1

Classical kinks

Kink solutions are special cases of "non-dissipative" solutions, for which the energy density at a given point does not vanish with time in the long time limit. On the contrary, a dissipative solution is one whose energy density at any given location tends to zero if we wait long enough [35],

$$\lim_{t\to\infty} \max_{\mathbf{x}}\{T_{00}(t,\mathbf{x})\} = 0, \quad \text{dissipative solution} \tag{1.1}$$

where $T_{00}(t,\mathbf{x})$ is the time-time component of the energy-momentum tensor, or the energy density, and is assumed to satisfy $T_{00} \geq 0$. Dissipationless solutions are special because they survive indefinitely in the system.

In this book we are interested in solutions that do not dissipate. In fact, for the most part, the solutions we discuss are static, though in a few cases we also discuss field configurations that dissipate. However, in these cases the dissipation is very slow and hence it is possible to treat the dissipation as a small perturbation. In addition to being dissipationless, kinks are also characterized by a topological charge. Just like electric charge, topological charge is conserved and this leads to important quantum properties.

In this chapter, we begin by studying kinks as classical solutions in certain field theories, and devise methods to find such solutions. The simplest field theories that have kink solutions are first described to gain intuition. These field theories are also realized in laboratory systems as we discuss in Chapter 9. The simple examples set the stage for the topological classification of kinks and similar objects in higher dimensions (Section 1.10), and are valuable signposts in our discussion of the more complicated systems of Chapter 2.

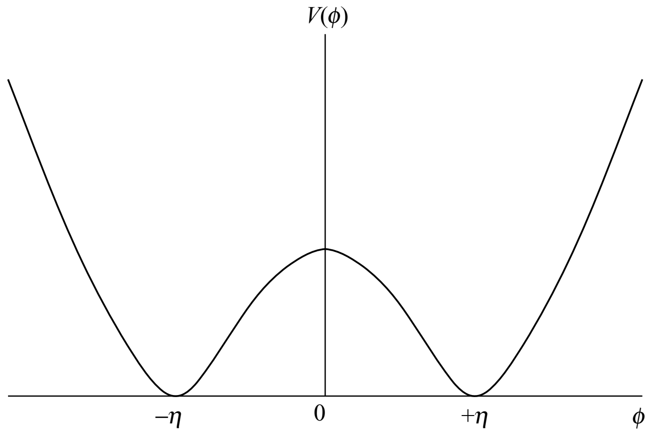

Figure 1.1 Shape of the $\lambda\phi^4$ potential.

1.1 Z_2 kink

The prototypical kink is the so-called "Z_2 kink." It is based on a field theory with a single real scalar field, ϕ, in $1+1$ dimensions. The action is

$$S = \int d^2x \left[\frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)\right]$$
$$= \int d^2x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4}(\phi^2-\eta^2)^2\right] \tag{1.2}$$

where $\mu = 0, 1$, and λ and η are parameters. The Lagrangian is invariant under the transformation $\phi \to -\phi$ and hence possesses a "reflectional" Z_2 symmetry. The potential for ϕ (see Fig. 1.1) is

$$V(\phi) = \frac{\lambda}{4}(\phi^2-\eta^2)^2 = -\frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{\lambda\eta^4}{4} \tag{1.3}$$

where $m^2 \equiv \lambda\eta^2$. The potential has two minima: $\phi = \pm\eta$, that are related by the reflectional symmetry. The "vacuum manifold," labeled by the classical field configurations with lowest energy, has two-fold degeneracy since $V(\phi) = V(-\phi)$.

The equations of motion can be derived from the action

$$\partial_t^2\phi - \partial_x^2\phi + \lambda(\phi^2-\eta^2)\phi = 0 \tag{1.4}$$

where $\partial_t \equiv \partial/\partial t$ and similarly for ∂_x. A solution is $\phi(t,x) = +\eta$, and another is $\phi(t,x) = -\eta$. These have vanishing energy density and are called the "trivial vacua." The action describing excitations (sometimes called "mesons") about one of the trivial vacua can be derived by setting, for example, $\phi = \eta + \psi$, where ψ is the excitation field. Then

$$S = \int d^2x \left[\frac{1}{2}(\partial_\mu \psi)^2 - \frac{m_\psi^2}{2}\psi^2 - \sqrt{\frac{\lambda}{2}} m_\psi \psi^3 - \frac{\lambda}{4}\psi^4 \right] \tag{1.5}$$

where

$$m_\psi = \sqrt{2}m \tag{1.6}$$

is the mass of the meson.

Next consider the situation in which different parts of space are in different vacua. For example, $\phi(t, -\infty) = -\eta$ and $\phi(t, +\infty) = +\eta$. In this case, the function $\phi(t, x)$ has to go from $-\eta$ to $+\eta$ as x goes from $-\infty$ to $+\infty$. By continuity of the field there must be at least one point in space, x_0, such that $\phi(t, x_0) = 0$. Since $V(0) \neq 0$, there is potential energy in the vicinity of x_0, and the energy of this state is not zero. The solution of the classical equation of motion that interpolates between the different boundary conditions related by Z_2 transformations is called the "Z_2 kink."

We might wonder why the Z_2 kink cannot evolve into the trivial vacuum? For this to happen, the boundary condition at, say, $x = +\infty$ would have to change in a continuous way from $+\eta$ to $-\eta$. However, a small deviation of the field at infinity from one of the two vacua costs an infinite amount of potential energy. This is because as ϕ is changed, the field in an infinite region of space lies at a non-zero value of the potential (see Fig. 1.1). Hence, there is an infinite energy barrier to changing the boundary condition.[1]

A way to characterize the Z_2 kink is to notice the presence of a conserved current

$$j^\mu = \frac{1}{2\eta}\epsilon^{\mu\nu}\partial_\nu \phi \tag{1.7}$$

where $\mu, \nu = 0, 1$ and $\epsilon^{\mu\nu}$ is the antisymmetric symbol in two dimensions ($\epsilon^{01} = 1$). By the antisymmetry of $\epsilon^{\mu\nu}$, it is clear that j^μ is conserved: $\partial_\mu j^\mu = 0$. Hence

$$Q = \int dx j^0 = \frac{1}{2\eta}[\phi(x = +\infty) - \phi(x = -\infty)] \tag{1.8}$$

is a conserved charge in the model. For the trivial vacua $Q = 0$, and for the kink configuration described above $Q = 1$. So the kink configuration cannot relax into the vacuum – it is in a sector that carries a different value of the conserved "topological charge."

