \in (G^c)^p_{k+1}$ such that $d - \Lambda = \tilde{g}_0 A_0$ near P with respect to the chosen frame.*

Proof Take the unitary frame used in the 'clutching'. As it is a genuine unitary frame for $\tilde{E}$ we need only check the $(0,1)$-part. Let g_0 be a weighted change of frame carrying the weighted unitary frame to a holomorphic one. Then $\tilde{g}_0 \in (G^c)^p_{k_0}$ by Lemma 6.4. Now $\tilde{g}_0$ carries the unitary frame to a frame which is no longer holomorphic because t is not (see (6.1)). However, the new $\bar{\partial}$-operator is given by $\bar{\partial} + t^{-1}\bar{\partial}t = \bar{\partial} + \Lambda^{0,1}$ in this frame. Finally, as $\tilde{g}_0 \in (G^c)^p_{k_0}$ and $\bar{\partial} + \Lambda^{0,1} \in A^p_{\Lambda,k}$ for all p and k, we have the first assertion. At $P \in \Sigma_i \cap \Sigma_j$ one argues in the same way.

The second claim follows as in [28].

Proposition 6.7 **[28, 5.1]** *Every $(G^c)^p_k$-orbit contains a model connexion, at least for $p = 2$ and $k \geq 3$, or $p > 2$ and $k \geq 2$.*

(This may be proved as in [28], but using the product of complex projective lines as a model. When $p = 2$ or in case $k \geq 3$ it is, maybe, easier to use the second proof there. That proof compares the holomorphic clutching c with the clutching t of 6.1. If F is a hermitian weighted bundle with a compatible holomorphic structure one notes that the automorphism $c^{-1}gt = c^{-1}t\tilde{g}$ on $\tilde{F}$ over X replaces the singular $\bar{\partial}$-operator by one, $\bar{\partial}_0$, without singularity. One shows that the orbit of $\bar{\partial}_0$ contains a smooth point. For this observe, using Hölder's inequality and Corollary 1 to 5.4, that the connection is in L_2^q, where $2 = p < q < 2/(1-\delta)$ and $\delta > 0$ is such that $|\varepsilon_i| < 1/2(1-\delta)$, where $\varepsilon_i = x_i - \lambda_i n$, for all i. By condition 6.2 (iv) this may be achieved since $|\varepsilon_i| < 1/2$. One now argues that if $g \in L_3^2$ then $c^{-1}gt = c^{-1}t\tilde{g} \in L_2^q$ and proceeds as in [2].)

Corollary *A $(G^c)_k^p$-orbit determines a holomorphic structure on E, unique up to the action of G^c, and conversely.*

As in [28] this follows from the preceding proposition because an L_k^p-orbit determines an L_2^q-one, where $2 = p < q < 2/(1-\delta)$.

In order to prove the existence of a Hermitian-Einstein connexion on E we need a weak compactness theorem like Uhlenbeck's. Such a theorem is proved in [17], [29] and for Riemann surfaces in [28]. As we are working over a compact orbifold with an elliptic operator there is no essential problem provided that $\|\Lambda\|_p$ is small, where $\|\Lambda\|_p$ denotes the operator norm of the map $\Lambda\colon L_k^p \to L_{k-1}^p$, defined by $a \to [a, \Lambda]$ as in Corollary 1 to Proposition 6.6. For $p = 2$ the operator norm tends to 0 as k_0 increases by clause (ii) of Corollary 1 to Proposition 6.6. For $p > 2$ this is not clear. However, we can always choose a p and k_0 such that the operator norm is as small as desired by the Riesz-Thorin convexity theorem [1], which says that the norm is continous near 2.

