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CATEGORICAL CONSTRUCTIONS IN C^*-ALGEBRA THEORY

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Abstract

The notions of limits and colimits are studied in the category of C^*-algebras. It is shown that limits and colimits of diagrams of C^*-algebras are stable under tensor product by a fixed C^*-algebra, and crossed product by a locally compact group.

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1. Introduction

In a recent paper, Pedersen has initiated a systematic study of pullback and pushout constructions and their intrinsic connection with extensions and free products in C^*-algebra theory ([6]). In this paper we investigate the more general notions of limits and colimits in the category of C^*-algebras, which include pullback and pushout as special cases. Our main results Theorem 3.3 and Theorem 4.2 are generalization of [6, Theorem 4.8 and Theorem 6.3] to colimits of diagrams of C^*-algebras. Namely, assuming some restrictions, colimits of diagrams of C^*-algebras are shown to be stable under tensoring by a fixed C^*-algebra, and under crossed product with a fixed group. To prove these results we need to assume that the diagrams are connected and that the connecting morphisms are proper. However, this restriction is not needed in the case of limits. Examples of connected diagrams are pullback, pushout, and direct limits. The paper is organized as follows. In Section 2, we gather notations, terminologies, and several constructions from category theory. In particular, the notions of limits

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and colimits which are the main subjects of the paper are discussed extensively. The colimit of a diagram is seen to be a generalized amalgamated free product of the C^*-algebras appearing in the diagram , while the limit is a C^*-subalgebra of the direct product of the family. Section 3, is devoted to tensor products and our main theorem in this section (Theorem 3.3) shows that if a diagram of C^*-algebras is tensored (maximal tensor product) by a C^*-algebra, then the colimit of the resulting diagram is obtained simply by taking the tensor product of the colimit of the original diagram with the tensoring C^*-algebra. In Section 4 we prove a similar theorem (Theorem 4.2) for the full crossed product by a locally compact group when the algebras in the diagram are equipped with an action of a locally compact group. The stability of the limit of a diagram under tensor product (minimal and maximal) and crossed product (full and reduced) is proved in Section 4. In the case of minimal tensor product or the reduced crossed product we need the tensoring C^*-algebra or the group to be exact (cf. [9]). Throughout, C^*-alg denotes the category of C^*-algebras. The maps in this category are $*$-homomorphisms of C^*-algebras.

As alluded to, this work was motivated by the recent work ([6]) of Pedersen on pullbacks and pushouts. Much of the ideas and techniques used in this paper are based on that article.

2. Constructions

In this section we introduce our notations, definitions and several constructions. The notions of limit and colimit in the category of C^*-algebras are given. As well, equalizers and coequalizers in C^*-alg and their relation with limits and colimits are studied. We show that in the category C^*-alg both limits and colimits exist. The existence of pullbacks and pushouts then follow as special cases. Our main reference for categorical results is ([4]).

DEFINITION 2.1. Let I be a small (indexed) category with a set of objects and a set of morphisms. A *diagram*, D, in C^*-alg, indexed by I, is a functor $I \to C^*$-alg. In other words, for each $i \in I$, $A(i) = A_i$ and for each $i \to j$ in I, there is a map $A_i \xrightarrow{\alpha_{ij}} A_j$ in C^*-alg. For $i, j \in I$, $i \leq j$ would mean that there exists a map $i \to j$ in the category I including the identity maps.

For example any C^*-algebra C determines a constant diagram, which has the same value C for all $i \in I$.

DEFINITION 2.2. A *cocone* is a map from a diagram D to a constant diagram C (called the vertext), which consists of a family of maps $\{A_i \xrightarrow{\iota_i} C\}_{i \in I}$, denoted by

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$\iota : D \to C$ such that the triangle

$$\begin{tikzcd} A_i \arrow[rr, "\alpha_{ij}"] \arrow[dr, "\iota_i"'] & & A_j \arrow[dl, "\iota_j"] \\ & C & \end{tikzcd}$$

commutes, where α_{ij} is the induced map from $i \to j$ in I.

A cocone $D \xrightarrow{\iota} C$ with vertex C is *universal* to D when for every other cocone $f : D \to C'$ there is a unique map $\gamma : C \to C'$ with $\gamma\iota_i = f_i$ for all $i \in I$ as in the following commutative diagram

$$\begin{tikzcd} A_i \arrow[rr, "\alpha_{ij}"] \arrow[dr, "\iota_i"] \arrow[ddr, "f_i"'] & & A_j \arrow[dl, "\iota_j"'] \arrow[ddl, "f_j"] \\ & C \arrow[d, "\gamma"] & \\ & \mathscr{C}' & \end{tikzcd}$$

The universal cocone, if it exists, is called the *colimit* of the diagram D and is denoted by $C = \varinjlim D$. For example, if I is the category $\leftarrow \bullet \rightarrow$, then the colimit is called a *pushout*, and when I is $\bullet \rightrightarrows \bullet$, the colimit is called a *coequalizer* (where we have indicated only the nonidentity maps). The dual notion of colimit is *limit*. A cone is a diagram map from a constant diagram C to some other diagram D, which consists of a family of maps $\{C \xrightarrow{f_i} A_i\}_{i \in I}$, denoted by $f : C \to D$, such that the triangle

$$\begin{tikzcd} & C \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ A_i \arrow[rr, "\alpha_{ij}"'] & & A_j \end{tikzcd}$$

commutes, where α_{ij} is the induced map from $i \to j$ in I.

The universal cone, if it exists, is called the *limit of the diagram D* and is denoted by $C = \varinjlim D$. For example, if I has two elements as a discrete category, then a diagram is just a pair of C^*-algebras and a limit of that diagram is the product of these C^*-algebras. The limits of the category $\rightarrow \bullet \leftarrow$ is called a pullback and that of $\bullet \rightrightarrows \bullet$ an *equalizer*.

We show that C^*-alg is closed under colimits. Given a family $\{A_i\}_{i \in I}$ of C^*-algebras, the *coproduct* of the family is the universal C^*-algebra A such that there

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exists a morphism from each A_i into A compatible with the connecting morphisms of the diagram and if B is another C^*-algebra with these properties, then there exists a unique morphism from A to B making the relevant diagrams commutative.