To obtain the field configuration with boundary conditions $\phi(\pm\infty) = \pm\eta$, we solve the equations of motion in Eq. (1.4). We set time derivatives to vanish since

[1] In Chapter 2 we will come across an example where the vacuum manifold is a continuum and correspondingly there is a continuum of boundary conditions that can be chosen as opposed to the discrete choice in the Z_2 case. This will lead to some new considerations.

we are looking for static solutions. Then, the kink solution is

$$\phi_{\rm k}(x) = \eta\ \tanh\left(\sqrt{\frac{\lambda}{2}}\eta x\right) \tag{1.9}$$

In fact, one can Lorentz boost this solution to get

$$\phi_{\rm k}(t,x) = \eta \tanh\left(\sqrt{\frac{\lambda}{2}}\eta X\right) \tag{1.10}$$

where

$$X \equiv \frac{x - vt}{\sqrt{1-v^2}} \tag{1.11}$$

(Recall that we are working in units in which the speed of light is unity i.e. $c = 1$.) The solution in Eq. (1.10) represents a kink moving at velocity v.

Another class of solutions is obtained by translating the solution in Eq. (1.9)

$$\phi_{\rm k}(x;a) = \eta\ \tanh\left(\sqrt{\frac{\lambda}{2}}\eta(x-a)\right) \tag{1.12}$$

It is easily checked that translations do not change the energy of the kink. This is often stated as saying that the kink has a zero energy fluctuation mode (or simply a "zero mode"). To explain this statement, we need to consider small fluctuations of the field about the kink solution, similar to Eq. (1.5). We now have

$$\phi = \phi_{\rm k}(x) + \psi(t,x) \tag{1.13}$$

where $\phi_{\rm k}$ denotes the kink solution. The fluctuation field, ψ, obeys the linearized equation

$$\partial_t^2\psi - \partial_x^2\psi + \lambda\left(3\phi_{\rm k}^2 - \eta^2\right)\psi = 0 \tag{1.14}$$

To find the fluctuation eigenmodes we set

$$\psi = {\rm e}^{-{\rm i}\omega t} f(x) \tag{1.15}$$

where $f(x)$ obeys

$$-\partial_x^2 f + \lambda\left(3\phi_{\rm k}^2 - \eta^2\right) = \omega^2 f \tag{1.16}$$

We will discuss all the solutions to this equation in Chapter 4. Here we focus on the translation mode. Since translations cost zero energy, there has to be an eigenmode with $\omega = 0$. This can be obtained by directly solving Eq. (1.16) or by noting that for small a, the solution in Eq. (1.12) can be Taylor expanded as

$$\phi_{\rm k}(x;a) = \phi_{\rm k}(x;a=0) + a\frac{{\rm d}\phi_{\rm k}}{{\rm d}x}\bigg|_{a=0} \tag{1.17}$$

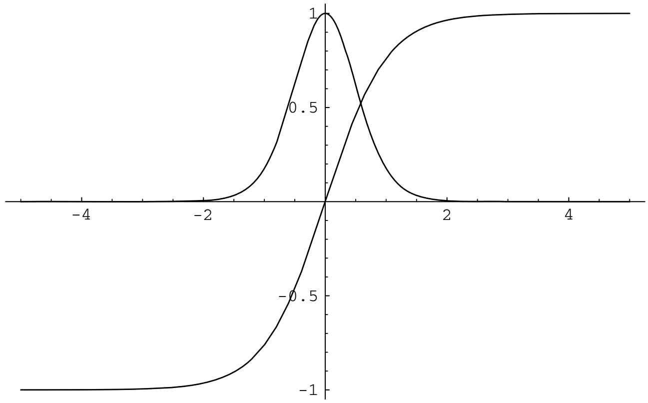

Figure 1.2 The curve ranging from -1 to $+1$ as x goes from $-\infty$ to $+\infty$ shows the Z_2 kink profile for $\lambda = 2$ and $\eta = 1$. The energy density of the kink has also been plotted on the same graph for convenience, and to show that all the energy is localized in the narrow region where the field has a gradient.

Comparing Eqs. (1.17) and (1.13), the zero mode solution is

$$f_0(x) = \left.\frac{\mathrm{d}\phi_\mathrm{k}}{\mathrm{d}x}\right|_{a=0} = \eta^2\sqrt{\frac{\lambda}{2}}\mathrm{sech}^2\left(\sqrt{\frac{\lambda}{2}}\eta x\right) \tag{1.18}$$

The solution in Eq. (1.9) can be used to calculate the energy density of the kink

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}(\partial_t\phi_\mathrm{k})^2 + \frac{1}{2}(\partial_x\phi_\mathrm{k})^2 + V(\phi_\mathrm{k}) \\ &= 0 + V(\phi_\mathrm{k}) + V(\phi_\mathrm{k}) \\ &= \frac{\lambda\eta^4}{2}\mathrm{sech}^4\left(\sqrt{\frac{\lambda}{2}}\eta x\right)\end{aligned} \tag{1.19}$$

where the second line is written to explicitly show that $(\partial_x\phi)^2 = 2V(\phi)$. The kink profile and the energy density are shown in Fig. 1.2. The total energy is

$$E = \int \mathrm{d}x\ \mathcal{E} = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} \tag{1.20}$$

As is apparent from the solution and also the energy density profile, the half-width of the kink is,

$$w = \sqrt{\frac{2}{\lambda}}\frac{1}{\eta} = \frac{\sqrt{2}}{m} = \frac{2}{m_\psi} \tag{1.21}$$

On the $x > 0$ side of the kink we have $\phi \sim +\eta$ while on the $x < 0$ side we have $\phi \sim -\eta$. At the center of the kink, $\phi = 0$, and hence the Z_2 symmetry is restored in the core of the kink. Therefore the interior of the kink is a relic of the symmetric phase of the system.