Theorem 6.8 *There is a $p > 2$ and a constant $k_1 \geq 4$ such that if E is a weighted Hermitian bundle, $\tilde{E}$ a k_0-approximation to E with $k_0 \geq k_1$ then $A_{\Lambda,k}^p$ has the standard weak compactness property: namely that if $\|F_{A_n}\|_{k-1}^p < C$ for some constant C and all A_n in the sequence then there is a finite set $\{P_1, \dots, P_s\}$ of points and a subsequence such that, modulo L_{k+1}^p gauge transformations, the subsequence converges over $X \setminus \{P_1, \dots, P_s\}$. (Here $k \geq 1$.)*

Theorem 6.9 *If E is a weighted Hermitian bundle and $\tilde{E}$ a k_0-approximation to E with $k_0 \geq k_1$, then if (A_n) is a sequence of connections in $A_{\Lambda,k}^p \times [-1/2, 1/2]^{ld}$ with $p = 2+\varepsilon$ and $\|F_{A_n}\|_{2+\varepsilon} < C$ then A_n has a weakly convergent subsequence modulo $L_2^{2+\varepsilon}$ gauge changes.*

(Unlike the case of a Riemann surface, we cannot deduce *a priori* that curvature does not blow up at points, even if the connexions are Hermitian-Einstein.) These are proved, as 4.9 and 4.10 of [28], using a local gauge-fixing theorem which is a direct analogue of 4.8 there. The local gauge-fixing theorem requires the singular exterior differential d_Λ to be Fredholm. (This, together with certain Sobolev constants, controls the value of the constant k_1.) We shall need such a result in the next section and state it here for reference.

Proposition 6.10 *For $k_0 \geq k_1$ the operators $\bar{\partial}_\Lambda$ and $\bar{\partial}_\Lambda^*$ are Fredholm.*

7

We prove the Kobayashi-Hitchin correspondence for all weights by using Theorem 6.9 following the arguments of Section 5 of [28]. Here we use almost exclusively the *balanced* convention.

Let $\mathcal{E}$ be a stable bundle over Y with at least one irrational weight and equipped with a parabolic hermitian metric $\tilde{h}$. Let A be the associated connexion. Let $\tilde{E}$ be a k_0-approximation (see 6.2) over $(X, D, \underline{n})$ to the underlying bundle E with k_0 so big that both Theorems 6.8 and 6.9 can be applied and also condition (4) below holds. For each Σ_i choose a sequence of rational approximations $(x_j^{(k)}/n^{(k)})_{k=1}^{\infty}$ to the weights λ_j. (Here we have dropped the index (i) corresponding to Σ_i.) Let this be done in such a way that

(1) $n^{(1)} = n$ and $x_j^{(1)} = x_j$, $1 \leq j \leq l$, each Σ_i;
(2) $1/2n^{(i)} < 1/2n - |x_j/n - \lambda_j|$ for each i and j;
(3) $n^{(k)}$ divides $n^{(k+1)}$ for all $k \geq 1$ and for each i;
(4) $|x_j^{(k)}/n^{(k)} - \lambda_j| < 1/2n^{(k)}$ for all k and each i, j;
(5) $\mathcal{E}$ remains parabolically stable when the λ_j are replaced by $x_j^{(k)}/n^{(k)}$.

The crucial condition (5) can be achieved by the argument of Mehta and Seshadri [22]. (It does not seem to be true—unlike the case of Riemann surfaces—that to any irrational weight there is a close by rational weight so that the moduli spaces of stable bundles are homeomorphic.) Let $\mathcal{E}^{(k)}$ be the parabolic bundle over Y with the same quasi-parabolic structure as $\mathcal{E}$ but with weights $x_j^{(k)}/n^{(k)}$. By condition (5) it is stable.

Write $k_j^{(k)} = x_j^{(k)} - \lambda_j n^{(k)}$ and $e^{(k)} = k_j^{(k)}/n^{(k)}$. Define an automorphism φ_k of E, singular along $D = \Sigma_1 \cup \cdots \cup \Sigma_d$, by setting

$$\varphi_k(u, v) = \begin{cases} \begin{pmatrix} |u|^{e_1^{(k)}} & & \\ & \ddots & \\ & & |u|^{e_l^{(k)}} \end{pmatrix} & \text{near } P \in \Sigma_i - D \cap D \\ \begin{pmatrix} |u|^{e_1^{(k)}}\,|v|^{e_1'^{(k)}} & & \\ & \ddots & \\ & & |u|^{e_l^{(k)}}\,|v|^{e_l'^{(k)}} \end{pmatrix} & \text{near } P \in \Sigma_i \cap \Sigma_j, \end{cases}$$

where the decomposition is with respect to that given by extending $E = E_1 \oplus \cdots \oplus E_l$ (given by the weighted metric) out along νD using the Kähler metric.

