LEVELS OF DIVISION ALGEBRAS

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Introduction. In [7] the level, sublevel, and product level of finite dimensional central division algebras D over a field F were calculated when F is a local or global field. In Theorem 1.4 of this paper we calculate the same quantities if all finite extensions K of F satisfy $\tilde{u}(K) \leq 2$, where $\tilde{u}$ is the Hasse number of a field as defined in [2]. This occurs, for example, if F is an algebraic extension of the function field $R(x)$ where R is a real closed field or hereditarily Euclidean field (see [4]).

We recall the main definitions here. The *level* of D, $s(D)$, is the least integer s such that -1 is a sum of s squares in D. The *sublevel* of D, $\underline{s}(D)$, is the least integer $\underline{s}$ such that 0 is a sum of $\underline{s}+1$ nonzero squares in D. The *product level* of D, $s_\pi(D)$, is the least integer s_π such that -1 is a sum of s_π elements which are products of squares in D. In each case, s, $\underline{s}$, or s_π is set equal to ∞ if no such representation exists. Clearly $s_\pi(D) \leq \underline{s}(D) \leq s(D)$ and if D is a field then all three quantities agree with the usual level of a field.

Section 1 deals with those properties of formally real fields that are useful in calculating levels of division algebras. In Sections 2 and 3 we restrict attention to the case of quaternion division algebras. Additional background to the problem of calculating levels of division algebras may be found in the introduction to [7]. The main references for Sections 2 and 3 are [**9**, **10**].

We use standard terminology from the theory of quadratic forms and ordered fields as found in [6] and [11]. Let $F^\times$ denote the nonzero elements of F. We shall assume throughout that char $F \neq 2$. We let $D_F(q)$ denote the nonzero elements of F represented by a quadratic form q over F. The topological space of orderings of a field F is denoted X_F. Basic properties of X_F and basic results on SAP fields can be found in [3] and [11].

1. Levels and sublevels of division algebras over formally real fields. In this section we shall assume $-1 \notin F^2$, since otherwise $s_\pi(D) = \underline{s}(D) = s(D) = 1$ for any division algebra D. We recall from [7] that for a cyclic extension K/F of odd degree, $t(K/F)$ is the least integer t for which there exist $a_1, \ldots, a_t \in K$ such that $N_{K/F}(a_i) = 1$, $1 \leq i \leq t$, and $-1 \in D_K(\langle a_1, \ldots, a_t \rangle)$. We will use the following two results from [7].

(1) [7, Proposition 2.4] If K/F is a cyclic extension of odd degree $n > 1$, then $t(K/F) \leq n-1$.

(2) [7, Proposition 2.5, 2.6] Let D be a division algebra of odd degree over its center F. Then $2 \leq \underline{s}(D)$ and $\min\{3, s(F)\} \leq s(D)$. If, in addition, D is a cyclic division algebra and K is a maximal subfield cyclic over F, then $\underline{s}(D) \leq t(K/F)$ and $s(D) \leq t(K/F)+1$.

Let K/F be a cyclic extension of degree $n > 1$ and let σ be an automorphism that generates $\mathrm{Gal}(K/F)$. If P is an ordering on K, then $P, \sigma(P), \ldots, \sigma^{n-1}(P)$ are distinct orderings of K and if $a \in K^\times$, then $\sigma(a) \in P$ if and only if $a \in \sigma^{-1}(P)$.

Let $\epsilon_{K/F}: X_K \to X_F$ be the continuous map defined by restricting an ordering on K to F. Then there exists a clopen set Y in X_K such that $\epsilon_{K/F}|_Y$ is a homeomorphism onto $\epsilon_{K/F}(X_K)$. (See [1, p. 139] or [8, Theorem 1.10].)

We claim $X_K = \bigcup_{i=0}^{n-1} \sigma^i(Y)$ is a disjoint union of clopen sets. Clearly each $\sigma^i(Y)$ is

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clopen. If $P \in \sigma^i(Y) \cap \sigma^j(Y)$ then $P = \sigma^i(P') = \sigma^j(P'')$, P', $P'' \in Y$. It follows that $\epsilon_{K/F}(\sigma^{-i}(P)) = \epsilon_{K/F}(\sigma^{-j}(P))$ and this implies $\sigma^{-i}(P) = \sigma^{-j}(P)$ since each lies in Y. Therefore $i = j$. Finally, if $P \in X_K$, then $\epsilon_{K/F}(P) = \epsilon_{K/F}(P')$ for some $P' \in Y$. It follows that $P = \sigma^i(P')$ for some i, and $P \in \bigcup_{i=0}^{n-1} \sigma^i(Y)$.