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1.2 Rescaling

It is convenient to rescale variables in the action in Eq. (1.2) as follows

$$\Phi = \frac{\phi}{\eta}, \qquad y^\mu = \sqrt{\lambda}\, \eta x^\mu \tag{1.22}$$

Then the rescaled action is

$$S = \eta^2 \int d^2 y \left[\frac{1}{2} (\partial_\mu \Phi)^2 - \frac{1}{4} (\Phi^2 - 1)^2 \right] \tag{1.23}$$

where derivatives are now with respect to y^μ. The overall multiplicative factor, η^2, does not enter the classical equations of motion. Hence the classical $\lambda\phi^4$ action is free of parameters.[2]

1.3 Derrick's argument

In the context of rescaling, we now give Derrick's result that there can be no static, finite energy solutions in scalar field theories in more than one spatial dimension [45]. Consider the general action in n spatial dimensions

$$S = \int d^{n+1} x \left[\frac{1}{2} \sum_a (\partial_\mu \phi^a)^2 - V(\phi^a) \right] \tag{1.24}$$

where the potential is assumed to satisfy $V(\phi^a) \geq 0$. The index on ϕ^a means that the model can contain an arbitrary number of scalar fields. Let a purported static, finite energy solution to the equations of motion be $\phi_0^a(x^\mu)$ and consider the rescaled field configuration

$$\Phi_0^a(x^\mu) = \phi_0^a(\alpha x^\mu) \tag{1.25}$$

where $\alpha \geq 0$ is the rescaling parameter. Then the energy of the rescaled field configuration is

$$E\left[\Phi_0^a\right] = \int d^n x \left[\frac{1}{2} \left(\nabla \Phi_0^a \right)^2 + V\left(\Phi_0^a\right) \right] \tag{1.26}$$

where the sum over a is implicit and ∇ denotes the spatial gradient. Now define $y^\mu = \alpha x^\mu$ and this gives

$$E\left[\Phi_0^a\right] = \int d^n y \left[\frac{\alpha^{-n+2}}{2} \left(\nabla \phi_0^a(y) \right)^2 + \alpha^{-n} \, V\left(\phi_0^a(y)\right) \right] \tag{1.27}$$

[2] In quantum theory, however, the value of the action enters the path integral evaluation of the transition amplitudes and this will depend on η^2. So the properties of the quantized kink also depend on the value of η^2 (see Chapter 4).

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Since the kinetic terms are non-negative, we find that with $n \geq 2$ and $\alpha > 1$ this gives

$$E\left[\Phi_0^a\right] < E\left[\phi_0^a\right] \tag{1.28}$$

and hence ϕ_0^a cannot be an extremum of the energy. Only if $n = 1$ can ϕ_0^a be a static, finite energy solution.

In more than one spatial dimension, Derrick's argument allows for static solutions of infinite energy. The next section describes one such static solution in three spatial dimensions.

1.4 Domain walls

When kink solutions are placed in more than one spatial dimension, they become extended planar structures called "domain walls." The field configuration for a Z_2 domain wall in the yz-plane in three spatial dimensions is

$$\phi(t, x, y, z) = \eta \tanh\left(\sqrt{\frac{\lambda}{2}}\eta x\right) \tag{1.29}$$

The energy density of the wall is concentrated over all the yz-plane and is given by Eq. (1.19). The new aspects of domain walls are that they can be curved and deformations can propagate along them. These will be discussed in detail in Chapter 7.

Another feature of the planar domain wall is that it is invariant under boosts in the plane parallel to the wall. This is simply because the solution is independent of t, y and z and any transformations of these coordinates do not affect the solution.

1.5 Bogomolnyi method for Z_2 kink

Rather than directly solve the equations of motion, as was done in Section 1.1, we can also obtain the kink solution by the clever method discovered by Bogomolnyi [20]. The method is to obtain a first-order differential equation by manipulating the energy functional into a "whole square" form

$$\begin{aligned}
E &= \int \mathrm{d}x \left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 + V(\phi)\right] \\
&= \int \mathrm{d}x \left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}\left(\partial_x\phi \mp \sqrt{2V(\phi)}\,\right)^2 \pm \sqrt{2V(\phi)}\partial_x\phi\right] \\
&= \int \mathrm{d}x \left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}\left(\partial_x\phi \mp \sqrt{2V(\phi)}\,\right)^2\right] \pm \int_{\phi(-\infty)}^{\phi(+\infty)} \mathrm{d}\phi' \sqrt{2V(\phi')}
\end{aligned}$$

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Then, for fixed values of ϕ at $\pm\infty$, the energy is minimized if

$$\partial_t \phi = 0 \tag{1.30}$$

and

$$\partial_x \phi \mp \sqrt{2V(\phi)} = 0. \tag{1.31}$$

Further, the minimum value of the energy is

$$E_{\text{min}} = \pm \int_{\phi(-\infty)}^{\phi(+\infty)} \mathrm{d}\phi' \sqrt{2V(\phi')}. \tag{1.32}$$

The energy can only be minimized provided a solution to Eq. (1.31) exists with the correct boundary conditions. This relates the choice of sign in Eq. (1.31) to the boundary conditions and to the sign in Eq. (1.32). In our case, for the Z_2 kink boundary conditions ($\phi(+\infty) > \phi(-\infty)$), we take the $-$ sign in Eq. (1.31). Inserting

$$\sqrt{V(\phi)} = \sqrt{\frac{\lambda}{4}}(\eta^2 - \phi^2) \tag{1.33}$$

in Eq. (1.31) we get the kink solution in Eq. (1.9).

