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Dr THIRD, Vice-President, in the Chair.

Notes on the Theorems of Menelaus and of Ceva.

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1. The remark of Carnot quoted by Dr Mackay at the last meeting, regarding the mutual dependence of the four equations derivable from the same figure in the case of each of these theorems, led me to consider more fully the mutual dependence of the equations in question; and the following are some of the results arrived at.

2. The equations of Menelaus and of Ceva can be, and usually are, written in terms of ratios of which both antecedent and consequent belong to the same line. Each line (being divided in three points, giving segments related by equations like $BD + DC + CB = 0$) furnishes only *one* independent ratio. The four lines in Menelaus' figure, then, give four variables. Now, for any given triangle, the transversal is fixed when we fix *two* of the ratios; so that we see from the geometrical point of view that two of the four ratios are independent. Hence we might expect that there would be two independent relations existing between the four ratios, and that of the four equations of Menelaus applying to one figure, two only would be independent.

3. To verify this, let us quote the four equations in Dr Mackay's notation (ABC being the triangle and DEF the transversal).

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = -1 \qquad (1)$$

$$\frac{ED}{DF}\cdot\frac{FB}{BA}\cdot\frac{AC}{CE} = -1 \qquad (2)$$

$$\frac{FE}{ED}\cdot\frac{DC}{CB}\cdot\frac{BA}{AF} = -1 \qquad (3)$$

$$\frac{DF}{FE}\cdot\frac{EA}{AC}\cdot\frac{CB}{BD} = -1 \qquad (4)$$

Choosing in (1) and (2) ratios which have a common segment, say CE, if we make use of the identity

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$$\frac{EA}{CE}+\frac{AC}{CE}=\frac{EC}{CE}=-1,$$

we get
$$-1=\frac{EA}{CE}+\frac{AC}{CE}=-\frac{BD}{DC}\cdot\frac{AF}{FB}-\frac{FD}{DE}\cdot\frac{AB}{BF};$$

$$\therefore\quad 1=\frac{BC+CD}{DC}\cdot\frac{AF}{FB}+\frac{FE+ED}{DE}\cdot\frac{AB}{BF}$$

$$=\left(-\frac{CB}{DC}-1\right)\frac{AF}{FB}+\left(-\frac{EF}{DE}-1\right)\frac{AB}{BF}$$

$$=-\frac{AF}{FB}\cdot\frac{BC}{CD}-\frac{AB}{BF}\cdot\frac{FE}{ED}-\frac{FA+AB}{BF};$$

$$\therefore\quad \frac{AF}{FB}\cdot\frac{BC}{CD}+\frac{AB}{BF}\cdot\frac{FE}{ED}=0;$$

$$\therefore\quad \frac{BC}{CD}\cdot\frac{DE}{EF}\cdot\frac{FA}{AB}=-1.$$

Thus we have derived equation (3) from (1) and (2); and as pointed out by Carnot, (4) is deducible from (1), (2), (3).

It is clear that any two of the equations being given, the other two can be deduced from them.

4. If we express the equations in terms of letters to represent the four ratios, taking α to denote BD : DC, β to denote CE : EA, γ to denote AF : FB, and δ to denote EF : FD, so that

$$FD:DE=-1/(1+\delta),\quad DE:EF=-(1+\delta)/\delta,\ \text{etc.},$$

the equations (1), (2), (3), (4) may now be written

$$\alpha\beta\gamma=-1 \qquad (1')$$

$$(1+\delta)\frac{1}{1+\gamma}\cdot\frac{1+\beta}{\beta}=1 \qquad (2')$$

$$\frac{\delta}{1+\delta}\cdot\frac{1}{1+\alpha}\cdot\frac{1+\gamma}{\gamma}=1 \qquad (3')$$

$$\frac{1}{\delta}\cdot\frac{1}{1+\beta}\cdot\frac{1+\alpha}{\alpha}=-1 \qquad (4')$$

In this form it is easy to deduce any one equation from any two of the others, by eliminating from them the letter not occurring in the required equation.

5. We can apply similar reasoning to Ceva's figure. The four equations of Ceva as written by Dr Mackay are

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$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1 \quad - \quad - \quad - \quad (5)$$

$$\frac{CD}{DB}\cdot\frac{BE}{EO}\cdot\frac{OF}{FC} = 1 \quad - \quad - \quad - \quad (6)$$

$$\frac{OD}{DA}\cdot\frac{AE}{EC}\cdot\frac{CF}{FO} = 1 \quad - \quad - \quad - \quad (7)$$

$$\frac{AD}{DO}\cdot\frac{OE}{EB}\cdot\frac{BF}{FA} = 1 \quad - \quad - \quad - \quad (8)$$

These involve ratios of the segments of 6 different lines. From the geometrical point of view we see that two only are independent variables, so that we might expect four independent equations connecting them. But, as we know, (5), (6), (7), (8) have one relation connecting them, so that any *one* is deducible from the other three.