LEMMA 2.3. *Let $\{A_i\}_{i\in I}$ be any family of C^*-algebras. Then the coproduct of this family exists.*

PROOF. Let $\mathscr{F}$ be the set of finite subsets of I. Then, for $F \in \mathscr{F}$, it is easy to see that the coproduct of $\{A_f\}_{f\in F}$ is just the free product of A_f for $f \in F$. If we denote these free products by $A_F = \coprod_{f\in F} A_f$, then we get a directed system of C^*-algebras $\{A_F\}_{F\in\mathscr{F}}$. Now the directed limit of this new family of C^*-algebras is the coproduct of the family $\{A_i\}_{i\in I}$ which we denote by $A = \coprod_{i\in I} A_i$. Clearly, there exists an embedding from each A_i into $A = \coprod_{i\in I} A_i$. On the other hand, if for each $i \in I$ there is a morphism α_i from A_i to another C^*-algebra B, then for each $F \in \mathscr{F}$, since A_F is finite coproduct, there exists a unique map β_F from A_F to B making the triangles commute:

$$\begin{tikzcd} A_f \arrow[r] \arrow[dr, "\alpha_i"'] & A_F \arrow[d, "\beta_F"] \\ & B \end{tikzcd}$$

Now since A is the directed limit of A_F's, there is a unique map γ from A to B making the following diagram commutative

$$\begin{tikzcd} A_f \arrow[r] \arrow[dr, "\alpha_i"'] & A_F \arrow[r] \arrow[d, "\beta_F"] & A \arrow[dl, "\gamma"] \\ & B & \end{tikzcd}$$

Therefore, the C^*-algebra A has the universal property of coproduct. $\square$

REMARK 2.4. The coproduct of a family $\{A_i\}_{i\in I}$ is denoted by $\coprod_i A_i$. If $D= \{A_i\}_{i\in I}$ is a diagram of C^*-algebras, $\coprod_{i\leq j} A_j$, will denote the coproduct of the family $\{A_j\}_{i\leq j}$ obtained by adding for each $i < j$ one copy of A_j to the original family. Similarly, $\prod_{i\leq j} A_i$ denotes the product of the family $\{A_i\}_{i\leq j}$ obtained by adding for each $i < j$ one copy of A_i to the original family.

Next we show that in C^*-alg every pair of parallel arrows have a coequalizer.

LEMMA 2.5. *Let $A \underset{g}{\overset{f}{\rightrightarrows}} B$ be parallel maps in C^*-alg. Then the coequalizer and the equalizer of f and g exist.*

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PROOF. We show that the diagram

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{\pi} B/\overline{\langle f(x) - g(x)\rangle}$$

where $\overline{\langle f(x) - g(x)\rangle}$ denotes the closed ideal of B generated by the differences $f(x) - g(x)$ and π is the canonical surjection is the coequalizer of the two maps. Obviously, $\pi f = \pi g$. Now if $B \xrightarrow{h} D$ is any map from B to D such that $hf = hg$, then we can define $\gamma : B/\overline{\langle f(x) - g(x)\rangle} \to D$ by $\gamma(b + I) = h(b)$, for all $b \in B$. It is easy to see that the map γ is well defined and unique. Therefore the above diagram is a coequalizer. It is clear that $E = \{x \in A : f(x) = g(x)\}$ is the equalizer of the two maps f and g. $\square$

EXAMPLE. The Calkin algebra is a coequalizer for the embedding $K(H) \to B(H)$ and the zero map. Similarly, each quotient is a coequalizer.

THEOREM 2.6. *In C^*-alg every diagram has a colimit.*

PROOF. Let $D = \{A_i\}_{i\in I}$ be a diagram in C^*-alg, where I is an indexed category. Consider the following diagram

$$\begin{tikzcd} A_i \arrow[r, "\mathrm{id}"] \arrow[d, "\iota_i"'] & A_i \arrow[d, "\iota_i"] \\ \coprod_{i\le j} A_i \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \coprod_{i\in I} A_i \\ A_i \arrow[r, "\alpha_{ij}"'] \arrow[u, "\iota_i"] & A_j \arrow[u, "\iota_j"'] \end{tikzcd}$$

Since $\coprod_{i\le j} A_j$ is a coproduct, there exists a unique map f making the upper square commute and a unique map g such that the lower square commutes. By Lemma 2.5, there is a coequalizer diagram for the two maps f and g,

$$\coprod_{i\le j} A_j \overset{f}{\underset{g}{\rightrightarrows}} \coprod_{i\in I} A_i \xrightarrow{\pi} C.$$

Now we will show that C is in fact the colimit of the diagram D. The map π composite with the injections ι_i gives maps $\sigma_i = \pi\iota_i : A_i \to C$ for each i. Since π is a coequalizer,

$$\sigma_j\alpha_{ij} = \pi\iota_j\alpha_{ij} = \pi g\iota_i = \pi f\iota_i = \pi\iota_i = \sigma_i.$$

Hence $D \xrightarrow{\sigma} C$ is a cocone. If $D \xrightarrow{\gamma} E$ is any other such cocone, its maps $A_i \xrightarrow{\gamma_i} E$ factor through a unique map $\coprod_{i\in I} A_i \xrightarrow{k} E$ from the coproduct. In other words,

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$k\iota_i = \gamma_i$ for all i. Since γ is a cocone, $kf = kg$. And π being a coequalizer, there exists a unique $C \xrightarrow{\delta} E$ such that $\delta\pi = k$. Therefore

$$\delta\sigma_i = \delta c\iota_i = k\iota_i = \gamma_i$$

for all i. Hence $\delta\sigma = \gamma$, that is, γ factors through σ. This proves that (C, σ) is a universal cocone. Therefore, (C, σ) is colimit for the diagram D, that is, $C = \varinjlim D$. □

COROLLARY 2.7. *In C^*-alg pushouts exist.*

PROOF. Since a pushout is the colimit of a diagram $\leftarrow \bullet \rightarrow$, by Theorem 2.6, it exists in C^*-alg. □