The energy of the kink follows from Eq. (1.32)

$$E = \frac{2\sqrt{2}}{3}\sqrt{\lambda}\eta^3 = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} \tag{1.34}$$

where $m = \sqrt{\lambda}\eta$ is the mass scale in the model (see Eq. (1.3)).

1.6 Z_2 antikink

In an identical manner, we can construct antikink solutions that have $Q = -1$. The boundary conditions necessary to get $Q = -1$ are $\phi(\pm\infty) = \mp\eta$ (see Eq. (1.8)). In the Bogomolnyi method, antikinks are obtained by taking the opposite choice of signs to the ones in the previous section

$$E = \int \mathrm{d}x \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}\left(\partial_x \phi + \sqrt{2V(\phi)}\,\right)^2 - \sqrt{2V(\phi)}\partial_x \phi \right] \tag{1.35}$$

This leads to the antikink solution

$$\bar{\phi}_{\text{k}} = -\eta \tanh\left(\sqrt{\frac{\lambda}{2}}\eta x\right) \tag{1.36}$$

1.7 Many kinks

The kink solution is well-localized and so it should be possible to write down field configurations with many kinks. However, a peculiarity of the Z_2 kink system is that a kink must necessarily be followed by an antikink since the asymptotic fields are restricted to lie in the vacuum: $\phi = \pm\eta$. It is not possible to have neighboring Z_2 kinks or a system with topological charge $|Q| > 1$.

There is a simple scheme, called the "product ansatz," to write down approximate multi-kink field configurations, i.e. alternating kinks and antikinks. Suppose we have kinks at locations $x = k_i$ and antikinks at $x = l_j$, where i, j label the various kinks and antikinks. The locations are assumed to be consistent with the requirement that kinks and antikinks alternate: $\ldots l_i < k_i < l_{i+1} \ldots$ Then an approximate field configuration that describes N kinks and N' antikinks is given by the product of the solutions for the individual objects with a normalization factor

$$\phi(x) = \frac{1}{\eta^{N+N'-1}} \prod_{i=1}^{N} \phi_{\mathrm{k}}(x - k_i) \prod_{j=1}^{N'} [-\phi_{\mathrm{k}}(x - l_j)] \tag{1.37}$$

where ϕ_{k} is the kink solution. Note that $|N - N'| \leq 1$ since kinks and antikinks must alternate.

The product ansatz is a good approximation as long as the kinks are separated by distances that are much larger than their widths. In that case, in the vicinity of a particular kink, say at $x = k_i$, only the factor $\phi(x - k_i)$ is non-trivial. All the other factors in Eq. (1.37) multiply together to give $+1$.

Another scheme to write down approximate multi-kink solutions is "additive" [109]. If ϕ_i denotes the ith kink (or antikink) in a sequence of N kinks and antikinks, we have

$$\phi(x) = \sum_{i=1}^{N} \phi_i \pm (N_2 - 1)\eta, \quad N_2 = N \ (\mathrm{mod}\ 2) \tag{1.38}$$

where the sign is $+$ if the leftmost object is a kink and $-$ if it is an antikink.

Neither the product or the additive ansatz yields a multi-kink solution to the equations of motion. Instead they give field configurations that resemble several widely spaced kinks that have been patched together in a smooth way. If the multi-kink configuration given by either of the ansätze is evolved using the equation of motion, the kinks will start moving due to forces exerted by the other kinks. In the next section we discuss the inter-kink forces.

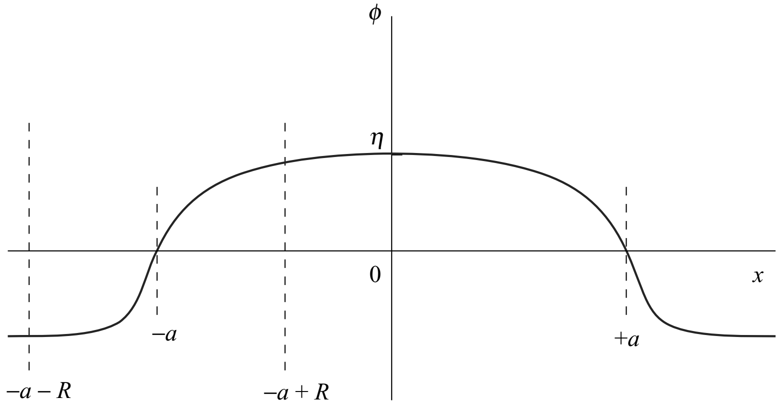

Figure 1.3 A widely separated kink-antikink.

1.8 Inter-kink force

Consider a kink at $x = -a$ and an antikink at $x = +a$ where the separation $2a$ is much larger than the kink width (see Fig. 1.3). We would like to evaluate the force on the kink owing to the antikink [109].

The energy-momentum tensor for the action Eq. (1.2) with a general potential $V(\phi)$ is

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left\{ \frac{1}{2}(\partial_\alpha \phi)^2 - V(\phi) \right\} \tag{1.39}$$

where $g_{\mu\nu}$ is the metric tensor that we take to be the flat metric, that is, $g_{\mu\nu} = \text{diag}(1, -1)$. The force exerted on a kink is given by Newton's second law, by the rate of change of its momentum. The momentum of a kink can be found by integrating the kink's momentum density, $T^{0i} = -T_{0i}$, in a large region around the kink. If the kink is located at $x = -a$, let us choose to look at the momentum, P, of the field in the region $(-a - R, -a + R)$

$$P = -\int_{-a-R}^{-a+R} \mathrm{d}x \ \partial_t \phi \partial_x \phi \tag{1.40}$$

After using the field equation of motion (for a general potential) and on performing the integration, the force on the field in this region is

$$F = \frac{\mathrm{d}P}{\mathrm{d}t} = \left[-\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 + V(\phi) \right]_{-a-R}^{-a+R} \tag{1.41}$$

To proceed further we need to know the field ϕ in the interval $(-a - R, -a + R)$. This may be obtained using the additive ansatz given in Eq. (1.38) which we take

