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# AN IDENTITY INVOLVING NÖRLUND POLYNOMIALS

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We prove an identity involving Nörlund polynomials, the proof of which is elementary and involves the enumeration of lattice points. The identity is slightly stronger than an identity of Carlitz which he obtained by using Apostol's transformation formula for Lambert series.

## 1. INTRODUCTION

Using a transformation formula for Lambert series developed by Apostol [1], Carlitz [2], proved an identity from which he was able to derive a number of properties of Dedekind sums. Let us elaborate.

Let $m$ be a fixed odd integer with $m > 1$ and let $j$ be an integer such that $0 \leqslant j \leqslant m+1$. Then the Dedekind sums of Apostol are defined as

$$c_j(h, k) = \sum_{\mu(k)} \overline{B}_{m+1-j}\left(\frac{\mu}{k}\right)\overline{B}_j\left(\frac{h\mu}{k}\right)$$

where $h$, $k$ are relatively prime integers, $\overline{B}_n(x)$ denotes the $n$th periodic Bernoulli polynomial, and the summation is over a complete set of residues modulo $k$.

More specifically, we define the $n$th Bernoulli polynomial by the series

$$\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = \frac{ze^{xz}}{e^z - 1}.$$

Then the $n$th periodic Bernoulli polynomial is given by

$$\overline{B}_n(x) = B_n(\{x\})$$

for any real number $x$, where $\{x\}$ denotes the fractional part of $x$, that is $x - [x]$.

For $h$, $k$, $m$ as above and $\tau \in \mathbf{C}$, the set of complex numbers, with $\operatorname{Im}\tau > 0$, define

$$f(h, k; \tau) = \sum_{j=0}^{m+1} \binom{m+1}{j}(k\tau - h)^{m-j}c_j(h, k).$$

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Carlitz then showed the identity

$$f(-k, h; \tau) = \tau^{m-1} f\left(h, k; -\frac{1}{\tau}\right) + \frac{1}{\tau}(B + \tau B)^{m+1}$$

where $(B + \tau B)^{m+1}$ is written symbolically for $\sum_{j=0}^{m+1} \binom{m+1}{j} \tau^j B_j B_{m+1-j}$ and $B_n$ is the $n$th Bernoulli number defined by $B_n(0)$ where $B_n(x)$ is the $n$th Bernoulli polynomial.

Carlitz' identity is a consequence of Apostol's transformation formulas for Lambert series.

In this note, we prove a slightly stronger identity by reformulating it as an identity involving Nörlund polynomials. Our proof is elementary and ultimately depends on showing that two sets of lattice points are equal.

## 2. Definition and basic properties of Nörlund polynomials

We now introduce the Nörlund polynomials and recall some of their properties; see [3] and [4] for some of the results.

Definition: The $m$th Nörlund polynomial in $n$ parameters $\omega_1, \ldots, \omega_n$, denoted $B_m^{(n)}(x; \omega_1, \ldots, \omega_n)$, is defined by

$$\frac{z^n e^{xz}}{(e^{\omega_1 z} - 1) \ldots (e^{\omega_n z} - 1)} = \sum_{m=0}^{\infty} B_m^{(n)}(x; \omega_1, \ldots, \omega_n) \frac{z^m}{m!}.$$

Here $\underline{\omega} = (\omega_1, \ldots, \omega_n) \in \mathbf{C}^n$ and $x$ is a variable.

In particular

$$\frac{z e^{xz}}{e^{\omega z} - 1} = \sum_{m=0}^{\infty} B_m^{(1)}(x; \omega) \frac{z^m}{m!}$$

from which we see that

$$B_m^{(1)}(x; \omega) = \omega^{m-1} B_m\left(\frac{x}{\omega}\right)$$

where $B_m(x)$ is the $m$th Bernoulli polynomial given by

$$\frac{z e^{xz}}{e^z - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{z^m}{m!}.$$

**Proposition 1.** *For all nonnegative integers $m$ and $a \in \mathbf{C}$,*

$$B_m^{(n)}(ax; a\omega_1, \ldots, a\omega_n) = a^{m-n} B_m^{(n)}(x; \omega_1, \ldots, \omega_n).$$