LEMMA 2.8. *In C^*-alg, every coequalizer is a pushout and conversely every pushout is a coequalizer.*

PROOF. Let $A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{\pi} C$ be a coequalizer diagram in C^*-alg. Then the following diagram is pushout

$$\begin{tikzcd}
A * B \arrow[r, "\binom{f}{1}"] \arrow[d, "\binom{g}{1}"'] & B \arrow[d, "\pi"] \arrow[ddr, "y"] & \\
B \arrow[r, "\pi"'] \arrow[drr, "x"'] & C & \\
 & & X
\end{tikzcd}$$

where $\binom{f}{1}$ and $\binom{g}{1}$ are the unique induced maps from the coproduct (free product) $A * B$. If $x, y : B \to X$ are two maps with $x\binom{f}{1} = y\binom{g}{1}$, then $xf = yg$ and $x = y$. But π is coequalizer, so there exists a unique map $C \xrightarrow{\gamma} X$ such that $\gamma\pi = x = y$. Hence the diagram is pushout. The converse follows dually. □

The following theorem is the dual of Theorem 2.6, which shows that C^*-alg is also closed under limits.

THEOREM 2.9. *Any diagram D in C^*-alg has a limit.*

PROOF. By Lemma 2.5, the equalizer of any two parallel maps exists in C^*-alg. If we show that the product of any family of C^*-algebras exists, then the limit of the diagram D would be the equalizer of the two parallel maps between $\prod_{i \in I} A_i$ and $\prod_{i \leq j} A_j$, and the proof is the dual of Theorem 2.6. Given $\{A_i\}_{i \in I}$, a family of C^*-algebras, the product of the family is simply the direct product denoted by $\prod_i A_i$.

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This is the C^*-algebra of functions defined on I such that $f(i) \in A_i$ and $i \to \|f(i)\|$ is bounded, under pointwise operations. This C^*-algebra has also universal property of product. For, if $\{C \xrightarrow{\pi_i} A_i\}_{i \in I}$ is a family of maps from a C^*-algebra C to A_i, then we define the unique map $C \xrightarrow{\gamma} \prod_i A_i$ by $\gamma(c) = f$, where $f(i) = \pi_i(c)$. $\square$

In fact, an explicit description of the limit of a diagram $D(\{A_i\}_{i \in I})$ as a C^*-subalgebra is given by

$$\varprojlim D = \left\{ f \in \prod_i A_i : f(j) = \alpha_{ij}(f(i)),\ \alpha_{ij} : A_i \to A_j \right\}.$$

COROLLARY 2.10. *In C^*-alg pullbacks exist.*

PROOF. Since a pullback is the limit of a diagram $\to \bullet \leftarrow$, by Theorem 2.9 it exists in C^*-alg. $\square$

LEMMA 2.11. *In C^*-alg every equalizer is a pullback and conversely a pullback is an equalizer.*

PROOF. Let $E \xrightarrow{e} A \underset{g}{\overset{f}{\rightrightarrows}} B$ be an equalizer diagram in C^*-alg. Then the following diagram is pullback

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, "e"'] & A \arrow[d, "{(I,g)}"] \\ A \arrow[r, "{(I,f)}"'] & A \times B \end{tikzcd}$$

where I is the identity map $A \xrightarrow{I} A$. If $a, b : Y \to A$ are two maps from Y to A such that $(I, g)a = (I, f)b$, then $ga = fb$ and $a = b$. Since E is an equalizer, there exists a unique map $Y \xrightarrow{\alpha} E$ such that $e\alpha = a = b$. Therefore the above diagram is pullback. The converse follows dually. $\square$

3. Tensor products

In this section we prove that colimit diagrams are stable under maximal tensor product by a fixed C^*-algebra Y. Throughout this paper $\otimes$ will denote the maximal tensor product. Recall that a morphism $\alpha : A \to B$ between C^*-algebras is said to be proper if for any approximate unit (u_i) of A, $(\alpha(u_i))$ is an approximate unit of B. For a C^*-algebra A, $M(A)$ denotes the multiplier algebra of A. It is easy to see that a proper morphism $\alpha : A \to B$ extends to a morphism from $M(A)$ into $M(B)$. Let $D = \{A_i\}_{i \in I}$ be a diagram of C^*-algebras. Then, $D \otimes Y$ denotes the diagram

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obtained by taking the maximal tensor product of the members of D by Y. Given a map $\alpha : A \to B$, the induced morphism, $\alpha \otimes i$ from $A \otimes Y$ into $B \otimes Y$ will always be denoted by $\overline{\alpha}$. The case of minimal tensor product is considered at the end of the next section along with the reduced crossed product.

DEFINITION 3.1. A diagram, D, of C^*-algebras is said to be connected if given $A_i, A_j \in D$, at least one of the following holds

(i) there exists a morphism $\alpha_{ij} : A_i \to A_j$;
(ii) there exists a morphism $\alpha_{ji} : A_j \to A_i$;
(iii) there exists $k \in I$ and morphisms $\alpha_{ik} : A_i \to A_k$ and $\alpha_{jk} : A_j \to A_k$;
(iv) there exists $k \in I$ and morphisms $\alpha_{ki} : A_k \to A_i$ and $\alpha_{kj} : A_k \to A_j$.

For A_i, A_j, A_k in D with morphisms $\alpha_{ki} : A_k \to A_i$, $\alpha_{kj} : A_k \to A_j$ let A_{ij} be the corresponding pushout given by the commutative diagram

$$\begin{tikzcd} A_k \arrow[r, "\alpha_{ki}"] \arrow[d, "\alpha_{kj}"'] & A_i \arrow[d] \\ A_j \arrow[r] & A_{ij} \end{tikzcd}$$

The C^*-algebra A_{ij} may not be in D. Denote by $\tilde{D}$ the diagram obtained by adding all such pushouts to D, that is, the pushout completion of D. The pullback completion is defined similarly and denoted by $\hat{D}$. With these conventions we have the following lemma whose proof is routine and omitted.