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as an initial condition

$$\phi(t=0,x) = \phi_{\rm k}(x) + \bar{\phi}_{\rm k}(x) - \phi_{\rm k}(\infty) \tag{1.42}$$

In addition, we assume that the kinks are initially at rest

$$\partial_t \phi \Big|_{t=0} = 0 \tag{1.43}$$

The expression for the force is further simplified by using the Bogomolnyi equation (Eq. (1.31)) which is satisfied by both $\phi_{\rm k}$ and $\bar{\phi}_{\rm k}$

$$(\partial_x \phi)^2 = 2V(\phi) \tag{1.44}$$

This gives

$$F = \Big[-\partial_x \phi_{\rm k} \partial_x \bar{\phi}_{\rm k} + V(\phi_{\rm k} + \bar{\phi}_{\rm k} - \phi_{\rm k}(\infty)) - V(\phi_{\rm k}) - V(\bar{\phi}_{\rm k}) \Big]_{-a-R}^{-a+R} \tag{1.45}$$

The terms involving the potential can be expanded since the field is nearly in the vacuum at $x = -a \pm R$. Let us define

$$\begin{aligned}
\phi_{\rm k}^{\pm} &= \phi_{\rm k}(-a \pm R), \quad \bar{\phi}_{\rm k}^{\pm} = \bar{\phi}_{\rm k}(-a \pm R) \\
\Delta\phi_{\rm k}^{\pm} &= \phi_{\rm k}(-a \pm R) - \phi_{\rm k}(\pm\infty) \\
\Delta\bar{\phi}_{\rm k}^{\pm} &= \bar{\phi}_{\rm k}(-a \pm R) - \bar{\phi}_{\rm k}(-\infty)
\end{aligned} \tag{1.46}$$

(Note that the argument in the very last term is $-\infty$, independent of the signs in the other terms. This is because both $x = -a \pm R$ lie to the left of the antikink.) Also define

$$m_\psi^2 \equiv V''(\phi_{\rm k}(\infty)) = V''(\bar{\phi}_{\rm k}(\infty)) \tag{1.47}$$

Then the force is

$$F = -(\partial_x \phi_{\rm k}^+ \partial_x \bar{\phi}_{\rm k}^+ - \partial_x \phi_{\rm k}^- \partial_x \bar{\phi}_{\rm k}^-) + m_\psi^2 (\Delta\phi_{\rm k}^+ \Delta\bar{\phi}_{\rm k}^+ - \Delta\phi_{\rm k}^- \Delta\bar{\phi}_{\rm k}^-) \tag{1.48}$$

Let us illustrate this formula for the Z_2 kink, where

$$\begin{aligned}
\phi_{\rm k}(x) &= \eta \tanh(\sigma(x+a)) \\
\bar{\phi}_{\rm k}(x) &= -\eta \tanh(\sigma(x-a))
\end{aligned} \tag{1.49}$$

with $\sigma = \sqrt{\lambda/2}\,\eta$. Inserting these expressions in Eq. (1.48) and retaining only the leading order behavior gives

$$F = \frac{4m_\psi^4}{\lambda} {\rm e}^{-m_\psi l} \tag{1.50}$$

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where $l \equiv 2a$ is the kink separation. The force is attractive since it is acting on the kink at $x = -a$ and points toward the antikink at $x = +a$.

The result for the force could have been guessed from other considerations. The kinks are interacting by the exchange of massive scalars of mass m_ψ. As described in many quantum field theory texts [119] the force mediated by scalar interactions is the Yukawa force which goes like $\exp(-m_\psi l)$. The dimensionful prefactor of the force can be deduced on dimensional grounds while the numerical coefficient requires more detailed analysis.

1.9 Sine-Gordon kink

The sine-Gordon model is a scalar field theory in $1+1$ space-time dimensions given by the Lagrangian

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\alpha}{\beta^2}(1 - \cos(\beta\phi)) \tag{1.51}$$

The model is invariant under $\phi \to \phi + 2\pi n$ where n is any integer and thus possesses Z symmetry. The vacua are given by $\phi = 2\pi n/\beta$ and are labeled by the integer n.

As in the Z_2 case, the classical kink solutions can be found directly from the second-order equations of motion or by using the Bogomolnyi method (see Section 1.5). The kinks are solutions that interpolate between neighboring vacua. The unit charge kink solution is

$$\phi_{\rm k} = \frac{4}{\beta}\tan^{-1}\left(e^{\sqrt{\alpha}x}\right) + \phi(-\infty) \tag{1.52}$$

where the inverse tangent is taken to lie in the interval $(-\pi/2, +\pi/2)$. The antikink with $\phi(-\infty) = 2\pi/\beta$ and $\phi(+\infty) = 0$ is obtained from Eq. (1.52) by replacing x by $-x$.

$$\bar{\phi}_{\rm k} = \frac{4}{\beta}\tan^{-1}\left(e^{-\sqrt{\alpha}x}\right) + \phi(+\infty) \tag{1.53}$$

The width of the kink follows directly from these solutions and is $\sim 1/\sqrt{\alpha}$.

The energy of the kink also follows from Bogomolnyi's method (Eq. (1.32))

$$E_{\rm sG} = \frac{8\sqrt{\alpha}}{\beta^2} \tag{1.54}$$

Defining $m_\psi = \sqrt{\alpha}$ – the mass of excitations of the true vacuum – and $\sqrt{\lambda} = \sqrt{\alpha}\beta$ we get

$$E_{\rm sG} = 8\frac{m_\psi^3}{\lambda} \tag{1.55}$$

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While the Z_2 and sine-Gordon kinks are similar as classical solutions, there are some notable differences. For example, it is possible to have consecutive sine-Gordon kinks whereas in the Z_2 case, kinks can only neighbor antikinks. In addition, the sine-Gordon system allows non-dissipative classical bound states of kink and antikink – the so-called "breather" solutions – while no such solutions are known in the Z_2 case (though see Section 3.1). The sine-Gordon kink is also much more amenable to a quantum analysis as we discuss in Chapter 4.

