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Abstract

Let $p > 2$ be prime, and let F be a totally real field in which p is unramified. We give a sufficient criterion for a mod p Galois representation to arise from a mod p Hilbert modular form of parallel weight one, by proving a 'companion forms' theorem in this case. The techniques used are a mixture of modularity lifting theorems and geometric methods. As an application, we show that Serre's conjecture for F implies Artin's conjecture for totally odd two-dimensional representations over F.

1. Introduction

The weight part of Serre's conjecture has been much-studied over the past two decades, and while the original problem has been resolved, a great deal remains to be proved in more general settings. In the present paper we address the question of the weight part of Serre's conjecture for totally real fields. Here one has the Buzzard–Diamond–Jarvis conjecture [BDJ10] and its various generalisations, of which a large part has now been established (see, e.g., [Gee11b, BGG13]). One case that has not (so far as we are aware) been considered at all over totally real fields is the case of forms of (partial) weight one. This case is markedly different from the case of higher weights, for the simple reason that mod p modular forms of weight one cannot necessarily be lifted to characteristic-zero modular forms of weight one; as the methods of [Gee11b] and [BGG13] are centred around modularity lifting theorems and, in particular, depend on the use of lifts to characteristic-zero modular forms, they cannot immediately tell us anything about the weight-one case.

In this paper, we generalise a result of Gross [Gro90] and prove a companion forms theorem for Hilbert modular forms of parallel weight one in the unramified p-distinguished case. To explain what this means and how it (mostly) resolves the weight-one part of Serre's conjecture for totally real fields, we return to the case of classical modular forms. Serre's original formulation [Ser87] of his conjecture considered only mod p modular forms which lift to characteristic zero and, in particular, ignored the weight-one case. However, Serre later observed that one could further refine his conjecture by using Katz's definition [Kat73] of mod p modular forms and thus include weight-one forms. He then conjectured that a modular Galois representation should arise from a weight-one form (of level coprime to p) if and only if the Galois representation is unramified at p. The harder direction is to prove that the Galois representation being unramified implies that there is a weight-one form; this was proved by Gross [Gro90], under the further hypothesis that the eigenvalues of a Frobenius element at p are distinct (i.e. we are in the p-distinguished case). It is this result that we generalise in this paper, proving the following theorem.

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THEOREM A. *Let $p > 2$ be prime, let F be a totally real field in which p is unramified, and let $\overline{\rho}: G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible modular representation such that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. If $p = 3$ (respectively, $p = 5$), assume further that the projective image of $\overline{\rho}(G_{F(\zeta_p)})$ is not conjugate to $\mathrm{PSL}_2(\mathbb{F}_3)$ (respectively, $\mathrm{PSL}_2(\mathbb{F}_5)$).*

Suppose that for each place $v \mid p$, $\overline{\rho}|_{G_{F_v}}$ is unramified and that the eigenvalues of $\overline{\rho}(\mathrm{Frob}_v)$ are distinct.

Then there is a mod p Hilbert modular form f of parallel weight 1 and level coprime to p such that $\overline{\rho}_f \cong \overline{\rho}$.

(See Theorem 3.1.1; any unfamiliar notation or terminology will be explained in the body of the paper.) The condition on $\overline{\rho}|_{G_{F(\zeta_p)}}$ is mild, and the only other hypothesis which is not expected to be necessary (other than that p is unramified in F, which we assume for our geometric arguments) is that the eigenvalues of $\overline{\rho}(\mathrm{Frob}_v)$ are distinct for all $v \mid p$. This condition appears to be essential to our method, as we explain below.

Our method of proof is a combination of modularity lifting theorem techniques and geometric methods.

The first part of the argument, using modularity lifting theorems to produce Hilbert modular forms of parallel weight p, was carried out in [Gee07] under some additional hypotheses (in particular, it was assumed that $\overline{\rho}$ arises from an ordinary Hilbert modular form), and in § 2 we shall use the techniques of [BGGT] (which involve the use of automorphy lifting theorems for rank 4 unitary groups) to remove these hypotheses. This gives us 2^n Hilbert modular forms of parallel weight p and level coprime to p, where there are n places of F above p, corresponding to the different possible choices of Frobenius eigenvalues at places above p. In § 3 we take a suitable linear combination of these forms and show that it is divisible by the Hasse invariant of parallel weight $p-1$, by explicitly calculating the pth power of the quotient. It is easy to show that the quotient is the sought-after Hilbert modular form of parallel weight one. If we do not assume that $\overline{\rho}$ has distinct Frobenius eigenvalues at each place dividing p, the weight-one form we obtain in this manner is actually zero.

In § 4 we give an application of our main theorem to Artin's conjecture, generalising the results (and arguments) of [Kha97] to prove the following result, which shows that the weak form of Serre's conjecture for totally real fields implies the strong form of Artin's conjecture for totally odd two-dimensional representations.