Applying φ_k manufactures a weighted hermitian metric h_k on E corresponding to the weights $x_j^{(k)}/n^{(k)}$: call this weighted Hermitian bundle $E^{(k)}$. By the hypothesis (5) $\mathcal{E}^{(k)}$, the parabolic bundle with those weights, is still stable and has a corresponding connexion A_k. (Again, this is a local formula for a global automorphism.)

Let $X_k = (X, D, \underline{n}^{(k)})$, the orbifold constructed as in Section 1 with $\underline{n}^{(k)} = (n_1^{(k)}, \ldots, n_d^{(k)})$. Then, as the denominators of $\mathcal{E}^{(k)}$ are the $n^{(k)}$'s, we have a corresponding holomorphic stable V-bundle $\tilde{\mathcal{E}}^{(k)}$ over X_k and ordinary Hermitian metric $\tilde{h}_k$—and corresponding connexion $A^{(k)}$—on $\tilde{\mathcal{E}}^{(k)}$ induced by φ_k (as in Proposition 4.2(ii)). As we intend to argue over $X = X_1$ we think of $\tilde{\mathcal{E}}^{(k)}$ as a parabolic V-bundle over X_1 and apply an adaptation of Theorem 4.5. (The notion of a parabolic V-bundle is briefly indicated in Section 3. The sole change in the definition is that the weights along Σ have to lie in $(-1/2n, 1/2n)$, where n is the isotropy around Σ.) Only the special case where the weights are constant on each isotypic component is needed. Suppose that $(X, D, \underline{n})$ is an abelian orbifold with $|X| = Y$ a compact Kähler surface—as in Section 1.

Exactly as in the standard case one can establish an equivalence between rational parabolic V-bundles and holomorphic V-bundles over another orbifold.

Theorem 7.1 *Suppose that $\mathcal{E}$ is a parabolic V-bundle over X with weights $\vartheta^i_j = d^i_j | l_i n_i$, where $|d^i_j| < l_i/2n_i$. Then there is associated a holomorphic V-bundle $\mathcal{E}'$ on X', the orbifold with isotropy $l_i n_i$ about Σ_i. Moreover, the correspondence is bijective between*

(a) parabolic bundles with rational denominators divisible by n_i and dividing $l_i n_i$, and
(b) holomorphic V-bundles on X'.

Also the correspondence preserves stability.

Thus, to each d-tuple $\underline{n}^{(k)}$ there is a parabolic bundle $\mathcal{E}^{(k)}_1$ over X_1 with weights $x^{(k)}_j/n^{(k)} - x_j/n$ of modulus $< 1/2n$ by condition (2). On $\tilde{\mathcal{E}}^{(1)}$ choose the Hermitian-Einstein metric given by Theorem 2.3. An adaptation of Theorem 4.5 would prove that there is an irreducible parabolic connexion $A^{(k)}$ for $\tilde{\mathcal{E}}^{(k)}_1$, Hermitian-Einstein with respect to the standard (orbifold) Kähler metric $\tilde{w}$ on $X = X_1$, corresponding to that $(\tilde{A}^{(k)})$ on $\tilde{\mathcal{E}}^{(k)}$. We know that both A_k and $\tilde{A}^{(k)}$ may be taken in L^p_2 on the orbifold X_1, but because the isotropies of X_1 are widely spaced we can do better.

