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ON THE SOLUBLE LENGTH OF GROUPS WITH PRIME-POWER ORDER

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To B. H. Neumann in his 90th year.

We show that for every integer $k \geqslant 3$ and every prime $p \geqslant 5$ there is a group with soluble length k and order p^{2^k-2}.

1. INTRODUCTION

There has been interest since the time of Burnside in the question: given a prime p and a positive integer k what is the smallest order of a group of p-power order with soluble length (exactly) k? Let $p^{\beta_p(k)}$ denote this smallest order.

The first paper which discussed problems like this is one by Burnside [4] in 1913. In that paper he observed that there are groups of order p^3 with soluble length 2; and groups of order p^6 with soluble length 3. Moreover he showed that a group with soluble length $k+1$ must have order at least p^{3k}; and said: but it seems probable that for greater values of k the actual lower limit for the order exceeds p^{3k}.

Burnside [5] confirmed this expectation by proving: $p^{(k+1)(k+2)/2}$ is a lower limit for the order of a prime power group whose k-th derived group is not the identity; and moreover: this lower limit is not attained except when k is 1 or 2. So, in particular, a p-group with soluble length 4 has order at least p^{11}.

The theme was taken up and a major advance was made by P. Hall and reported in his now famous paper [8] of 1934. In it he proved that $2^{k-1}+k-1 \leqslant \beta_p(k) \leqslant 2^{k-2}(2^{k-1}-1)$ (pp. 56-7). For $k \leqslant 4$ this lower bound is no better than Burnside's bound. Hall also established that $\beta_2(k) \leqslant 2^k - 1$.

In 1950 Itô [12] refined the upper bound given by Hall. He showed that $\beta_p(k) \leqslant 3 \cdot 2^{k-1}$. Further progress was made by Blackburn in his thesis [1] in 1956. [The relevant part of that work has never been otherwise published or announced. We are indebted to Professor Blackburn for recently supplying us with a copy of his thesis.] Blackburn

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proved [**1**, Theorem 10] for every $p > 3$ every group of order p^{13} has soluble length at most 3; therefore $\beta_p(4) \geqslant 14$. He also described examples of Hall, later published in [**9**], of groups with order p^{2^k-1} and soluble length k (for p odd). (Details of the Hall examples can be found conveniently in Huppert [**11**, Satz III.17.7].)

In this paper we present examples which complete the story for $k = 4$. Namely, there are p-groups with soluble length 4 and order p^{14} for all $p \geqslant 5$. More generally, in Section 3 we describe a series of finite p-groups with soluble length k and order p^{2^k-2} for all $p \geqslant 5$. In the context of this problem the primes 2, 3 behave differently already for soluble length 3 (see Blackburn [**2**, pp. 89–91]).

One of the contributions that Burnside and Hall made was to show the significance of commutators in the study of groups of prime-power order. Hence some results about p-groups come as corollaries of results about nilpotent groups. In Section 2 we show that there is a 2-generator torsion-free nilpotent group with soluble length 4 and Hirsch length 14. (The Hirsch length is the number of infinite cyclic factors in a polycyclic series.) This group has nilpotency class 11. Factoring out the p-th powers of the generators gives that for all $p \geqslant 11$ there is p-group with soluble length 4, order p^{14} and exponent p. Examples for the primes 5 and 7 are also described in Section 2.

While, in some sense, the upper and lower bounds have the same order of magnitude, there is still a gap to fill even at this level. The quoted results give $k-1 < \log_2 \beta_p(k) < k$. Does $\log_2 \beta_p(k) - k$ have a limit and, if so, what is it?

2. Soluble length 4

Consider the pro-p-presentation

$$\{\, a, b \mid a^p = 1,\ b^p = 1,\ [b,a,b] = 1,\ [b,a,a,a,a,a] = 1,\ [b,a,a,a,b,a,a,b,a,a,a] = 1 \,\}.$$

With the help of the p-Quotient Program (Havas et al. [**10**]) it is easy to establish that the pro-p-group G_p defined by this presentation is a finite p-group of order p^{14} for $p = 5, 7, 11$ – and other individual primes as far as resources allow. Moreover, using the same program, one can see that the presentation

$$\{\, a, b \mid a^p = 1,\ b^p = 1,\ [b,a,b] = 1,\ [b,a,a,a,a,a] = 1,\ [b,a,a,a,b,a,a,b,a,a,a] = 1,\ [[[b,a,a,a],[b,a]],[[b,a,a],[b,a]]] = 1 \,\}$$

defines a finite p-group of order p^{13} for the same primes. Thus G_p must have soluble length at least 4.

A more general result can be obtained by using the Nilpotent Quotient Program (Nickel [**14**]) as follows. Let G denote the group defined by the presentation

$$\{\, a, b \mid [b,a,b] = 1,\ [b,a,a,a,a,a] = 1,\ [b,a,a,a,b,a,a,b,a,a,a] = 1 \,\}.$$

Then the program shows that the largest class 11 quotient of G has Hirsch length 14. Moreover the largest class 11 quotient of the group H defined by the presentation

$$\{ a,b \mid [b,a,b]=1,\ [b,a,a,a,a,a]=1,\ [b,a,a,a,b,a,a,b,a,a,a]=1,\ [[[b,a,a,a],[b,a]],[[b,a,a],[b,a]]]=1 \}$$

has Hirsch length 13.