We can use the additive ansatz described in Section 1.7 to construct field configurations for many kinks. Specializing to a kink-kink pair ($\phi(-\infty) = 0$ to $\phi(+\infty) = 4\pi/\beta$) and a kink-antikink pair ($\phi(-\infty) = 0$ and back to $\phi(+\infty) = 0$), we have

$$\phi_{\rm kk}(t,x) = \frac{4}{\beta}\left[\tan^{-1}\left(e^{\sqrt{\alpha}(x-a)}\right) + \tan^{-1}\left(e^{\sqrt{\alpha}(x-b)}\right)\right] \tag{1.56}$$

$$\phi_{\rm k\bar{k}}(t,x) = \frac{4}{\beta}\left[\tan^{-1}\left(e^{\sqrt{\alpha}(x-a)}\right) + \tan^{-1}\left(e^{-\sqrt{\alpha}(x-b)}\right)\right] - \frac{2\pi}{\beta} \tag{1.57}$$

with $b > a$.

The additive ansatz described above gives approximate solutions to the equations of motion for widely separated kinks ($b - a >> 1/\sqrt{\alpha}$). A one-parameter family of exact, non-dissipative, breather solutions composed of a kink and an antikink is

$$\phi_{\rm b}(t,x) = \frac{4}{\beta}\tan^{-1}\left[\frac{\eta\sin(\omega t)}{\cosh(\eta\omega x)}\right] \tag{1.58}$$

where $\eta = \sqrt{\alpha - \omega^2}/\omega$ and the $\tan^{-1}$ function is taken to lie in the range $(-\pi/2, +\pi/2)$. The frequency of oscillation, ω, is the parameter that labels the different breathers of the one-parameter family.

To see the breather as a bound state of a kink and an antikink, note that $\phi(t, \pm\infty) = 0$. Also, if $\eta \gg 1$, then $\phi(t, 0) \approx 2\pi/\beta$ during the time when $\eta\sin(\omega t) \gg 1$. Hence the breather splits up into a kink and an antikink for part of the oscillation period. For the remainder of the oscillation period, the kink and an antikink overlap and a clear separation cannot be made.

The constant energy of the breather is evaluated by substituting the solution at $t = 0$ (for convenience) in the sine-Gordon Hamiltonian with the result

$$E_{\rm b} = \frac{16}{\beta^2}\sqrt{\alpha - \omega^2} = 2E_{\rm sG}\sqrt{1 - \frac{\omega^2}{\alpha}} \tag{1.59}$$

As expected, when $\omega \to 0$, the breather energy is twice the kink energy.

As in Section 1.8 we can find the force on a kink owing to an antikink: from Eq. (1.48) the leading order behavior of the force is

$$F = \frac{20m_\psi^2}{\beta^2}e^{-m_\psi l} \tag{1.60}$$

where l is the kink separation.

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1.10 Topology: π_0

The kinks in the Z_2 and sine-Gordon models can be viewed as arising purely for topological reasons, as we now explain. A very important advantage of the topological viewpoint is that it is generalizable to a wide variety of models and can be used to classify a large set of solutions. When applied to field theories in higher spatial dimensions, topological considerations are convenient in order to demonstrate the existence of solutions such as strings and monopoles.

Consider a field theory for a set of fields denoted by $\mathbf{\Phi}$ that is invariant under transformations belonging to a group G. This means that the Hamiltonian of the theory is invariant under G:

$$\mathcal{H}[\mathbf{\Phi}] = \mathcal{H}[\mathbf{\Phi}^g] \tag{1.61}$$

where $g \in G$ and $\mathbf{\Phi}^g$ represents $\mathbf{\Phi}$ after it has been transformed by the action of g. The group G is a symmetry of the system, if Eq. (1.61) holds for every $g \in G$ and for every possible $\mathbf{\Phi}$. Now, let the Hamiltonian be minimized when $\mathbf{\Phi} = \mathbf{\Phi}_0$. Then, from Eq. (1.61), it is also minimized with $\mathbf{\Phi} = \mathbf{\Phi}_0^g$ for any $g \in G$, and the manifold of lowest energy states – "vacuum manifold" – is labeled by the set of field configurations $\mathbf{\Phi}_0^g$. However, there will exist a subgroup (sometimes trivial), H of G, whose elements do not move $\mathbf{\Phi}_0$:

$$\mathbf{\Phi}_0^h = \mathbf{\Phi}_0 \tag{1.62}$$

Hence, a group element $gh \in G$ acting on $\mathbf{\Phi}_0$ has the same result as g acting on $\mathbf{\Phi}_0$ (because h acts first and does not move $\mathbf{\Phi}_0$). So, while the configuration $\mathbf{\Phi}_0^g$ has the same energy as $\mathbf{\Phi}_0$ for any g, not all the $\mathbf{\Phi}_0^g$s are distinct from each other. The distinct $\mathbf{\Phi}_0^g$s are labeled by the set of elements $\{gh : h \in H\} \equiv gH$. The set of elements $\{gH : g \in G\}$ are said to form a "coset space" and the set is denoted by G/H; each element of the space is a coset (more precisely a "left coset" since g multiplies H from the left). Therefore the vacuum manifold is isomorphic to the coset space G/H.

We have so far connected the symmetries of the model to the vacuum manifold. Now we discuss the tools for describing the topology of the vacuum manifold. This will lead to a description of the topology of the vacuum manifold directly in terms of the symmetries of the model.