Theorem 7.2 *Associated to $\tilde{\mathcal{E}}^{(k)}$ there is a $(k_0 - 1)$-times differentiable connexion $\tilde{A}^{(k)}$ Hermitian-Einstein with respect to the (orbifold) Kähler metric on X_1.*

Proof We could follow through the argument of [11] as in Section 2, checking at each stage that it extends, but it is simpler to use Theorem 2.3 and the argument of Section 6. (If we used Theorem 5.4 we should not have so much regularity. It is not surprising [18], [28] that we get much better regularity when the isotropies or weights are far apart.) From the choice of $\tilde{\mathcal{E}}^{(1)}$ and the inequalities at the beginning of the section we have

$$(n^{(k)}/n)(k_0 - 1) - 1 < |x^{(k)}_i - x^{(k)}_j| < (n^{(k)}/n)(n - k_0 + 1) - 1.$$

Let k denote the metric on $\tilde{\mathcal{E}}^{(k)}$ Hermitian-Einstein with respect to $\tilde{w}^{(k)}$. We need to solve the equation $\tilde{\Lambda}\big(F_K + \partial(\eta^{-1}\partial\eta)\big) = \lambda$. Again we take unitary clutching so as to work over X_1 with the (now) singular operators $\bar{\partial}_\Delta$ and ∂_Δ, where $\Delta = 1/2\,\mathrm{diag}(nx^{(k)}_j/n^{(k)} - x_j)$. Applying $\tilde{*}$, the star operator on X_1, we have to solve the equation

$$(F_K - \lambda\tilde{w}) \wedge \tilde{w} = \partial^*_\Delta \eta^{-1} \partial_\Delta \eta,$$

where now we know that the operators are Fredholm on $L^p_{k_0}$ by Proposition 6.10 since we have chosen p close to 2 and k_0 sufficiently big. On the left-hand side, the term involving the curvature may not be very small on the diagonal, so first we take a scalar automorphism, say ζI, as on p. 162 of [28], to remove completely the diagonal terms. The inequalities above and equivariance now imply that $F_{\zeta K} \wedge \tilde{w} = O(|u|^{k_0-1})$, where $u^n = z^{n^{(k)}}$. We solve the equation with $F_{\zeta K} \wedge \tilde{w}$ replacing $F_K \wedge \tilde{w}$. The equation has a Fredholm linearization and so we have a solution as required. (The operators depend solely on X_1 so that we can always choose $\tilde{w}^{(k)}$ close enough to $\tilde{w}$.)

(If we do wish to follow [11], then we note that Proposition 7.2 of [18] or the analogue of 4.8 of [28] provides the local gauge-fixing; we have chosen k_0 large enough for this to apply. The functional $J(A)$ is well defined on singular connexions A with fixed singularity. The identities

$$\bar{\partial}_A^* = i[\partial_A, \Lambda], \quad \partial_A^* = -i[\bar{\partial}_A, \Lambda],$$
$$\bar{\partial}_A^* \bar{\partial}_A = \frac{1}{2}(\nabla_A \nabla_A^* - i\hat{F}_A),$$

where here Λ denotes the Hodge operator, still hold. Consequently, the argument of Section 6.2 of [10] goes through up to the L_2^2 bound on the connexions. But here we may argue locally as in 2.35 of [11]. So we have L_2^2 convergence and a limit connexion as there. The limit is smooth away from D and on D we must then "bootstrap" to get more derivatives.)

Naturally we expect that the parabolic Hermitian-Einstein connexions $A^{(k)}$ on $\mathcal{E}^{(k)}$ will converge to one on $\mathcal{E}$. To prove this we argue over the orbifold X_1, recalling that $n^{(1)}$ is so big that $\tilde{\mathcal{E}}^{(1)}$ is a k_0-approximation with $k_0 \geq k_1$, where k_1 is the constant of Theorem 6.8. We prove that the singular connections on the bundle $\tilde{E}^{(1)}$ corresponding under Theorem 7.2 to the Hermitian-Einstein one on $\tilde{\mathcal{E}}^{(k)}$ converge to one with singularity and which is thus a Hermitian-Einstein parabolic one on the parabolic bundle over X_1 associated to E. We use L_k^p-spaces taken for the moment with respect to the *orbifold* Kähler metric $\tilde{w}$ on X_1; and Hermitian-Einstein will also be with respect to the *orbifold* Kähler metric $\tilde{w}$.