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PROOF:

$$\sum_{m=0}^{\infty} B_m^{(n)}(ax;\, a\underline{\omega})\frac{z^m}{m!} = \frac{z^n e^{axz}}{(e^{a\omega_1 z}-1)\dots(e^{a\omega_n z}-1)}$$

$$= \frac{a^{-n}(az)^n e^{x(az)}}{(e^{\omega_1(az)}-1)\dots(e^{\omega_n(az)}-1)} = \sum_{m=0}^{\infty} a^{m-n} B_m^{(n)}(x;\, \underline{\omega})\frac{z^m}{m!}.$$

□

PROPOSITION 2. *For $m$, $n$, $\underline{\omega}$ as above, and $1 \leqslant j \leqslant n$,*

$$B_m^{(n)}(x;\, \underline{\omega}) = -B_m^{(n)}(x-\omega_j;\, \omega_1, \dots, \omega_{j-1}, -\omega_j, \omega_{j+1}, \dots, \omega_n).$$

PROOF:

$$\sum_{m=0}^{\infty} B_m^{(n)}(x;\, \underline{\omega})\frac{z^m}{m!} = \frac{z^n e^{xz}}{(e^{\omega_1 z}-1)\dots(e^{\omega_n z}-1)}$$

$$= \frac{z^n e^{xz} e^{-\omega_j z}}{(e^{\omega_1 z}-1)\dots(e^{\omega_j z}-1)e^{-\omega_j z}\dots(e^{\omega_n z}-1)}$$

$$= \frac{z^n e^{(x-\omega_j)z}}{(e^{\omega_1 z}-1)\dots(1-e^{-\omega_j z})\dots(e^{\omega_n z}-1)}$$

$$= \frac{-z^n e^{(x-\omega_j)z}}{(e^{\omega_1 z}-1)\dots(e^{-\omega_j z}-1)\dots(e^{\omega_n z}-1)} = -\sum_{m=0}^{\infty} B_m^{(n)}(x-\omega_j;\, \tilde{\underline{\omega}})\frac{z^m}{m!}$$

where $\tilde{\underline{\omega}} = (\omega_1, \dots, \omega_{j-1}, -\omega_j, \omega_{j+1}, \dots, \omega_n)$. □

PROPOSITION 3. *Let $(n_1, \dots, n_k)$ be a partition of $n$ and let $\underline{\omega} = (\omega_1, \dots, \omega_n) = (\underline{\omega}_1, \dots, \underline{\omega}_k)$ where $\underline{\omega}_1 = (\omega_1, \dots, \omega_{n_1}), \dots, \underline{\omega}_k = (\omega_{n-n_k+1}, \dots, \omega_n)$. Then*

$$B_m^{(n)}(x_1 + \dots + x_k;\, \underline{\omega}) = \left(B^{(n_1)}(x_1;\, \underline{\omega}_1) + \dots + B^{(n_k)}(x_k;\, \underline{\omega}_k)\right)^m$$

*where the last expression is written symbolically for*

$$\sum_{\substack{j_1,\dots,j_k \geqslant 0 \\ j_1+\dots+j_k=m}} \frac{m!}{j_1!\dots j_k!} B_{j_1}^{(n_1)}(x_1;\underline{\omega}_1) \dots B_{j_k}^{(n_k)}(x_k;\underline{\omega}_k).$$

PROOF:

$$\sum_{m=0}^{\infty} B_m^{(n)}(x_1 + \dots + x_k;\, \underline{\omega})\frac{z^m}{m!} = \frac{z^n e^{(x_1+\dots+x_k)z}}{(e^{\omega_1 z}-1)\dots(e^{\omega_n z}-1)}$$

$$= \frac{z^{n_1} e^{x_1 z}}{(e^{\omega_1 z}-1)\dots(e^{\omega_{n_1} z}-1)} \dots \frac{z^{n_k} e^{x_k z}}{(e^{\omega_{n-n_k+1} z}-1)\dots(e^{\omega_n z}-1)}$$

$$= \sum_{m=0}^{\infty} \left(B^{(n_1)}(x_1;\, \underline{\omega}_1) + \dots + B^{(n_k)}(x_k;\, \underline{\omega}_k)\right)^m \frac{z^m}{m!}.$$