1.1. Lemma. *Using the notation above, assume also that K is a SAP field. Then there exist $\alpha, \beta \in K^\times$ such that $\left\langle 1, \frac{\alpha}{\sigma(\alpha)}, \frac{\beta}{\sigma(\beta)} \right\rangle$ is totally indefinite over K.*

Proof. If n is even, the lemma is trivial since $N_{K/F}(-1) = 1$ implies $-1 = \frac{\alpha}{\sigma(\alpha)}$ for some $\alpha \in K^\times$ by Hilbert's Satz 90. Now assume n is odd. Let $D_1 = \bigcup_{i=0}^{(n-1)/2} \sigma^{2i}(Y)$ and let $D_2 = \bigcup_{i=0}^{(n-1)/2} \sigma^{2i+1}(Y)$. Note that $X_K = D_1 \cup D_2$, $D_1 \cap D_2 = Y$ and D_1, D_2 are clopen sets. Using the SAP property of K, choose $\alpha, \beta \in K^\times$ such that

$$\begin{array}{llll} \alpha >_P 0 & \text{if} \quad P \in D_1 & \beta >_P 0 & \text{if} \quad P \in D_2, \\ \alpha <_P 0 & \text{if} \quad P \notin D_1 & \beta <_P 0 & \text{if} \quad P \notin D_2. \end{array}$$

Then $\frac{\alpha}{\sigma(\alpha)} <_P 0$ if $P \in \bigcup_{i=1}^{n-1} \sigma^i(Y)$ and $\frac{\beta}{\sigma(\beta)} <_P 0$ if $P \in \bigcup_{i=2}^{n} \sigma^i(Y)$. Therefore $\left\langle 1, \frac{\alpha}{\sigma(\alpha)}, \frac{\beta}{\sigma(\beta)} \right\rangle$ is totally indefinite over K.

1.2. Proposition. *Suppose K/F is a cyclic extension of odd degree $n > 1$. If K is a SAP field and K satisfies property A_m, $m \geq 2$, (every torsion m-fold Pfister form defined over K is hyperbolic), then $t(K/F) \leq \min\{n-1, 2^{m-1}\}$.*

Proof. In general $t(K/F) \leq n - 1$ [**7**, Proposition 2.4]. Since K is a SAP field we may choose $\alpha, \beta \in K^\times$ as in Proposition 1.1. Then $q = \left\langle 1, \frac{\alpha}{\sigma(\alpha)}, \frac{\beta}{\sigma(\beta)} \right\rangle$ is a totally indefinite quadratic form defined over K and $\tau = q \perp \left\langle \frac{\alpha\beta}{\sigma(\alpha\beta)} \right\rangle$ is a torsion 2-fold Pfister form over K. Therefore $2^{m-2}\tau$ is hyperbolic over K, since K satisfies A_m, and it follows that the subform $\langle 1 \rangle \perp 2^{m-3}\tau$ is isotropic over K if $m \geq 3$ and q is isotropic over K if $m = 2$. This implies $-1 \in D_K(2^{m-3}\tau)$ if $m \geq 3$ and $-1 \in D_K\left(\left\langle \frac{\alpha}{\sigma(\alpha)}, \frac{\beta}{\sigma(\beta)} \right\rangle\right)$ if $m = 2$. In each case $t(K/F) \leq 2^{m-1}$.

1.3. Corollary. *Suppose K/F is a cyclic extension of odd degree $n > 1$. If $\bar{u}(K) < 2^m$, $m \geq 2$, then $t(K/F) \leq \min\{n-1, 2^{m-1}\}$.*

Proof. This follows from Proposition 1.2 since K satisfies property A_m, $m \geq 2$. Note that K is a SAP field by [**4**, Theorems B, C].

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1.4. THEOREM. *Let $D \neq F$ be a finite-dimensional crossed product division algebra over a field F. Suppose K is a maximal subfield of D, K/F Galois, and $\tilde{u}(K) \leq 2$. Then*
(1) $s(D) = \underset{\sim}{s}(D) = s_\pi(D) = 1$ *if* $\deg D$ *is even,*
(2) $s(D) = \min\{3, s(F)\}$ *if* $\deg D$ *is odd,*
(3) $\underset{\sim}{s}(D) = s_\pi(D) = 2$ *if* $\deg D$ *is odd.*
(*We are still assuming* $-1 \notin F^2$.)