The argument can be carried out, as in [28] for a Riemann surface, by splitting the construction φ into two steps *or* one may argue slightly differently as follows. The bundle $\mathcal{E}_1^{(k)}$ is the one over X_1 corresponding to $\tilde{\mathcal{E}}^{(k)}$ in Theorem 7.1. From Theorem 7.2 we get a singular Hermitian-Einstein connexion, to be written also $\tilde{A}^{(k)}$, on $\mathcal{E}_1^{(k)}$. It is $(k_0 - 1)$-times differentiable on X_1 and so lies in $L_{k_0-1}^p$ for any $p > 1$. Since the underlying bundle to $\mathcal{E}_1^{(k)}$ is the Hermitian bundle underlying $\tilde{\mathcal{E}}^{(1)}$ and since $k_0 \geq k_1$, the constant of Theorem 6.9, that theorem may be applied once it is known that $|F(\tilde{A}^{(k)})|$ is bounded in $L^{2+\varepsilon}$ for some $\varepsilon > 0$ and all k. (In this section we write $F(A)$ rather than F_A for the curvature of A.) This is essentially because all the connexions are manufactured out of one given at the start.

To check this argue locally. Suppose first that $P \in \Sigma_i - D \cdot D$ and let $s_1, \ldots, s_l$ be a local holomorphic frame with $e_1, \ldots, e_l$ an orthonormal one (with respect to the Hermitian-Einstein metric on $\tilde{\mathcal{E}}^{(1)}$), chosen so that $g(e_i) = s_i$ where g is lower triangular. Then $\tilde{A}^{(1)} = (g^* g)^{-1} \partial(g^* g)$. On $\tilde{\mathcal{E}}^{(1)}$ take the weighted metric h_k given by $h_k(e_i, e_i) = \delta_{i,j} |z|^{2d_j^k}$, where $d_j^k = n(x_j^{(k)}/n^{(k)} - x_j/n)$. The parabolic unitary connexion associated to the holomorphic structure and this metric is

$$A_k = d + (g^* h_k g)^{-1} \partial (g^* h_k g) = g^{-1} h_k^{-1} (g^*)^{-1} \partial (g^* h_k g).$$

Therefore,

$$A_k = A^{(1)} = g^{-1}(\delta_{i,j} d_j^k)(dz/z) g + g^{-1}(\alpha_{i,j})\{|z|^{2(d_i^k - d_j^k)} - 1\} g,$$

where round brackets denote the matrix whose (i, j)-th entry is exhibited and where $\alpha_{i,j} = \big((g^*)^{-1}\partial g^*\big)_{i,j}$. Because g is differentiable, equivariant and triangular, and because we have $k_0 \geq 4$, clause (ii) of Proposition 6.6 says that each entry of the matrix expansion of the first term is in L_2^p independently of k, whilst the second is bounded in L_1^p by clause (v) of the same proposition since $2|d_i^k - d_j^k| < 2$. A similar calculation checks this at points of $D \cdot D$. The passage from an arbitrary connexion to a Hermitian-Einstein one reduces curvature [9] so that we know that $|F(\tilde{A}^{(k)})|$ is a bounded sequence in $L^{2+\varepsilon}$ and hence the sequence $\tilde{A}^{(k)}$ converges modulo gauge over $X - \{P_1, \ldots, P_r\}$—for some points $P_1, \ldots, P_r$—to a connexion A^∞ by Theorem 6.9, and A^∞ is Hermitian-Einstein with singularities $\Lambda_1(dz/z)$ and may be supposed twice differentiable because $k_0 \geq 4$. The problem is that A^∞ may be a connexion on a different bundle $\mathcal{E}'$ so that to complete the proof we must show that there is a non-trivial map $j\colon \mathcal{E}^{(1)} \to \mathcal{E}'$ between them. Then, from the stability of each, we have that j is an isomorphism. (Since A^∞ is Hermitian-Einstein, the bundle $\mathcal{E}'$ is polystable.) Argue as in [11, Section 6.25]. There is a series of smooth complex automorphisms (g_k) of $\tilde{\mathcal{E}}^{(1)}$ such that $\bar{\partial}_{A_k^*, \tilde{A}^{(k)}} g_k = 0$ works in local holomorphic coordinates on X_1. In such coordinates, although the connexions do have singularities the $\bar{\partial}$-operators do not, so that standard elliptic theory may be applied. Normalize so that $\det(g_k) = 1$ and $\|g_k\|_{L^2} = 1$. Then