□

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## 3. An identity involving Nörlund Polynomials

**Lemma.** *Let $h$, $k$ be relatively prime integers with $k > 0$. Then*

$$f(h, k; \tau) = \sum_{\mu=0}^{k-1} B_{m+1}^{(2)}\left(\tau\mu - \left[\frac{h\mu}{k}\right]; k\tau - h, 1\right)$$

and

$$f(0, 1; \tau) = B_{m+1}^{(2)}(0; \tau, 1) = \frac{1}{\tau}(B + \tau B)^{m+1}.$$

*Here $[x]$ is the largest integer less than or equal to $x$.*

Proof:

$$f(h, k; \tau) = \sum_{j=0}^{m+1} \binom{m+1}{j} (k\tau - h)^{m-j} c_j(h, k)$$

$$= \sum_{j=0}^{m+1} \binom{m+1}{j} (k\tau - h)^{m-j} \sum_{\mu=0}^{k-1} B_{m+1-j}\left(\frac{\mu}{k}\right) B_j\left(\left\{\frac{h\mu}{k}\right\}\right)$$

where $\{x\}$ denotes the fractional part of $x$, that is, $x - [x]$. Since $B_m(x) = B_m^{(1)}(x; 1)$, we see by Proposition 1 that

$$f(h, k; \tau) = \sum_{\mu=0}^{k-1} \sum_{j=0}^{m+1} \binom{m+1}{j} B_{m+1-j}^{(1)}\left((k\tau - h)\frac{\mu}{k}; k\tau - h\right) B_j^{(1)}\left(\left\{\frac{h\mu}{k}\right\}; 1\right)$$

$$= \sum_{\mu=0}^{k-1} \sum_{j=0}^{m+1} \binom{m+1}{j} B_{m+1-j}^{(1)}\left(\tau\mu - \frac{h\mu}{k}; k\tau - h\right) B_j^{(1)}\left(\left\{\frac{h\mu}{k}\right\}; 1\right).$$

Then by Proposition 3 we have

$$f(h, k; \tau) = \sum_{\mu=0}^{k-1} B_{m+1}^{(2)}\left(\tau\mu - \frac{h\mu}{k} + \left\{\frac{h\mu}{k}\right\}; k\tau - h, 1\right)$$

$$= \sum_{\mu=0}^{k-1} B_{m+1}^{(2)}\left(\tau\mu - \left[\frac{h\mu}{k}\right]; k\tau - h, 1\right).$$

In particular

$$f(0, 1; \tau) = B_{m+1}^{(2)}(0; \tau, 1).$$

But by Propositions 1 and 3 and the fact that $B_m = B_m(0)$, we see

$$B_{m+1}^{(2)}(0; \tau, 1) = \sum_{j=0}^{m+1} \binom{m+1}{j} B_{m+1-j}^{(1)}(0; \tau) B_j^{(1)}(0; 1)$$

$$= \sum_{j=0}^{m+1} \binom{m+1}{j} \tau^{m-j} B_{m+1-j} B_j = \frac{1}{\tau}(B + \tau B)^{m+1}.$$

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The lemma is now established. □

By the lemma Carlitz' identity takes the form

$$\sum_{\nu=0}^{h-1} B_{m+1}^{(2)}\left(\tau\nu - \left[\frac{-k\nu}{h}\right]; h\tau + k, 1\right)$$
$$= \tau^{m-1} \sum_{\mu=0}^{k-1} B_{m+1}^{(2)}\left(-\frac{1}{\tau}\mu - \left[\frac{h\mu}{k}\right]; k\left(-\frac{1}{\tau}\right) - h, 1\right) + B_{m+1}^{(2)}(0; \tau, 1),$$

for all odd $m > 1$. By using Proposition 1, this identity becomes

$$\sum_{\nu=0}^{h-1} B_{m+1}^{(2)}\left(\tau\nu - \left[\frac{-k\nu}{h}\right]; h\tau + k, 1\right)$$
$$= \sum_{\mu=0}^{k-1} B_{m+1}^{(2)}\left(-\mu - \left[\frac{h\mu}{k}\right]\tau; -(h\tau + k), \tau\right) + B_{m+1}^{(2)}(0; \tau, 1),$$

for all odd $m > 1$.