LEMMA 3.2. *Let D be a diagram of C^*-algebras. Then,*

(i) $\varinjlim D = \varinjlim \tilde{D}$;
(ii) $\varprojlim D = \varprojlim \hat{D}$.

THEOREM 3.3. *Let D be a connected diagram of C^*-algebras such that the connecting morphisms are proper. Then, for any C^*-algebra Y, $\varinjlim(D \otimes Y) = \varinjlim D \otimes Y$.*

PROOF. Let $A = \varinjlim D$. We want to prove that $A \otimes Y = \varinjlim(D \otimes Y)$. First assume that Y is a unital C^*-algebra. For each $i \in I$, let $\overline{\psi_i} : A_i \otimes Y \to Z$ be a morphism into the C^* algebra Z such that for all i, j the diagram

$$\begin{tikzcd} A_i \otimes Y \arrow[rr, "\overline{\alpha}_{ij}"] \arrow[dr, "\overline{\psi}_i"'] & & A_j \otimes Y \arrow[dl, "\overline{\psi}_j"] \\ & Z & \end{tikzcd}$$

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commutes, where $\overline{\alpha}_{ij} = \alpha_{ij} \otimes I$. Restrict each $\overline{\psi}_i$ to A_i to get commuting diagrams

$$\begin{tikzcd} A_i \arrow[rr, "\alpha_{ij}"] \arrow[dr, "\psi_i"'] & & A_j \arrow[dl, "\psi_j"] \\ & Z & \end{tikzcd}$$

Hence, there exists a unique morphism $\sigma : A \to Z$ such that the triangles

$$\begin{tikzcd} & Z & \\ & A \arrow[u, "\sigma"] & \\ A_i \arrow[uur, "\psi_i"] \arrow[ur, "\varphi_i"] \arrow[rr, "\alpha_{ij}"'] & & A_j \arrow[uul, "\psi_j"'] \arrow[ul, "\varphi_j"'] \end{tikzcd}$$

are commutative, that is $\psi_j = \sigma \circ \varphi_j$ for all $j \in I$, where $\varphi_i : A_1 \to A$ is the injection. Next, consider the diagrams

$$\begin{tikzcd} A_i \otimes Y \arrow[rr, "\overline{\alpha}_{ij}"] \arrow[dr, "\overline{\psi}_i"'] & & A_j \otimes Y \arrow[dl, "\overline{\psi}_j"] \\ & Z_{ij} & \end{tikzcd}$$

where Z_{ij} is the C^*-subalgebra of Z generated by $\overline{\psi}_j(A_j \otimes Y) \bigcup \overline{\psi}_i(A_i \otimes Y)$. Since $\overline{\alpha}_{ij}$ is proper, an application of ([6, Lemma 4.4]) shows that there exists a map $\overline{\psi}_i : M(A_i \otimes Y) \to M(Z_{ij}) \subset M(Z)$, an extension of $\overline{\psi}_i$, where the last inclusion follows from ([5, 3.12.12]). Now, in view of the inclusion $Y \cong I \otimes Y \subset M(A_i \otimes Y)$ we obtain morphisms $\rho_i = \overline{\psi}_i|_Y : Y \to M(Z)$. We show that $\rho_i(y)$ is independent of i for each $y \in Y$. Given $i, j \in I$, using the connectedness of the diagram D, there are three cases. There exists $\alpha_{ij} : A_i \to A_j$ (or from A_j to A_i), or there exists $k \in I$ and morphisms $\alpha_{ik} : A_i \to A_k$ and $\alpha_{jk} : A_j \to A_k$, or $\alpha_{ki} : A_k \to A_i$ and $\alpha_{kj} : A_k \to A_j$. Since the morphisms α_{ij}'s are proper we obtain commuting diagrams

$$\begin{tikzcd} M(A_i \otimes Y) \arrow[rr, "\overline{\alpha}_{ij}"] \arrow[dr, "\overline{\psi}_i"'] & & M(A_j \otimes Y) \arrow[dl, "\overline{\psi}_j"] \\ & M(Z) & \end{tikzcd}$$

and

$$\begin{tikzcd} M(A_i \otimes Y) \arrow[r, "\overline{\alpha}_{ik}"] \arrow[dr, "\overline{\psi}_i"'] & M(A_k \otimes Y) \arrow[r, "\overline{\alpha}_{kj}"] \arrow[d, "\overline{\psi}_k"] & M(A_j \otimes Y) \arrow[dl, "\overline{\psi}_j"] \\ & M(Z) & \end{tikzcd}$$

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The claim in the first two cases follows easily from the commutativity of the above diagrams. The third case is reduced to the previous cases by using Lemma 3.2. To see that $\rho(y)$ commutes with $\sigma(a)$ for each $y \in Y$ and $a \in A$, it suffices to verify this for $a \in A_i$. But,

$$\sigma(\varphi(a))\rho(y) = \psi_i(a \otimes 1)\psi_i(1 \otimes y) = \psi_i(a \otimes y).$$

On the other hand,

$$\rho(y)\sigma(\varphi(a)) = (\sigma(\varphi(a^*))\rho(y^*))^* = (\psi_i(a^* \otimes y^*))^* = \psi_i(a \otimes y).$$

Hence, $\rho(y)$ commutes with $\sigma(\varphi(A_i))$. Since $\bigcup_i \varphi_i(A_i)$ generates A it follows that for each $y \in Y$, $\rho(y)$ commutes with $\sigma(A)$. There exists, by ([7, Proposition 4.7]), a unique morphism $\tau : A \otimes Y \to Z$ such that $\tau(a \otimes y) = \sigma(a)\rho(y)$. To complete the proof we must show that the triangles

$$\begin{tikzcd}
 & Z & \\
 & A \otimes Y \arrow[u, "\tau"'] & \\
A_i \otimes Y \arrow[uur, "\overline{\psi}_i"] \arrow[ur, "\overline{\varphi}_i"'] \arrow[rr, "\overline{\alpha}_{ij}"'] & & A_j \arrow[uul, "\overline{\psi}_j"'] \arrow[ul, "\overline{\varphi}_j"]
\end{tikzcd}$$

commute, that is, $\psi_i(x) = \tau(\overline{\varphi}_i(x))$. It is enough to check this for the elements of the form $a \otimes y \in A_i \otimes Y$.