Proof. First assume $\deg D$ is even. Then $[K:F]$ is even and Galois theory implies there exists a subfield L with $[K:L] = 2$ and $\tilde{u}(L) \leq 2$ by [**5**, Proposition 3.3]. The centralizer of L in D is then a quaternion algebra $(a, b)_L$ with $a, b \in L$. The quadratic form $q = \langle 1, a, b, -ab\rangle$ is isotropic over L since q is totally indefinite over L and $\tilde{u}(L) \leq 2$. Therefore $-1 = a\alpha^2 + b\beta^2 - ab\gamma^2 = (\alpha i + \beta j + \gamma k)^2$ for some $\alpha, \beta, \gamma \in L$ and where $i, j, k = ij$ is the standard basis of $(a, b)_L$. This shows $s(D) = 1$ and hence $\underset{\sim}{s}(D) = s_\pi(D) = 1$.

Now assume $\deg D$ is odd. Then $\mathrm{Gal}(K/F)$ has odd order and K contains a subfield L corresponding to a subgroup of prime order. Thus K/L is a cyclic extension of odd degree > 1. The centralizer of L in D is a cyclic algebra E of odd degree over its center L. From [**7**, Proposition 2.6] we have $2 \leq \underset{\sim}{s}(D)$ and $\min\{3, s(F)\} \leq s(D)$. From [**7**, Proposition 2.5] and Corollary 1.3 we have $\underset{\sim}{s}(D) \leq \underset{\sim}{s}(E) \leq t(K/L) \leq 2$ and $s(D) \leq s(E) \leq t(K/L) + 1 \leq 3$. Since $s(D) \leq s(F)$ we conclude that $s(D) = \min\{3, s(F)\}$ and $\underset{\sim}{s}(D) = 2$. We have $2 \leq s_\pi(D) \leq \underset{\sim}{s}(D)$ since $\deg D$ is odd [**7**, Proposition 1.1] and therefore $s_\pi(D) = 2$.

2. Levels and sublevels of quaternion algebras. Levels and sublevels of quaternion algebras were considered in [**9**] and [**10**]. We give several additional results in this section. For convenience we list some of Lewis's results below in Proposition 2.1.

Let $D = \left(\frac{a, b}{F}\right)$ be a quaternion algebra with standard basis $\{1, i, j, k = ij\}$ where $i^2 = a$, $j^2 = b$, $ji = -ij$. Following the notation in [**9, 10**], let $T_D = \langle 1, a, b, -ab\rangle$ and $T_P = \langle a, b, -ab\rangle$. We will consider the equation $c = \sum_{\lambda=1}^{n} (x_\lambda + y_\lambda i + z_\lambda j + w_\lambda k)^2$ with $c = 0$ or -1. Let $\vec{x} = (x_1, \ldots, x_n), \ldots, \vec{w} = (w_1, \ldots, w_n)$. Then this equation is equivalent to $c = \sum x_\lambda^2 + a \sum y_\lambda^2 + b \sum z_\lambda^2 - ab \sum w_\lambda^2$ and $\vec{x} \cdot \vec{y} = \vec{x} \cdot \vec{z} = \vec{x} \cdot \vec{w} = 0$.

Note that $s_\pi(D) = 1$ for all quaternion algebras D since $i^2 j^2 (ij)^{-2} = -1$. Also note that if D is a split algebra, then we may assume $a = 1$. In this case T_P is isotropic and the next result shows $s(D) = 1$.

2.1. PROPOSITION. (1) [**9**, Lemmas 2, 4] *If $\langle 1\rangle \perp nT_P$ is isotropic over F, then $s(D) \leq n$. The converse holds if $n = 2^k - 1$, $k \geq 2$. If $k = 1$, then $s(D) = 1$ if and only if either T_D is isotropic or $-1 \in F^2$.*

(2) [**10**, Proposition 2] *If either $\langle 1\rangle \perp nT_P$ or $(n+1)T_P$ is isotropic over F, then $\underset{\sim}{s}(D) \leq n$. The converse holds if $n = 2^k - 1$, $k \geq 1$.*

Lewis proved the "only if" direction of the following result in [**9**, Lemma 3].