Hence the largest torsion-free and class 11 quotient of G has Hirsch length 14 and soluble length 4.

The presentations used in this section were suggested by some in Caranti et al. [**6**] and the considerations in the next section.

3. The general case

Our starting point is the p-adic Lie algebra $\mathcal{T}$ of dimension 8 described in Caranti et al. [**6**]. Note that $\mathcal{T}$ is a free $\mathbb{Z}_p$-module with $\mathbb{Z}_p$-basis $\{x,y,c,d,v,s,t,w\}$ and the multiplication can be described by the following table:

$$\begin{array}{lllll} [y,x]=c, & [c,x]=d, & [c,y]=0, & [d,x]=v, & [d,y]=0, \\ [v,x]=s, & [v,y]=t, & [s,x]=0, & [s,y]=2w, & [t,x]=w, \\ & [t,y]=0, & [w,x]=px, & [w,y]=-2py. & \end{array}$$

It is shown in Caranti et al. that the quotients $\mathcal{T}_j/\mathcal{T}_{j+1}$ of the lower central series have characteristic p. The dimensions (over the field of p elements) of the $\mathcal{T}_j/\mathcal{T}_{j+1}$ are periodic with period $2,1,1,1,2,1$. The terms $\mathcal{D}_k$ of the derived series are easy to calculate and this gives that $\mathcal{T}_{j+1} < \mathcal{D}_k < \mathcal{T}_j$, for $k \geqslant 2$, where $j = 3 \cdot 2^{k-1} - 1$. (Thus $\mathcal{T}_j/\mathcal{T}_{j+1}$ is the second 2-dimensional lower central factor of the appropriate period.) This means that $\mathcal{D}_k$ is one of the $p+1$ ideals between $\mathcal{T}_{j+1}$ and $\mathcal{T}_j$. Factoring out one of the p other ideals gives an algebra of class j, soluble length $k+1$ and dimension $2^{k+1}-2$. The same p-adic algebra appears in Klaas et al. [**13**, p. 51] (taking $\Pi = \sqrt{p/3}$). Using the Cayley map $x \mapsto (1-x)(1+x)^{-1}$ whose properties are described in [**13**, pp. 31–7] gives a pro-p-group with the corresponding lattice of normal subgroups. The corresponding quotient is a group with soluble length $k+1$ and order $p^{2^{k+1}-2}$.

References

[1] N. Blackburn, *Problems on the theory of finite groups of prime-power order*, (Ph.D. Thesis) (University of Cambridge, Cambridge, U.K., 1956).

[2] N. Blackburn, 'On a special class of p-groups', *Acta Math.* **100** (1958), 45–92.

[3] W. Bosma, J. Cannon and C. Playoust, 'The Magma algebra system I: The user language', *J. Symbolic Comput.* **24** (1997), 235–265.

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[4] W. Burnside, 'On some properties of groups whose orders are powers of primes', *Proc. London Math. Soc.* (2) **11** (1913), 225-243.

[5] W. Burnside, 'On some properties of groups whose orders are powers of primes', (second paper), *Proc. London Math. Soc.* (2) **13** (1914), 6–12.

[6] A. Caranti, S. Mattarei, M.F. Newman and C.M. Scoppola, 'Thin groups of prime-power order and thin Lie algebras', *Quart. J. Math. Oxford* (2) **47** (1996), 279–296.

[7] K.W. Gruenberg and J.E. Roseblade (eds), *The collected works of Philip Hall* (Clarendon Press, Oxford, 1988).

[8] P. Hall, 'A contribution to the theory of groups of prime-power order', *Proc. London Math. Soc.* (2) **36** (1934), 29–95; or collected works [**7**] pp. 57–125.

[9] P. Hall, 'A note on $\overline{SI}$-groups', *J. London Math. Soc.* **39** (1964), 338–344; or collected works [**7**] pp. 653–661.

[10] G. Havas, M.F. Newman and E.A. O'Brien, 'The ANU p-Quotient Program (version 1.4)', The Australian National University, Canberra, (1996); Available as part of GAP [**15**] and MAGMA [**3**]; also via `ftp` from `ftpmaths.anu.edu.au/pub/algebra/PQ/`.

[11] B. Huppert, *Endliche Gruppen I*, Die Grundlehren der Mathematischen Wissenschaften **134** (Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[12] N. Itô, 'Note on p-groups', *Nagoya Math. J.* **1** (1950), 113–116.

[13] G. Klaas, C.R. Leedham-Green and W. Plesken, *Linear pro-p-groups of finite width*, Lecture Notes in Mathematics **1674** (Springer-Verlag, Berlin, Heidelberg, New York, 1997).

[14] W. Nickel, 'The ANU Nilpotent Quotient Program (version 1.2)', Available as part of GAP [**15**]; also via `ftp` from `ftpmaths.anu.edu.au/pub/algebra/NQ/`.

[15] M. Schönert et al., *GAP – Groups, Algorithms and Programming*, (fifth edition) (Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, 1995).

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