$$\begin{aligned}
\tau(\overline{\varphi}_i(a \otimes y)) &= \tau(\overline{\varphi}_i(a) \otimes y) = \sigma(\varphi_i(a))\rho(y) \\
&= \psi_i(a \otimes 1)\psi_i(1 \otimes y) = \psi_i(a \otimes y).
\end{aligned}$$

Finally, we consider the non-unital case. Given coherent morphisms $\psi_i : A_i \otimes Y \to Z$, first as before we extend them to get morphisms $\overline{\psi_i} : M(A_i \otimes Y) \to M(Z)$. Since $A_i \otimes Y$ is an essential ideal in $A_i \otimes \tilde{Y}$, where $\tilde{Y}$ is the unitization of Y, we have that $A_i \otimes \tilde{Y} \subset M(A_i \otimes Y)$. Hence, by restriction we obtain morphisms $\overline{\psi_i} : A_i \otimes \tilde{Y} \to M(Z)$. By the first part of the proof there exists a morphism $\overline{\sigma} : A \otimes \tilde{Y} \to M(Z)$ such that for each $i \in I$, $\overline{\psi_i} = \overline{\sigma} \circ \overline{\varphi_i}$. Let $\sigma = \overline{\sigma}|_{A \otimes Y}$. Then, $\psi_i = \sigma \circ \overline{\varphi_i}$ for all $i \in I$. This completes the proof. $\square$

REMARK 3.4. The above theorem is false if the connectedness is not assumed. For the simplest nonconnected diagram consisting of two points A and B and no morphism, the theorem implies that $(A * B) \otimes D \cong (A \otimes D) * (B \otimes D)$ which is false. Any disconnected diagram can be reduced to the discrete case of points and no morphisms by taking the colimits of its components.

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4. Crossed products

Let G be a locally compact group. An action of G on a C^*-algebra A is a homomorphism $\alpha : G \to \mathrm{Aut}\,(A)$, $\mathrm{Aut}(A)$ being the automorphism group of A, such that $g \to \alpha(g)a$ is continuous for each $a \in A$. The *full* and *reduced crossed products* $A \rtimes G$ and $A \rtimes_r G$ are the C^*-closures of the involutive algebra $L^1(G, A)$ under certain norms. See [5, 7.6] for details of crossed product constructions. Recall that if the group G acts on the C^*-algebras A and B, then a morphism $f : A \to B$ is said to be equivariant if it commutes with the action of G, that is, $f(ga) = gf(a)$ for all $a \in A$ and $g \in G$.

PROPOSITION 4.1. *Let $D = \{A_i\}_{i\in I}$ be a diagram of C^*-algebras with a group G acting on each A_i such that the connecting morphisms are G-equivariant. Let $A = \varinjlim D$ and $\tilde{A} = \varprojlim D$. Then there exists a unique action of G on A, and a unique action on $\tilde{A}$ such that the morphisms $\varphi_i : A_i \to A$ and $\pi_i : \tilde{A} \to A_i$ are equivariant for all $i \in I$.*

PROOF. For each $i \in I$ and $g \in G$ we have a morphism $\varphi_i \circ g : A_i \to A$ defined by $\varphi_i \circ g(x) = \varphi_i(gx)$. Moreover, if $\alpha_{ij} : A_i \to A_j$, then

$$\varphi_i \circ g(\alpha_{ij}(x)) = \varphi_i(g\alpha_{ij}(x)) = \varphi_i(\alpha_{ij}(gx)) = \varphi_i(gx) = \varphi_i \circ g(x).$$

Hence, by the universal property of colimit, there exists a unique morphism

$$\sigma_g : A \to A$$

such that $\varphi_i \circ g = \sigma_g \circ \varphi_i$. It is routine to check that $g \to \sigma_g$ defines an action of G on A. The limit case is similar or one may use the remarks following Theorem 3.3 and define the action by $(sf)(i) = s.f(i)$ for each $f \in \varprojlim D$, $i \in I$, and $s \in G$. $\square$

THEOREM 4.2. *Let D be a connected diagram of C^*-algebras equipped with an action of a locally compact group G. If, the connecting maps are proper and equivariant, then*

$$\varinjlim(D \rtimes G) = \varinjlim D \rtimes G,$$

where the action of G on $\varinjlim D$ is given by Proposition 4.1.

PROOF. Let $A = \varinjlim D$ and let $\varphi_i : A_i \to A$ be the injection. Let $\overline{\pi}_i : A_i \times G \to Y$ be coherent morphisms into a C^*-algebra Y. This means that $\overline{\pi}_i \circ \overline{\alpha}_{ij} = \overline{\pi}_j$, where $\overline{\alpha}_{ij} : A_i \times G \to A_j \times G$ is induced from the G-invariant morphism $\alpha_{ij} : A_i \to A_j$. Assume that $Y \subset B(H)$ for some Hilbert space H. By [5, Proposition 7.6.4] there exists a covariant representation (π_i, u_i) of (A_i, G) such that $\overline{\pi}_i = \pi_i \times u_i$. First we show that the diagram

$$\begin{tikzcd} A_i \arrow[rr, "\alpha_{ij}"] \arrow[dr, "\pi_i"'] & & A_j \arrow[dl, "\pi_j"] \\ & B(H) & \end{tikzcd}$$

commutes whenever there exists a morphism $\alpha_{ij} : A_i \to A_j$. Let $\{f_k\}$ be an approximate unit of $L^1(G)$. Then,

$$\begin{aligned} (1) \qquad \pi_j(\alpha_{ij}(a))\tilde{u}_j(f_k) &= \overline{\pi}_j(\alpha_{ij}(a) \otimes f_k) = \overline{\pi}_j(\overline{\alpha}_{ij}(a \otimes f_k)) \\ &= \overline{\pi}_i(a \otimes f) = \pi_i(a)\tilde{u}_i(f_k), \end{aligned}$$