6. Let us consider these four equations in conjunction with the six equations derivable from the figure by Menelaus' Theorem. Taking the further notation $\lambda \equiv AO:OD$, etc., the equations may be written

$$\alpha.\beta.\gamma = 1 \quad - \quad - \quad - \quad - \quad (O)$$

$$\frac{1}{\alpha}\cdot\frac{1+\mu}{1}\cdot\frac{1}{1+\nu} = 1 \quad - \quad - \quad (A)$$

$$\frac{1}{\beta}\cdot\frac{1+\nu}{1}\cdot\frac{1}{1+\lambda} = 1 \quad - \quad - \quad (B)$$

$$\frac{1}{\gamma}\cdot\frac{1+\lambda}{1}\cdot\frac{1}{1+\mu} = 1 \quad - \quad - \quad (C)$$

$$\frac{1+\alpha}{1}\cdot\frac{1+\lambda}{\lambda}\cdot\frac{1}{1+\mu} = 1 \quad - \quad - \quad (A')$$

$$\frac{1+\beta}{1}\cdot\frac{1+\mu}{\mu}\cdot\frac{1}{1+\nu} = 1 \quad - \quad - \quad (B')$$

$$\frac{1+\gamma}{1}\cdot\frac{1+\nu}{\nu}\cdot\frac{1}{1+\lambda} = 1 \quad - \quad - \quad (C')$$

$$\frac{1+\alpha}{\alpha}\cdot\frac{1+\lambda}{\lambda}\cdot\frac{1}{1+\nu} = 1 \quad - \quad - \quad (A'')$$

$$\frac{1+\beta}{\beta}\cdot\frac{1+\mu}{\mu}\cdot\frac{1}{1+\lambda} = 1 \quad - \quad - \quad (B'')$$

$$\frac{1+\gamma}{\gamma}\cdot\frac{1+\nu}{\nu}\cdot\frac{1}{1+\mu} = 1 \quad - \quad - \quad (C'')$$

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Here we have ten equations connecting the six quantities

$$\alpha,\ \beta,\ \gamma,\ \lambda,\ \mu,\ \nu.$$

We note that (A) and (A′) being multiplied together give (A″); (B) and (B′) give (B″); (C) and (C′) give (C″); also, (A), (B), and (C) give (O).

Rejecting as dependent the equations (A″), (B″), (C″), (O), we see further that on eliminating α from (A) and (A′) we get the symmetrical equation

$$2+\lambda+\mu+\nu=\lambda\mu\nu \qquad \text{(O′)}$$

which can be got equally from (B) and (B′), or from (C) and (C′). This shows that, given (A), (B), (C) and *one* other, all the remaining six equations can be deduced.

7. We may note the following arrangement of equations deducible from the above:

$$\alpha=\frac{1+\mu}{1+\nu}, \quad \beta=\frac{1+\nu}{1+\lambda}, \quad \gamma=\frac{1+\lambda}{1+\mu},$$

$$\lambda=\gamma+\frac{1}{\beta}, \quad \mu=\alpha+\frac{1}{\gamma}, \quad \nu=\beta+\frac{1}{\alpha}.$$

The last three show that λ, μ, ν are expressible in terms of α, β, γ, which again are connected by a single relation, $\alpha\beta\gamma=1$. This analysis confirms the proposition put forward on geometrical grounds, that of the six quantities only two are independent.

Another method of arriving at the same result would be to express α, β, γ, λ, μ, ν in terms of x, y, z, the areal coordinates of O referred to ABC, thus:—

$$\alpha=\frac{z}{y},\ \beta=\frac{x}{z},\ \gamma=\frac{y}{x},$$

$$\lambda=\frac{y+z}{x}\left(=\frac{1}{x}-1\right),\ \mu=\frac{z+x}{y}\left(=\frac{1}{y}-1\right),\ \nu=\frac{x+y}{z}\left(=\frac{1}{z}-1\right).$$

Here the six ratios are expressed in terms of $x:y:z$, *i.e.*, in terms of two independent variables.

8. In comparing the figures for the two theorems, we note that the figure of Menelaus has 4 lines intersecting in 6 points, and that of Ceva, 4 points joined by 6 lines; but the theorems are not precisely correlative by the principle of duality, as perhaps might be supposed, for each refers to ratios of line segments.

The dualistic correlative to Menelaus' Theorem in Ceva's figure would be found by naming the lines AB, BC, CA, AD, BC, EF by the letters a, b, c, d, e, f, and writing

$$\frac{\sin(bd)}{\sin(dc)} \cdot \frac{\sin(ce)}{\sin(ea)} \cdot \frac{\sin(af)}{\sin(fb)} = -1 \quad - \quad - \quad - \quad (9)$$

where (bd) denotes the angle got by rotating b till it coincides in direction and sense with d. As each line occurs in two angles, the particular choice of the *sense* of a line will not affect the result, provided the choice, once made, is adhered to; nor will it matter whether we reckon as positive the clockwise or the counter-clockwise rotation, since the number of the angles is even.

Reducing (9) to the usual notation we get

$$\frac{\sin \text{BAD}}{\sin \text{DAC}} \cdot \frac{\sin \text{CBE}}{\sin \text{EBA}} \cdot \frac{\sin \text{ACF}}{\sin \text{FCB}} = +1 \quad - \quad - \quad (10)$$

from which again, using the sine rule for triangles, equation (5) could be deduced, as well as all the six equations got by applying Menelaus' equation to Ceva's figure.

9. Again, to get the dual correlative of Ceva's Theorem, with reference to the four lines of Menelaus' figure, we note that in Ceva's figure we make use of the intersections of AO, BO, CO with the opposite sides. Hence in Menelaus' figure we must make use of the joins of D, E, F, to ABC, whereby we get a figure of 7 lines and 6 points, in place of 7 points and 6 lines, as in Ceva's figure.

We might by reciprocating the ten theorems we have discussed in connection with Ceva's figure obtain ten theorems connecting the sines of the 18 angles whose vertices are at one or other of the six points of the extended figure of Menelaus. Of these theorems, four would be independent.