The topology of a manifold, M, is classified by the homotopy groups, $\pi_n(M; x_0)$, $n = 0, 1, 2, \ldots$ The idea is to consider maps from n-spheres to M, with the image of an n-sphere in M containing one common base point, x_0 (see Fig. 1.4). If two maps can be continuously deformed into each other, they are considered to be topologically equivalent. In this way, the set of maps is divided into equivalence classes of maps, where each equivalence class contains the set of maps that are continuously deformable into each other. The elements of $\pi_n(M; x_0)$ are the equivalence classes of maps from S^n to M with fixed base point. It is also possible to define (except for $n = 0$ as explained below) a suitable "product" of two maps: essentially the product of maps f and g (denoted by $g \cdot f$) and is defined to be "f composed with g" or "f followed by g." Then it is easily verified that the product is closed, associative, an identity map exists, and every map has an inverse. In mathematical language, $\forall\ f,\ g,\ h \in G$,

$$\begin{aligned} & f \cdot g \in G \\ & f \cdot (g \cdot h) = (f \cdot g) \cdot h \\ & \exists\ e \in G \text{ such that } f \cdot e = e \cdot f = f \\ & \exists\ f^{-1} \in G \text{ such that } f \cdot f^{-1} = f^{-1} \cdot f = e \end{aligned} \tag{1.63}$$

Thus all the group properties are satisfied and $\pi_n(M; x_0)$ is a group.

Two homotopy groups with different base points, say $\pi_n(M; x_0)$ and $\pi_n(M; x_0')$, can be shown to be isomorphic and hence the reference to the base point is often dropped and the homotopy group simply written as $\pi_n(M)$. Mathematicians have calculated the homotopy groups for a wide variety of manifolds and this makes it very convenient to determine if a given symmetry breaking leads to a topologically non-trivial vacuum manifold [145, 3, 171].

In the case of kinks or domain walls, the field defines a mapping from the points $x = \pm\infty$ to the vacuum manifold. Hence the relevant homotopy "group" is $\pi_0(M; x_0)$, which contains maps from S^0 (a point) to M. Since the base point is fixed, the image of either of the two possible S^0s ($x = \pm\infty$) has to be x_0, and $\pi_0(M; x_0)$ is trivial. Even if we do not impose the restriction that the maps should have a fixed base point, it is not possible to define a suitable composition of maps. Therefore π_0 does not have the right group structure and should merely be considered as a set of maps from S^0 to the vacuum manifold. The exception occurs if $M = G/H$ is itself a group, which occurs when H is a normal subgroup of G, because then $\pi_0(M)$ can inherit the group structure of M. In this case, the product of two maps from S^0 to M can be defined to be the map from S^0 to the product of the two image points in M. Generally, however, $\pi_0(M)$ should simply be thought of as a set of maps from S^0 to the various disconnected pieces of M.

To connect the elements of the homotopy groups to topological field configurations assume that the field, $\mathbf{\Phi}$, is in the vacuum manifold on S^n_∞. Therefore, $\mathbf{\Phi}_\infty \equiv \mathbf{\Phi}(\mathbf{x} \in S^n_\infty)$ defines a map from S^n to the vacuum manifold and this map can be topologically non-trivial if $\pi_n(M)$ is non-trivial. We want to show that if the map $\mathbf{\Phi}_\infty$ is topologically non-trivial, $\mathbf{\Phi}$ cannot be in the vacuum manifold at all points in the interior of S^n_∞. Consider what happens as the radius of S^n_∞ is continuously decreased. If the field remains on the vacuum manifold, continuity implies that the

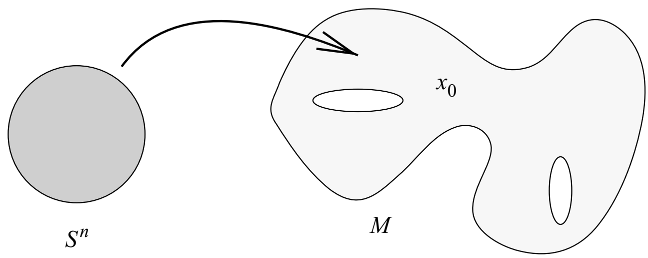

Figure 1.4 The nth homotopy group consists of maps from the n-dimensional sphere, S^n, to the vacuum manifold, M, such that the image of any map contains one common base point $x_0 \in M$. If two maps can be continuously deformed into each other, they are identified, and correspond to the same element of π_n. If two maps cannot be continuously deformed into each other, then they correspond to distinct elements of π_n. For example, this can happen if one of the maps encloses a "hole" in M, while the other encloses the hole a different number of times.

map $\mathbf{\Phi}_R$ from a sphere of radius R to M must also be non-trivial. Then as $R \to 0$, the map would still be non-trivial, implying that the field is multivalued at the origin since the field must continue to map S^n_R non-trivially as $R \to 0$. However, a field (by definition) cannot be multivalued. The only way out is if the field does not lie on the vacuum manifold everywhere. Therefore non-trivial topology at infinity implies that the energy density does not vanish at some points in space. The distribution of energy density is the topological defect which, depending on dimensionality, can manifest itself as a domain wall ($n = 0$), string ($n = 1$) or monopole ($n = 2$) or texture ($n = 3$).

The above argument establishes that topologically non-trivial boundary conditions imply non-vanishing energy in the field. However, it does not establish that a static solution exists with those boundary conditions. These must be found on a case-by-case basis. Indeed there are examples of topologically non-trivial boundary conditions where no static solution exists.[3] Also distinct elements of $\pi_n(M)$ ($n \geq 1$) need not lead to distinct field solutions. Only those solutions that correspond to elements of $\pi_n(M)$ that cannot be continuously deformed into each other, *if the maps are released from the base point*, are distinct. The italicized remark is in recognition of the fact that there can be two maps that are mathematically distinct (i.e. cannot be deformed into each other) only because they are fixed at the base point. However, the analog of a "common base point" in field theory would be to restrict attention to field configurations for which the fields attain a certain fixed value at some point on S^n_∞. Such a restriction is generally unphysical and hence, we are interested in

[3] For example, in three dimensions, the boundary conditions corresponding to a charge two 't Hooft-Polyakov magnetic monopole [124, 79] do not lead to any solution (for all but one value of model parameters). This is because any field configuration with those boundary conditions breaks up into two magnetic monopoles, each of unit charge, that repel each other and are never static.

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maps that cannot be deformed into each other even if we release the restriction that all maps have a common base point (for a more detailed discussion, see [171]).