where $a \otimes f$ denotes the function defined by $a \otimes f(g) = f(g)a$ and $\tilde{u}_i$ denotes the representation of $L^1(G)$ induced by u_i. Taking limit as $k \to \infty$ we obtain $\pi_j(\alpha_{ij}(a)) = \pi_i(a)$ when $a \in A_i$. This shows that the morphisms π_i are coherent. Hence, there exists a unique morphism $\pi : A \to B(H)$ such that the triangles

$$\begin{tikzcd} & B(H) & \\ & A \arrow[u, "\pi"] & \\ A_i \arrow[uur, "\pi_i"] \arrow[ur, "\varphi_i"'] \arrow[rr, "\alpha_{ij}"'] & & A_j \arrow[uul, "\pi_j"'] \arrow[ul, "\varphi_j"] \end{tikzcd}$$

commute. Next, we show that if there exists a morphism $\alpha_{ij} : A_i \to A_j$, and $y = \pi(\varphi_j(\alpha_{ij}(a)))$, then

$$(2) \qquad u_i(g)y = u_j(g)y.$$

This fact is needed later in the proof. First, if $f \in C_c(G)$, where $C_c(G)$ denotes the algebra of compactly supported continuous complex valued functions on G, and $g \in G$, then $u_i(g)y\tilde{u}_i(f) = u_j(g)y\tilde{u}_j(f)$. To see this observe that, $y = \pi(\varphi_j(\alpha_{ij}(a)) = \pi_j(\alpha_{ij}(a)) = \pi_i(a)$. If $f \in L^1(G)$, then by (1), $\pi_j(\alpha_{ij}(a))\tilde{u}_j(f) = \pi_i(a)\tilde{u}_i(f)$. Moreover, when $g \in G$ it is easy to see that $u_i(g)\tilde{u}_i(f) = \tilde{u}_i(f_g)$, where $f_g(h) = f(g^{-1}h)$ for $h \in G$. Now

$$\begin{aligned} (3) \qquad u_i(g)y\tilde{u}_i(f) &= u_i(g)\pi_i(a)\tilde{u}_i(f) = \pi_i(ga)u_i(g)\tilde{u}_i(f) \\ &= \pi_i(ga)\tilde{u}_i(f_g) = \pi_j(\alpha_{ij}(ga))\tilde{u}_j(f_g) \\ &= \pi_j(\alpha_{ij}(ga))u_j(g)\tilde{u}_j(f) = \pi_j(g\alpha_{ij}(a))u_j(g)\tilde{u}_j(f) \\ &= u_j(g)\pi_j(\alpha_{ij}(a))\tilde{u}_j(f) = u_j(g)y\tilde{u}_j(f). \end{aligned}$$

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Now (2) follows from (3) by taking the limit over an approximate unit of $L^1(G)$. To define the representation $u : G \to B(H)$ such that (π, u) is a covariant pair for the C^*-dynamical system (A, σ, G) we proceed as follows. Without loss of generality we assume that π is nondegenerate. That is the linear span of the set $\tilde{H} = \{\pi(a)\xi : a \in A, \xi \in H\}$ is dense in H. Since $\bigcup_i \varphi_i(A_i)$ generates A and the diagram D is connected, it suffices to define $u(g)$ on vectors of the form $\pi(\varphi_i(a))\xi$ for $a \in A_i$ and $\xi \in H$. Let

$$u(g)(\pi(\varphi_i(a))\xi) = u_i(g)\pi_i(a)\xi.$$

To show that u_g is well defined we must prove that if $\pi(\varphi_i(a))\xi = \pi(\varphi_j(b))\eta$, then $u_g(\pi(\varphi_i(a)))\xi = u_g(\pi(\varphi_j(a)))\eta$. If there is a map $\alpha_{ij} : A_i \to A_j$, then by (2), $u_i(g)\pi_i(a) = u_j(g)\pi_i(a)$ and hence

$$\begin{aligned} u(g)(\pi(\varphi_i(a))\xi) &= u_i(g)\pi_i(a) = u_j(g)\pi_i(a)\xi \\ &= u_j(g)\pi_j(a)\eta = u(g)(\pi(\varphi_j(a))\eta). \end{aligned}$$

Next, suppose there is no morphism between A_i and A_j but there exist maps $\alpha_{ik} : A_i \to A_k$ and $\alpha_{jk} : A_j \to A_k$ for some $k \in I$. Then, using (2) we get

$$\begin{aligned} u(g)(\pi(\varphi_i(a))\xi) &= u_i(g)\pi_i(a) = u_k(g)\pi_i(a)\xi = u_k(g)\pi_j(a)\eta \\ &= u_j(g)\pi_j(a)\eta = u(g)(\pi(\varphi_j(a))\eta). \end{aligned}$$

Finally, suppose there exist maps $\alpha_{ki} : A_k \to A_i$ and $\alpha_{kj} : A_k \to A_j$ for some $k \in I$. Let A_{ij} be the resulting pushout of α_{ki} and α_{kj} (see Proposition 4.1) added to the diagram D. Since the maps are proper, by [6, Theorem 6.3] there exists a covariant pair (π_{ij}, u_{ij}) for the C^*-dynamical system (A_{ij}, G) which brings us back to the previous case, and hence u is well defined. Clearly u_g is bounded and hence extends to H. To show that $u_g : H \to H$ is a unitary operator again we must consider three cases. Consider the generating vectors $\varphi_i(a)\xi$, $\varphi_j(b)\eta$ in H. If $\alpha_{ij} : A_i \to A_j$, $a \in A_i$, and $b \in A_j$, then using

$$\begin{aligned} &\langle u(g)\pi(\varphi_i(a))\xi, u_g\pi(\varphi_j((b)))\eta\rangle \\ &\quad = \langle u_i(g)\pi_i(a)\xi, u_j(g)\pi_j(b)\eta\rangle = \langle u_j(g)\pi_i(a)\xi, u_j(g)\pi_j(b)\eta\rangle \\ &\quad = \langle \pi_i(a)\xi, \pi_j(b)\eta\rangle = \langle \pi(\varphi_i(a))\xi, \pi(\varphi_j(b))\eta\rangle, \end{aligned}$$