$$\|g_k\|_{L_3^2} \leq K(\|\bar{\partial}_k g_k\|_{L_2^2} + \|g_k\|_{L^2}) \leq K';$$

writing $\bar{\partial}_k = \bar{\partial}_{A_k^* \tilde{A}^{(k)}}$. Consequently $g_k \to g$ over $X - \{P_1, \ldots, P_r\}$ and we get $g\colon \mathcal{E}^{(1)} \to \mathcal{E}'$ by Hartog's theorem. So if $g \neq 0$ we are done. A formal argument with curvature as in [11] establishes this, and thus the following theorem.

Theorem 7.3 *If it is stable, the parabolic bundle $\mathcal{E}^{(1)}$ over $X = (X, D, \underline{n}^{(1)})$ with weights $\lambda_j^i - (x_j^i/n_i)$ along Σ_i has a parabolic connexion Hermitian-Einstein with respect to $\tilde{w}$. Moreover this connexion may be supposed to be $(k_0 - 1)$-times differentiable. (Recall that $\underline{n} = \underline{n}^{(1)}$.)*

The two final steps are to exchange this connexion for one Hermitian-Einstein with respect to the Kähler metric w on Y and then to carry it down from X to $Y = |X|$ as in Lemma 4.3. The first step is the same as in Section 5 because we know that for enough weights the operator is Fredholm. On the other hand, the second is perhaps a little more complicated because of the singularity. Take local holomorphic coordinates on X with z transverse and w along Σ, and choose a local holomorphic frame for $\mathcal{E}$. Then one can write

$$A^\infty = d + \Lambda(dz/z) + G dz + H dw,$$

where $(\Lambda)_{p,q} = \left(\lambda_p - (x_p/n)\right)\delta_{p,q}$, $n = n^{(1)}$; and the Kähler metric

$$w = \gamma|z|^{2n-2}dz \wedge d\bar{z} + \beta z^{n-1}dz \wedge d\bar{w} + \beta'\bar{z}^{n-1}d\bar{z} \wedge dw + \delta dw \wedge d\bar{w},$$

with $\alpha, \beta, \beta', \gamma, \delta$ functions of (z, w) such that neither δ nor $\gamma\delta + \beta\beta'$ vanish on Σ. The volume form is $(\gamma\delta + \beta\beta')|z|^{2n-2}dz \wedge d\bar{z} \wedge dw \wedge d\bar{w}$. Write $F = F(A^\infty)$. Because $F \wedge w = \lambda$ (volume form), as in Section 4, one deduces that $F \wedge w = O(|z|^{2n-2})$ and hence that

$$\partial G/\partial\bar{z} = O(|z|^{n-1}).$$

The explicit formula for $\partial G/\partial\bar{z}$ tells us that for fixed w the function $\partial G/\partial\bar{z}$ is $(k_0 + n - 1)$-times differentiable in z and $\bar{z}$. As there, one now finds a change of frame such that

$$A^\infty = d + \Lambda(dz/z) + O(|z|^{n-1}dz) + b'dw + c'd\bar{w},$$

so that, in fact, we may suppose that $A^\infty \in L_1^p(Y)$ and the transformation in L_2^p, $p = 2 + \varepsilon$ for some tiny ε which is bounded above by various constants of Section 6. This proves the final theorem.

Theorem 7.4 *If $\mathcal{E}$ is a stable parabolic bundle over Y, parabolic with respect to the divisor $D = \Sigma_1 \cup \cdots \cup \Sigma_d$, where each Σ_i is smoothly embedded and if $i \neq j$ then Σ_i meets Σ_j in at most one point and there transervally, then there is for some $p = 2 + \varepsilon$, $\varepsilon > 0$, an L_2^p parabolic Hermitian-Einstein metric with curvature in L^p.*

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