THEOREM B. *Let F be a totally real field. Assume that every irreducible, continuous and totally odd representation $\overline{\rho}: G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular, for every prime p. Then every irreducible, continuous and totally odd representation $\rho: G_F \to \mathrm{GL}_2(\mathbb{C})$ is modular.*

(See Theorem 4.1.3.) Finally, we remark that it is possible that our results can be applied to Artin's conjecture more directly (that is, without assuming Serre's conjecture), as an input to modularity lifting theorems in parallel weight one; see the forthcoming work of Calegari and Geraghty for some results in this direction. Additionally, it would be of interest to generalise our results to forms of partial weight one; the geometric techniques we use in this paper amount to determining the intersection of the kernels of the Θ-operators of [AG05], and it is possible that a determination of the kernels of the individual Θ-operators could shed some light on this.

1.1 Notation

If M is a field, we let $\overline{M}$ denote an algebraic closure of M, and we let $G_M := \mathrm{Gal}\,(\overline{M}/M)$ denote its absolute Galois group. Let p be a prime number, and let ε denote the p-adic cyclotomic character; our choice of convention for Hodge–Tate weights is that ε has all Hodge–Tate weights equal to 1. Let F be a totally real field and f a cuspidal Hilbert modular eigenform of parallel weight k. If v is a finite place of F which is coprime to the level of f, then we have, in particular, the usual Hecke operator T_v corresponding to the double coset

$$\mathrm{GL}_2(\mathcal{O}_{F_v}) \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_v}),$$

where ϖ_v is a uniformiser of $\mathcal{O}_{F_v}$, the ring of integers of F_v.

There is a Galois representation $\rho_f : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ associated to f; we adopt the convention that if $v \nmid p$ is as above and Frob_v is an *arithmetic* Frobenius element of G_{F_v}, then $\mathrm{tr}\,\rho_f(\mathrm{Frob}_v)$ is the T_v-eigenvalue of f, so that, in particular, the determinant of ρ_f is a finite-order character times ε^{k-1}.

We say that $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is *modular* if it arises as the reduction mod p of the Galois representation $\rho_f : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ for some f.

2. Modularity lifting in weight p

2.1 In this section we apply the modularity lifting techniques first used in [Gee07] to produce companion forms in (parallel) weight p. We make use of the techniques of [BGGT] to weaken the hypotheses (for example, to avoid the necessity of an assumption of ordinarity). In this section, we do not assume that p is unramified in F. Note that the definition of a mod p modular form will be recalled in § 3 below.

THEOREM 2.1.1. *Let $p > 2$ be prime, let F be a totally real field, and let $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible modular representation such that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. If $p = 3$ (respectively, $p = 5$), assume further that the projective image of $\overline{\rho}(G_{F(\zeta_p)})$ is not conjugate to $\mathrm{PSL}_2(\mathbb{F}_3)$ (respectively, $\mathrm{PSL}_2(\mathbb{F}_5)$).*

Suppose that for each place $v \mid p$, $\overline{\rho}|_{G_{F_v}}$ is unramified; in fact, suppose that

$$\overline{\rho}|_{G_{F_v}} \cong \begin{pmatrix} \lambda_{\alpha_{1,v}} & 0 \\ 0 & \lambda_{\alpha_{2,v}} \end{pmatrix}$$

where λ_x is the unramified character sending an arithmetic Frobenius element to x. For each place $v \mid p$, let γ_v be a choice of one of $\alpha_{1,v}$ and $\alpha_{2,v}$. Let N be an integer coprime to p and divisible by the Artin conductor of $\overline{\rho}$. Then there is a Hilbert modular eigenform f of parallel weight p such that:

- *f has level $\Gamma_1(N)$; in particular, f has level coprime to p;*
- *$\overline{\rho}_f \cong \overline{\rho}$;*
- *for each place $v \mid p$, we have $T_v f = \widetilde{\gamma}_v f$ for some lift $\widetilde{\gamma}_v$ of γ_v.*

Proof. We remark that in the argument below, we will feel free to let $\widetilde{\gamma}_v$ denote *any* lift of γ_v, rather than a fixed lift. By local–global compatibility at places dividing p (cf. [Kis08, Theorem 4.3]), it is enough to find a lift ρ of $\overline{\rho}$ which is automorphic, is minimally ramified outside p, and has the further property that for each place $v \mid p$ we have

$$\rho|_{G_{F_v}} \cong \begin{pmatrix} \varepsilon^{p-1}\lambda_{x_v} & * \\ 0 & \lambda_{\widetilde{\gamma}_v} \end{pmatrix}$$

for some x_v and some $\widetilde{\gamma}_v$ (such a representation is automatically crystalline with Hodge–Tate weights 0 and $p-1$ at each place $v \mid p$, and thus corresponds to a Hilbert modular form of parallel weight p and level coprime to p).