where the second equality follows by using (2). The other cases can be dealt with by using the connectedness of the diagram. This proves that u_g is a unitary operator. Now we prove that $g \to u_g$ is a representation of G. Let $g, h \in G$ and $a \in A_i$. Then,

$$\begin{aligned} u(gh)\pi(\varphi_i(a))\xi &= u_i(gh)(\pi_i(a)\xi) = u_i(g)u_i(h)\pi_i(a)\xi \\ &= u_i(g)(\pi_i(ha)u_i(h)\xi) = u_g(\pi(\varphi_i(ha))u_i(h)\xi) \\ &= u_g(\pi_i(ha)u_i(h)\xi) = u_g(u_i(h)\pi_i(a)\xi) \\ &= u_g(u_h(\pi(\varphi_i(a))\xi)). \end{aligned}$$

Hence, $u(gh) = u(g)u(h)$. It is also easy to show that if $g_i \to g$ in G, then $u(g_i)\xi \to u(g)\xi$ for each $\xi \in H$. Finally, to show that (π, u) is a covariant pair, one checks that $u(g)\pi(\varphi_i(a)) = \pi(g\varphi_i(a))u(g)$ for $g \in G$, and $a \in A_i$ for all $i \in I$. It suffices to verify this identity for vectors of the form $\pi_j(b)\xi$ for $b \in A_j$, $\xi \in H$, and $j \in I$. For this consider three cases.

Case I. There exists a map between A_i and A_j. From the commutativity of the first diagram on page 108, we have

$$\begin{aligned} u(g)\pi(\varphi_i(a))(\pi_j(b)\xi) &= u_g\pi_i(a)(\pi_i(b)\xi) = u(g)\pi_j(\alpha_{ij}(a)b)\xi \\ &= u_j(g)\pi_j(\alpha_{ij}(a))\pi_j(b)\xi = \pi_j(\alpha_{ij}(ga))u_j(g)\pi_j(b)\xi \\ &= \pi_i(ga)u_j(g)\pi_j(b)\xi = \pi(\varphi_i(ga))u(g)\pi(\varphi_j(b))\xi \\ &= \pi(g\varphi_i(a))u(g)\pi(\varphi_j(b))\xi. \end{aligned}$$

Case II. There is no morphism between A_i and A_j but for some $k \in I$ we have morphisms $\alpha_{ik} : A_i \to A_k$ and $\alpha_{jk} : A_j \to A_k$. Using the corresponding commuting diagram we have

$$\begin{aligned} &u(g)\pi(\varphi_i(a))(\pi_j(b)\xi) \\ &\quad = u(g)\pi(\varphi_k(\alpha_{ik}(a)))(\pi_k(\alpha_{jk}(b))\xi) = u(g)\pi_k(\alpha_{ik}(a))\pi_k(\alpha_{jk}(b))\xi \\ &\quad = \pi_k(\alpha_{ik}(ga))u_k(g)\pi_k(\alpha_{jk}(b))\xi = \pi_i(ga)u_k(g)\pi_k(\alpha_{jk}(b))\xi \\ &\quad = \pi_i(ga)u(g)\pi_j(b)\xi = \pi(g\varphi_i(a))u(g)\pi_j(b)\xi. \end{aligned}$$

Case III. For some $k \in I$ there exists morphisms $\alpha_{ki} : A_k \to A_i$ and $\alpha_{kj} : A_k \to A_j$. Let A_{ij}, π_{ij}, and u_{ij} be as before. Then, we are back in the previous case. Since $\bigcup_i \varphi_i(A_i)$ generates A, $u_g\pi(a) = \pi(ga)u_g$ for all $a \in A$ and $g \in G$. This proves that (π, u) is a covariant pair. It is straightforward to show that $\pi \times u : A \times G \to B(H)$ is the desired map. □

EXAMPLE. Let $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{\pi} B/\overline{\langle f(x) - g(x)\rangle}$ be a coequalizer situation (see Lemma 2.5) with a group G acting on the C^*-algebras A and B, and equivariant morphisms f and g. Note that the ideal $I = \overline{\langle f(x) - g(x)\rangle}$ is G invariant. Then, the above theorem says that

$$A \rtimes G \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} B \rtimes G \xrightarrow{\hat{\pi}} \left(B/\overline{\langle f(x) - g(x)\rangle}\right) \rtimes G$$

is also a coequalizer. This means that $B \rtimes G/J \cong B/I \rtimes G$, where $J = \overline{\langle \hat{f}(t) - \hat{g}(t)\rangle}$. On the other hand, by the general properties of the crossed product, $B/I \rtimes G \cong B \rtimes G/I \rtimes G$. From these we have conclude the relation

$$\overline{\langle \hat{f}(t) - \hat{g}(t)\rangle} \cong \overline{\langle f(x) - g(x)\rangle} \rtimes G.$$

We end this section by proving the analogue of Theorem 3.3 and Theorem 4.2 for limits. It turns out that the minimal tensor product and the reduced crossed product must be considered in the case of limits. Recall that a C^* -algebra Y is said to be exact if whenever a short exact sequence is minimal tensored by Y it remains short exact. On the other hand a group G is said to be exact if given a short exact sequence of C^*-algebras equipped with actions of G and if the maps are equivariant, then the sequence remains short exact upon taking reduced crossed product by G. See, for example [1, 2, 3, 9] for more on the notion of exactness. We will denote by $A \otimes_{\min} B$ the minimal tensor tensor product of the C^*-algebras A and B.