We establish a little more.

THEOREM. *Let $h$, $k$ be relatively prime positive integers. Then for all integers $m \geqslant 0$,*

$$\sum_{\nu=0}^{h-1} B_m^{(2)}\left(\tau\nu - \left[\frac{-k\nu}{h}\right]; h\tau + k, 1\right)$$
$$= \sum_{\mu=0}^{k-1} B_m^{(2)}\left(-\mu - \left[\frac{h\mu}{k}\right]\tau; -(h\tau + k), \tau\right) + B_m^{(2)}(0; \tau, 1)$$
$$+ B_m^{(2)}(0; h\tau + k, -\tau) - B_m^{(2)}((h-1)\tau + k; h\tau + k, -\tau).$$

COROLLARY. *Let $h$, $k$ and $m$ be as in the theorem. Then*

$$\sum_{\nu=0}^{h-1} B_m^{(2)}\left(\tau\nu - \left[\frac{-k\nu}{h}\right]; h\tau + k, 1\right)$$
$$= \sum_{\mu=0}^{k-1} B_m^{(2)}\left(-\mu - \left[\frac{h\mu}{k}\right]\tau; -(h\tau + k), \tau\right) + B_m^{(2)}(0; \tau, 1)$$
$$+ (1 - (-1)^m) B_m^{(2)}(0; h\tau + k, -\tau).$$

REMARK. Notice that Carlitz' version is a special case when $m$ is even. ($m$ above is $m + 1$ in the Carlitz identity.)

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PROOF OF COROLLARY: By two applications of Proposition 2, we have

$$\begin{aligned} B_m^{(2)}((h-1)\tau+k;\, h\tau+k,\, -\tau) &= -B_m^{(2)}(-\tau;\, -(h\tau+k),\, -\tau) \\ &= B_m^{(2)}(0;\, -(h\tau+k),\, \tau). \end{aligned}$$

Now apply Proposition 1 with $a=-1$ to obtain

$$B_m^{(2)}(0;\, -(h\tau+k),\, \tau) = (-1)^{m-2} B_m^{(2)}(0;\, h\tau+k,\, -\tau).$$

□

PROOF OF THEOREM: We reformulate the theorem into a sequence of equivalent statements. We then prove the last of these statements by a lattice counting argument.

By the definition of the Nörlund polynomials, we need to show that

$$\begin{aligned} &\sum_{\nu=0}^{h-1} \frac{z^2 e^{(\tau\nu-[-k\nu/h])z}}{\left(e^{(h\tau+k)z}-1\right)\left(e^z-1\right)} - \sum_{\mu=0}^{k-1} \frac{z^2 e^{(-\mu-[h\mu/k]\tau)z}}{\left(e^{-(h\tau+k)z}-1\right)\left(e^{\tau z}-1\right)} \\ &\quad - \frac{z^2}{\left(e^{\tau z}-1\right)\left(e^z-1\right)} + \frac{z^2\left(e^{((h-1)\tau+k)z}-1\right)}{\left(e^{(h\tau+k)z}-1\right)\left(e^{-\tau z}-1\right)} = 0. \end{aligned}$$

We factor out

$$\frac{z^2}{\left(e^{(h\tau+k)z}-1\right)\left(e^{\tau z}-1\right)\left(e^z-1\right)}$$

of the left-hand side of the equation and then divide both sides by this factor obtaining the equivalent statement

$$\begin{aligned} &\left(e^{\tau z}-1\right)\sum_{\nu=0}^{h-1} e^{(\tau\nu-[-k\nu/h])z} + \left(e^z-1\right)e^{(h\tau+k)z}\sum_{\mu=0}^{k-1} e^{(-\mu-[h\mu/k]\tau)z} \\ &\quad - \left(e^{(h\tau+k)z}-1\right) - e^{\tau z}\left(e^z-1\right)\left(e^{((h-1)\tau+k)z}-1\right) = 0. \end{aligned}$$