2.2. THEOREM. *For $k \geq 0$, $s(D) \leq 2^k$ if and only if either* (1) *or* (2) *below holds.*
(1) $(2^k + 1)\langle 1\rangle \perp (2^k - 1)T_P$ *is isotropic over F.*
(2) $\langle 1\rangle \perp 2^k T_P$ *is isotropic over F.*

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Proof. We prove the "if" direction here. If (2) holds then $s(D) \leqq 2^k$ by Proposition 2.1 (1). Now assume (1) holds. Then there exists $-A \in D_F(\langle 1\rangle \perp (2^k-1)T_P)$ for some nonzero $A \in D_F(2^k\langle 1\rangle)$. Hence for some $\alpha \in F$ and $B, C, D \in D_F((2^k-1)\langle 1\rangle) \cup \{0\}$ we have

$$-A = \alpha^2 + aB + bC - abD, \quad \text{i.e.,} \quad -1 = \frac{1}{A^2}(\alpha^2 A + aAB + bAC - abAD).$$

Let $A = \sum_{\lambda=1}^{2^k} x_\lambda^2$. We show now there exist $y_\lambda \in F$ such that $\sum_{\lambda=1}^{2^k} y_\lambda^2 = AB$ and $\vec{x} \cdot \vec{y} = 0$. If $B = 0$, let each $y_\lambda = 0$. If $B \neq 0$, then $\langle A, AB\rangle \cong A\langle 1, B\rangle$ is a subform of $A \cdot 2^k\langle 1\rangle \cong 2^k\langle 1\rangle$ since $B \in D_F((2^k-1)\langle 1\rangle)$ and $A \in D_F(2^k\langle 1\rangle)$. Therefore such a $\vec{y}$ exists. Similarly $\vec{z}$, $\vec{w}$ exist such that $\sum_{\lambda=1}^{2^k} z_\lambda^2 = AC$, $\sum_{\lambda=1}^{2^k} w_\lambda^2 = AD$ and $\vec{x} \cdot \vec{z} = \vec{x} \cdot \vec{w} = 0$. It follows that

$$\sum_{\lambda=1}^{2^k}\left(\frac{\alpha x_\lambda}{A} + \frac{y_\lambda}{A} i + \frac{z_\lambda}{A} j + \frac{w_\lambda}{A} k\right)^2 = \frac{1}{A^2}(\alpha^2 A + aAB + bAC - abAD) = -1.$$

Therefore $s(D) \leq 2^k$.

2.3. Lemma. *Suppose $2^k T_P$ is isotropic, $k \geq 0$. Then $(1 + [\frac{2}{3} \cdot 2^k])T_P$ is isotropic.* ([ ] *is the greatest integer function.*)

Proof. If $2^k T_P$ is isotropic then $2^k\langle -a, -b, ab\rangle$ is isotropic and $2^k\langle\langle -a, -b\rangle\rangle$ is hyperbolic. After multiplying by -1 we see that any subform of $2^k\langle -1, a, b, -ab\rangle$ of dimension greater than $2 \cdot 2^k$ is isotropic. The conclusion follows since $3(1 + [\frac{2}{3} \cdot 2^k]) > 3(\frac{2}{3} \cdot 2^k) = 2 \cdot 2^k$.

2.4. Proposition. *If $k \geq 2$, then $\underset{\sim}{s}(D) \leq 2^k - 1$ implies $s(D) \leq 2^k - 1$.*

Proof. If $\underset{\sim}{s}(D) \leq 2^k - 1$ then by Proposition 2.1(2), either $\langle 1\rangle \perp (2^k-1)T_P$ or $2^k T_P$ is isotropic. If $\langle 1\rangle \perp (2^k-1)T_P$ is isotropic, then $s(D) \leq 2^k - 1$ by Proposition 2.1(1). If $2^k T_P$ is isotropic, then $(1 + [\frac{2}{3} \cdot 2^k])T_P$ is isotropic by Lemma 2.3. Then Proposition 2.1(1) implies $s(D) \leq 1 + [\frac{2}{3} \cdot 2^k] \leq 2^k - 1$ since $k \geq 2$.

Remark. This result is a slight improvement of [**10**, Proposition 4].

2.5. Theorem. (1) *If $\underset{\sim}{s}(D) = 1$, then $s(D) \leq 2$ and if $2 \leq \underset{\sim}{s}(D) < \infty$, then $s(D) < 2\underset{\sim}{s}(D)$.*

(2) *If $\underset{\sim}{s}(D) = 2^k - 1$, $k \geq 2$, then $s(D) = 2^k - 1$.*

(3) *If $s(D) = 2^k$, $k \geq 2$, then $\underset{\sim}{s}(D) = 2^k$.*

(4) *If $s(D) = 2^k + 1$, $k \geq 1$, then $\underset{\sim}{s}(D) = 2^k$ or $2^k + 1$.*

Proof. (1) *If* $\underset{\sim}{s}(D) = 1$, then $s(D) \leqq 2$ by Proposition 2.1(2) and Theorem 2.2. If $2 \leq \underset{\sim}{s}(D) < \infty$, then $s(D) < 2\underset{\sim}{s}(D)$ by Proposition 2.4.