THEOREM 4.3. *Let $D = \{A_i\}_{i\in I}$ be a diagram of C^*-algebras.*

(a) *If Y is exact, then $\varprojlim(D \otimes_{\min} Y) = \varprojlim D \otimes_{\min} Y$.*

(b) *If the exact group G acts on D such that the connecting morphisms are equivariant, then $\varprojlim(D \rtimes_r G) = \varprojlim D \rtimes_r G$.*

PROOF. Let $X = \varprojlim D$. Then, by Theorem 2.9 $X \xrightarrow{\rho} \prod_i A_i \overset{f}{\underset{g}{\rightrightarrows}} \prod_{i\le j} A_j$ is an equalizer. Where f and g are as mentioned in Theorem 2.9. Moreover, $f^{-1}(g(\prod_i A_i)) = \rho(X)$. To see this, clearly $\rho(X) \subset f^{-1}(g(\prod_i A_i))$. If $x \in f^{-1}(g(\prod_i A_i))$, then there exists $a' \in \prod_{i\in I} A_i$ such that $f(a) = g(a')$. From the definition of f and g

$$\pi_j f(a) = \pi_j(a) = a_j, \quad \pi_j g(a') = \alpha_{ij}\pi_i(a') = a'_j.$$

Hence, $a = a'$ and $f(a) = g(a') = g(a)$. Therefore, $a \in \rho(X)$ and hence $\rho(X) = f^{-1}(g(\prod_{i\in I} A_i))$. This implies that

$$0 \to \ker f \cap \rho(X) \xrightarrow{\rho} \rho(X) \xrightarrow{g} g\left(\prod_{i\in I} A_i\right) \to 0$$

is short exact. Since Y is exact

$$0 \to \ker f \cap \rho(X) \otimes_{\min} Y \to \rho(E) \otimes_{\min} Y \to g(A) \otimes_{\min} Y \to 0$$

is also short exact. As $g(A) \otimes_{\min} Y = \bar{g}(A \otimes_{\min} Y)$, we conclude from the above short exact sequence that $\bar{f}^{-1}(\bar{g}(A \otimes_{\min} Y) = \bar{\rho}(X \otimes_{\min} Y)$. Using this we show that

$$X \otimes_{\min} Y \xrightarrow{\rho} \prod_i A_i \otimes_{\min} Y \overset{f}{\underset{g}{\rightrightarrows}} \prod_{i\le j} A_j \otimes_{\min} Y$$

is an equalizer. First, by [7, Proposition 4.22] $\bar{\rho}$ is injective. Let $\alpha : Z \to \prod_i A_i \otimes_{\min} Y$ be such that $\bar{f} \circ \alpha = \bar{g} \circ \alpha$. We must show that α factors through $\bar{\rho}$ uniquely. For $z \in Z$, we have $\bar{f}\alpha(z) = \bar{g}\alpha(z)$, or $\alpha(z) \in \bar{f}^{-1}(g(\prod_{i\in I} A_i)) = \bar{\rho}(X \otimes_{\min} Y)$. Therefore, there exists $x \in X \otimes_{\min} Y$ such that $\rho(x) = \alpha(z)$. Define, $\delta : Z \to X \otimes_{\min} Y$ by

$\delta(z) = x$. Since, $\bar{\rho}$ is 1-1, δ is well defined and it is clear that δ is unique with respect to the relation $\bar{\rho}\delta = \alpha$. Now (a) follows from Theorem 2.9.

To prove part (b) using the exactness of G and the above short exact sequence, we obtain

$$0 \to (\ker f \cap \rho(X)) \rtimes_{\mathrm{r}} G \to \rho(X) \rtimes_{\mathrm{r}} G \to g\left(\prod_{i \in I} A_i\right) \rtimes_{\mathrm{r}} G \to 0$$

a short exact sequence. Now the proof of part (a) may be repeated. □

REMARK 4.4. The analogue of Theorem 4.3 for $\otimes_{\max}$ and full crossed product follows from [6, Theorem 6.3 and Remark 3.10]. In this case exactness of Y or G is not needed. We summarize here how this goes for the tensor product. The proof for crossed product is similar. Using the equalizer stated at the beginning of the proof of Theorem 4.3 and using Lemma 2.11 we obtain the pullback

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, "\rho"'] & \prod_i A_i \arrow[d, "(I,g)"] \\ \prod_i A_i \arrow[r, "(I,f)"'] & \prod_i A_i \times \prod_{i \le j} A_j \end{tikzcd}$$

It follows from [6, Remark 3.10] and $\prod_i A_i \otimes Y \cong \prod_i (A_i \otimes Y)$ that

$$\begin{tikzcd} X \otimes Y \arrow[r, "\bar{\rho}"] \arrow[d, "\bar{\rho}"'] & \prod_i A_i \otimes Y \arrow[d, "(I,\bar{g})"] \\ \prod_i A_i \otimes Y \arrow[r, "(I,\bar{f})"'] & \prod_i A_i \otimes Y \times \prod_{i \le j} A_j \otimes Y \end{tikzcd}$$

is a pullback. Now, one checks that

$$\begin{tikzcd} X \otimes Y \arrow[r, "\bar{\rho}"] & \prod_i (A_i \otimes Y) \arrow[r, shift left, "\bar{f}"] \arrow[r, shift right, "\bar{g}"'] & \prod_{i \le j} (A_j \otimes Y) \end{tikzcd}$$

is also an equalizer and hence $X \otimes Y = \varinjlim (D \otimes Y)$ by Theorem 2.9.

References

[1] K. Dykema, 'Exactness of reduced amalgamated free product C^*-algebras', preprint.

[2] E. Kirchberg and A. Wassermann, 'Exact groups and continuous bundles of C^*-algebras', preprint.

[3] ——, 'Permanence properties of C^*-exact groups', preprint.

[4] S. MacLane, *Categories for working mathematicians* (Springer, New York, 1972).

[5] G. Pedersen, *C^*-algebras and their automorphism groups* (Academic Press, New York, 1979).

[6] ——, 'Pullback and pushout constructions in C^*-algebra theory', *J. Funct. Anal.* **167** (1999), 243–344.

[7] M. Takesaki, *Theory of operator algebras I* (Springer, New York, 1979).
[8] D. Voiculescu, K. Dykema and A. Nica, *Free random variables*, CRM Monogr. Ser. 1 (AMS, Providence, RI, 1992).
[9] S. Wassermann, 'On tensor products of free-group C^*-algebras', *Bull. London Math. Soc.* **222** (1990), 375–380.

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