We now multiply out all the terms obtaining

$$\begin{aligned} &\sum_{\nu=0}^{h-1} e^{(\tau(\nu+1)-[-k\nu/h])z} - \sum_{\nu=0}^{h-1} e^{(\tau\nu-[-k\nu/h])z} \\ &\quad + \sum_{\mu=0}^{k-1} e^{((h-[h\mu/k])\tau+k+1-\mu)z} - \sum_{\mu=0}^{k-1} e^{((h-[h\mu/k])\tau+k-\mu)z} \\ &\quad - e^{(h\tau+k)z} + 1 - e^{(h\tau+k+1)z} + e^{(h\tau+k)z} + e^{(\tau+1)z} - e^{\tau z} = 0. \end{aligned}$$

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We simplify the left-hand side and then divide both sides by $e^{\tau z}$ obtaining

$$\sum_{\nu=1}^{h-1} e^{(\tau\nu-[-k\nu/h])z} - \sum_{\nu=1}^{h-1} e^{(\tau(\nu-1)-[-k\nu/h])z} + \sum_{\mu=1}^{k-1} e^{((h-1-[h\mu/k])\tau+k+1-\mu)z}$$
$$- \sum_{\mu=1}^{k-1} e^{((h-1-[h\mu/k])\tau+k-\mu)z} - e^{((h-1)\tau+k)z} + e^{z} = 0.$$

Using $-[-x] = \begin{cases} [x]+1 & \text{if } x \notin \mathbf{Z} \\ [x] & \text{if } x \in \mathbf{Z} \end{cases}$ we get

$$\sum_{\nu=0}^{h-1} e^{(\tau\nu+[h\nu/k]+1)z} - \sum_{\nu=1}^{h-1} e^{(\tau(\nu-1)+[h\nu/k]+1)z} + \sum_{\mu=1}^{k-1} e^{((h-1-[(h/\mu)k])\tau+k+1-\mu)z}$$
$$- \sum_{\mu=1}^{k-1} e^{(((h-1)-[h\mu/k])\tau+k-\mu)z} - e^{((h-1)\tau+k)z} = 0.$$

In the last two sums on the left-hand side, we change variable of summation $\mu \to k-\mu$ and use $h-1-[h\mu/k] \to h-1-[h(k-\mu)/k] = h-1-h-[-(h\mu)/k] = h-1-h+[h\mu/k]+1 = [h\mu/k]$ for $1 \leqslant \mu \leqslant k-1$. Hence the above identity becomes

$$\sum_{\nu=0}^{h-1} e^{(\tau\nu+[k\nu/h]+1)z} - \sum_{\nu-1}^{h-1} e^{(\tau(\nu-1)+[k\nu/h]+1)z}$$
$$+ \sum_{\mu=1}^{k-1} e^{([h\mu/k]\tau+\mu+1)z} - \sum_{\mu=1}^{k-1} e^{([h\mu/k]\tau+\mu)z} - e^{((h-1)\tau+k)z} = 0.$$

We finally have an equivalent form of the theorem that we shall prove. Comparing terms shows that it suffices to prove

$$S_1 \dot{\cup} S_2 = T_1 \dot{\cup} T_2 \dot{\cup} T_3$$

where

$$S_1 = \left\{ \left( \nu, \left[ \frac{k\nu}{h} \right] + 1 \right) \mid 0 \leqslant \nu \leqslant h-1 \right\},$$
$$S_2 = \left\{ \left( \left[ \frac{h\mu}{k} \right], \mu+1 \right) \mid 1 \leqslant \mu \leqslant k-1 \right\},$$
$$T_1 = \left\{ \left( \nu-1, \left[ \frac{k\nu}{h} \right] + 1 \right) \mid 1 \leqslant \nu \leqslant h-1 \right\},$$
$$T_2 = \left\{ \left( \left[ \frac{h\mu}{k} \right], \mu \right) \mid 1 \leqslant \mu \leqslant k-1 \right\},$$
$$T_3 = \{(h-1, k)\}.$$