If $k \geq 2$, then (2), (3), (4) all follow from Proposition 2.4 and the estimate $\underset{\sim}{s}(D) \leq s(D)$. If $k = 1$ in (4) and $s(D) = 3$, then $\underset{\sim}{s}(D) \leq 3$. Since $\underset{\sim}{s}(D) = 1$ implies $s(D) \leq 2$ by (1), it follows that $s(D) = 3$ implies $\underset{\sim}{s}(D) = 2$ or 3.

The next section deals with examples where the ordered pair $(\underset{\sim}{s}(D), s(D))$ has been computed.

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3. Examples of levels and sublevels of quaternion division algebras. We continue the notation of Section 2. If F is a local or global field then $(\underset{\sim}{s}(D), s(D))$ was calculated in [7] and we had $\underset{\sim}{s}(D) \le s(D) \le 2$ in all cases. If $\tilde{u}(F) \le 2$, then $(\underset{\sim}{s}(D), s(D))$ was calculated in Theorem 1.4 and we found $(\underset{\sim}{s}(D), s(D)) = (1, 1)$. (The maximal subfield K satisfies $\tilde{u}(K) \le 2$ by [**5**, Proposition 3.3].) In [**12**], $s(D)$ was calculated if $D = \left(\frac{a, t}{F((t))}\right)$ where a is a nonsquare in $F^\times$ and $F((t))$ is the field of formal power series over F. We review this result below and also calculate $\underset{\sim}{s}(D)$.

If $a \in F^\times$, let $g(a)$ be the least integer such that $g(a)\langle 1, -a\rangle$ is isotropic over F and set $g(a) = \infty$ if no such integer exists.

3.1. THEOREM [**12**]. (1) *Let* $D = \left(\frac{a, t}{F((t))}\right)$ *where a is a nonsquare in* $F^\times$. *Then* $s(D) = \min\{g(a), s(F(\sqrt{a}))\}$.

(2) *Let* $F = \mathbb{R}(x_1, \dots, x_n)$, *the rational function field in n variables over the real numbers, and assume* $n \ge 2$. *Let* $a = \sum_{i=1}^{n} x_i^2$ *and let* $D = \left(\frac{a, t}{F((t))}\right)$, $D' = \left(\frac{-a, t}{F((t))}\right)$. *Then* $s(D) = 2^k + 1$ *if* $2^k < n \le 2^{k+1}$ *and* $s(D') = 2^k$ *if* $2^k \le n < 2^{k+1}$.

It was shown in [**12**] that $g(a)$ always has the form $2^k + 1$ if $g(a)$ is finite. Since the level of a field is always a power of 2 if finite, we see that for D as in Theorem 3.1(1), $s(D) = 2^k$ or $2^k + 1$ if $s(D) < \infty$.

3.2. THEOREM. (1) *Let D be as in Theorem* 3.1(1). *Then*

$$\underset{\sim}{s}(D) = \min\{g(a) - 1, s(F(\sqrt{a}))\}.$$

(2) *Let* D, D' *be as in Theorem* 3.1(2). *Then* $\underset{\sim}{s}(D) = 2^k$ *if* $2^k < n \le 2^{k+1}$ *and* $\underset{\sim}{s}(D') = 2^k$ *if* $2^k \le n < 2^{k+1}$.

We omit the proof of Theorem 3.2 since it is so similar to the proof in [**12**]. Note that the result for D' in (2) with $k \ge 2$ follows from Theorems 2.5(3) and 3.1(2).

The examples mentioned here show that the following values of the ordered pair $(\underset{\sim}{s}(D), s(D))$ can occur when D is a quaternion algebra:

$$(\underset{\sim}{s}(D), s(D)) = (2^k, 2^k) \quad \text{or} \quad (2^k, 2^k + 1), \quad k \ge 0.$$

The main questions to consider on levels and sublevels of quaternion algebras D are the following.

(1) Is it always true that $\underset{\sim}{s}(D) = 2^k$?

(2) Is it always true that $s(D) = 2^k$ or $2^k + 1$?

(3) Is it always true that $s(D) \le \underset{\sim}{s}(D) + 1$?

Added in proof. Krüskemper and Wadsworth have constructed a quaternion division algebra D with $\underset{\sim}{s}(D) = 3$. By Theorem 2.5(2), it follows that $s(D) = 3$. Thus the answer to question (1) is no.

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