In the case when the vacuum manifold has disconnected components, $\pi_0(G/H)$ is non-trivial since there are points (zero-dimensional spheres) that lie in different components that cannot be continuously deformed into one another. Therefore kinks occur whenever $\pi_0(G/H)$ is non-trivial. In the $\lambda\boldsymbol{\Phi}^4$ model, $G = Z_2$, $H = 1$ and $\pi_0(G/H) = Z_2$. In the sine-Gordon model $G = Z$, $H = 1$ and $\pi_0(G/H) = Z$. If $\pi_0 = Z_N$, we name the resulting kinks "Z_N kinks." In these simple examples, π_0 forms a group because G is Abelian and so G/H itself is a group. An example in which π_0 is not a group can be constructed by choosing $G = S_3$ (S_n is the permutation group of n elements) broken down to $H = S_2$.

The kinks in a model with disconnected elements in M can now be classified. Every element of $\pi_0(M)$ corresponds to a mapping from a point at spatial infinity to M and hence specifies a domain at infinity. Kinks occur if the domains at $\pm\infty$ are distinct. Therefore pairs of elements of $\pi_0(M)$ classify domain walls.

1.11 Bogomolnyi method revisited

The Bogomolnyi method can be extended to include a large class of systems. Let us start with the general energy functional for a matrix-valued complex scalar field $\boldsymbol{\Phi}$

$$
\begin{aligned}
E &= \int \mathrm{d}x\left[\mathrm{Tr}|\partial_t \Phi|^2 + \mathrm{Tr}|\partial_x \Phi|^2 + V(\Phi, \Phi^*)\right] \\
&= \int \mathrm{d}x\left[\partial_t \Phi^*_{ab}\partial_t \Phi_{ba} + \partial_x \Phi^*_{ab}\partial_x \Phi_{ba} + V(\Phi, \Phi^*)\right] \qquad (1.64)
\end{aligned}
$$

where a sum over matrix components labeled by a, b is understood. As in Section 1.5, we would like to write the energy density in "whole square" form

$$
E = \int \mathrm{d}x\left[\mathrm{Tr}\{|\partial_t \Phi|^2 + |\partial_x \Phi \mp U(\Phi)|^2 \pm (\partial_x \Phi^\dagger U) \pm (U^\dagger \partial_x \Phi)\}\right] \qquad (1.65)
$$

where we are restricting ourselves to static solutions and U is some matrix-valued function of $\boldsymbol{\Phi}$ such that

$$
\mathrm{Tr}(U^\dagger U) = V(\Phi, \Phi^*) \qquad (1.66)
$$

The energy is minimized if

$$
\partial_t \Phi = 0 \qquad (1.67)
$$

and

$$
\mathrm{Tr}|\partial_x \Phi \mp U(\Phi, \Phi^*)|^2 = 0 \qquad (1.68)
$$

which in turn gives

$$\partial_x \Phi \mp U(\Phi, \Phi^*) = 0 \tag{1.69}$$

The energy of the kink is

$$E = \pm \int_{-\infty}^{+\infty} \mathrm{d}x \ \mathrm{Tr}(\partial_x \Phi^\dagger \ U + U^\dagger \partial_x \Phi) \tag{1.70}$$

There is a further special case – the "supersymmetric" case – in which the energy integral can be performed explicitly. This is if U is a total derivative

$$U^* = \frac{\partial W}{\partial \Phi} \tag{1.71}$$

where $W(\mathbf{\Phi}, \mathbf{\Phi}^*)$ is the "superpotential," assumed to be real. Then

$$\begin{aligned} E &= \pm \int_{-\infty}^{+\infty} \mathrm{d}x \ \mathrm{Tr}\left(\partial_x \Phi^\dagger \frac{\partial W}{\partial \Phi^*} + \partial_x \Phi^T \frac{\partial W}{\partial \Phi}\right) \\ &= \pm \int_{-\infty}^{+\infty} \mathrm{d}x \ \partial_x W \\ &= \pm[W(\Phi(+\infty)) - W(\Phi(-\infty))] \end{aligned} \tag{1.72}$$

Therefore we see that the Bogomolnyi method allows for first-order equations of motion provided that V can be written as $\mathrm{Tr}(U^\dagger U)$. The method also provides an explicit expression for the kink energy if V is given in terms of a superpotential W as

$$V(\Phi) = \mathrm{Tr}(U^\dagger U) = \mathrm{Tr}\left|\frac{\mathrm{d}W}{\mathrm{d}\Phi}\right|^2 \tag{1.73}$$

1.12 On more techniques

The kink solutions we have been discussing fall under the more general category of "solitary waves," often discussed under the soliton heading. Strictly speaking, for a solution to classify as a "soliton," it also has to satisfy certain conditions on its scattering with other solitons. The subject is incredibly rich, and has led to the development of very sophisticated mathematical techniques such as Backlund transformations, inverse scattering methods, Lax heirarchy, etc. In addition, solitons have found tremendous importance in physical applications, especially non-linear optics and communication. Readers interested in the mathematics and physics of solitons might wish to consult [1, 48, 56].

Strict solitons are usually discussed in one spatial dimension and have limited application in the context of particle physics. Nonetheless, there are equally sophisticated techniques to study solitary wave solutions in higher dimensions. In

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particular, the ADHM construction [12] is used to find instanton solutions in four spatial dimensions and the Nahm equations lead to magnetic monopole solutions in three dimensions [114].

The soliton analyses mentioned above consider equations with complicated nonlinear terms and higher derivatives. In the context of particle physics, such terms and derivatives are rarely encountered. However, one complication that arises is due to larger (non-Abelian) symmetry groups. In the next chapter we will take the analysis of this section to such particle-physics motivated models. There we will find a spectrum of kink solutions with unusual interactions. As we proceed to further chapters, we will learn that the physics of such non-Abelian kinks can be quite different from that of the simple kinks discussed in this chapter.

1.13 Open questions

1. Discuss the conditions needed for a breather solution to exist. If an exact breather does not exist, can there be an approximate breather (see Section 3.1)? What is the approximation?