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Let

$$B = \mathbf{Z} \times \mathbf{Z} \cap ([0, h] \times [0, k]),$$
$$B_0 = \mathbf{Z} \times \mathbf{Z} \cap ((0, h) \times (0, k)),$$
$$B_1 = \mathbf{Z} \times \mathbf{Z} \cap ((0, h) \times [0, k]),$$
$$B_2 = \mathbf{Z} \times \mathbf{Z} \cap ([0, h] \times (0, k)).$$

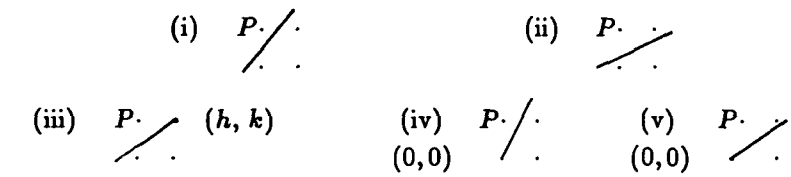
Also let $L$ denote the line in $\mathbf{R} \times \mathbf{R}$ with equation $y = (k/h)x$. Then geometrically we have

$$S_1 = \{P \in B \mid P \text{ lies just above } L\},$$
$$S_2 = \{P \in B \mid (\exists Q \in B_2)(Q \text{ lies just to the left of } L \text{ and } P = Q + (0, 1))\},$$
$$T_1 = \{P \in B \mid (\exists Q \in B_1)(Q \text{ lies just above } L \text{ and } P = Q + (-1, 0))\},$$
$$T_2 = \{P \in B_2 \mid P \text{ lies just to the left of } L\}.$$

Notice that from this description $S_1$ and $S_2$ are disjoint and $T_1$, $T_2$, $T_3$ are pairwise disjoint.

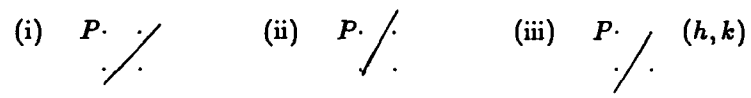
We now show that $S_1 \cup S_2 = T_1 \cup T_2 \cup T_3$. To this end, suppose $P \in S_1 \cup S_2$.

CASE 1. Suppose $P \in S_1$. Then there are five possibilities which are depicted below. The points represent points in $B$ and the line segment represents a portion of $L$.




In (i) $P \in T_2$, in (ii) $P \in T_1$, in (iii) $P \in T_3$, in (iv) $P \in T_2$ and in (v) $P \in T_1$.

CASE 2. Suppose $P \in S_2$. Then there are three possibilities given by




In (i) $P \in T_1$, in (ii) $P \in T_2$ and in (iii) $P \in T_3$.
Thus $S_1 \cup S_2 \subseteq T_1 \cup T_2 \cup T_3$.

Now suppose $P \in T_1 \cup T_2 \cup T_3$.

CASE 1. Suppose $P \in T_1$. Then we have two possibilities:


In (i) $P \in S_2$ and in (ii) $P \in S_1$.

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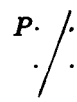
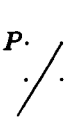
CASE 2. Suppose $P \in T_2$. Then we have two possibilities:

(i)  (ii) 

In (i) $P \in S_1$ and in (ii) $P \in S_2$.

CASE 3. Suppose $P \in T_3$, that is, $P = (h-1, k)$. Then there are two possibilities:

(i)  (ii) 

In (i) $P \in S_1$ and in (ii) $P \in S_2$.

Therefore $T_1 \cup T_2 \cup T_3 \subseteq S_1 \cup S_2$.

The theorem is now established. □

It would be of interest to see if there are similar identities involving Nörlund polynomials in more than two parameters. This might lead to the study of Lambert series in several variables as well as reciprocity laws for "multidimensional" Dedekind sums.

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