The existence of such a representation ρ is a straightforward application of the results of [BGG13], as we now explain. This argument is very similar to that in [BGG12, proof of Proposition 6.1.3]. Firstly, choose a quadratic imaginary CM extension F_1/F that splits at all places dividing p and all places where $\overline{\rho}$ is ramified, and which is linearly disjoint from $\overline{F}^{\ker\overline{\rho}}(\zeta_p)$ over F. Let S denote a finite set of places of F, consisting of the infinite places and the union of the set of places of F at which $\overline{\rho}$ is ramified and the places that divide p. From now on we will consider $\overline{\rho}$ as a representation of $G_{F,S}$, the Galois group of the maximal extension of F unramified outside of S. Let χ be the Teichmüller lift of $\overline{\varepsilon}^{1-p}\det\overline{\rho}$.

Fix a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$ such that $\overline{\rho}$ is valued in $\mathrm{GL}_2(\mathbb{F})$. For each finite place v of F, let $R^{\chi}_{G_{F_v}}$ denote the universal $\mathcal{O}$-lifting ring for lifts of $\overline{\rho}|_{G_{F_v}}$ of determinant $\chi\varepsilon^{p-1}$. By [GG12, Lemma 3.1.8] (for example), for each place $v \mid p$ of F there is a quotient $R^{\chi,\gamma}_{G_{F_v}}$ of $R^{\chi}_{G_{F_v}}$ whose $\overline{\mathbb{Q}}_p$-points correspond precisely to those lifts of $\overline{\rho}|_{G_{F_v}}$ which are conjugate to a representation of the form

$$\begin{pmatrix} \varepsilon^{p-1}\chi/\lambda_{\widetilde{\gamma}_v} & * \\ 0 & \lambda_{\widetilde{\gamma}_v} \end{pmatrix}$$

for some $\widetilde{\gamma}_v$ lifting γ_v. For each finite place $v \in S$ with $v \nmid p$, let $\overline{R}^{\chi}_{G_{F_v}}$ be a quotient of $R^{\chi}_{G_{F_v}}$ corresponding to an irreducible component of $R_{G_{F_v}}[1/p]$, the points of which correspond to lifts of $\overline{\rho}|_{G_{F_v}}$ with the same Artin conductor as $\overline{\rho}|_{G_{F_v}}$. Let $R^{\chi,\gamma}$ denote the universal deformation ring for deformations of $\overline{\rho}$ of determinant $\chi\varepsilon^{p-1}$, which have the additional property that for each place $v \mid p$, the deformation corresponds to a point of $R^{\chi,\gamma}_{G_{F_v}}$, while for each finite place $v \in S$ with $v \nmid p$, it corresponds to a point of $\overline{R}^{\chi}_{G_{F_v}}$. In order to construct the representation r that we seek, it is enough to find a $\overline{\mathbb{Q}}_p$-point of $R^{\chi,\gamma}$ that is automorphic. We will do this by showing that $R^{\chi,\gamma}$ is a finite $\mathcal{O}$-algebra of dimension at least one (so that it has $\overline{\mathbb{Q}}_p$-points) and that all its $\overline{\mathbb{Q}}_p$-points are automorphic.

We can and do extend $\overline{\rho}|_{G_{F_1}}$ to a representation $\overline{\rho} : G_F \to \mathcal{G}_2(\mathbb{F})$, where $\mathcal{G}_2$ is the group scheme introduced in [CHT08, § 2.1] (cf. [BGG12, § 3.1.1]). In the notation of [CHT08, § 2.3], we let $\widetilde{S}$ be a set of places of F_1 consisting of exactly one place $\widetilde{v}$ above each place v of S, and we let $\mathcal{S}$ denote the deformation problem

$$(F_1/F, S, \widetilde{S}, \mathcal{O}, \overline{\rho}, \varepsilon^{p-1}\chi, \{\overline{R}^{\chi}_{G_{F_{1,\widetilde{v}}}}\}_{v\in S, v\nmid p} \cup \{R^{\chi,\gamma}_{G_{F_{1,\widetilde{v}}}}\}_{v \mid p}).$$

Let $R_{\mathcal{S}}$ denote the corresponding universal deformation ring. Exactly as in [GG12, § 7.4], the process of 'restriction to G_{F_1} and extension to $\mathcal{G}_2$' makes $R^{\chi,\gamma}$ a finite $R_{\mathcal{S}}$-module in a natural way. By [Gee11a, Proposition 3.1.4] we have $\dim R^{\chi,\gamma} \geqslant 1$, so that (by cyclic base change for GL_2) it suffices to show that $R_{\mathcal{S}}$ is finite over $\mathcal{O}$ and that all its $\overline{\mathbb{Q}}_p$-points are automorphic.

By [BGG13, Proposition A.2.1] and our assumptions on $\overline{\rho}|_{G_{F(\zeta_p)}}$, $\overline{\rho}(G_{F(\zeta_p)})$ is adequate in the sense of [Tho12]. By [Tho12, Theorems 7.1 and 10.1], it is enough to check that $R_{\mathcal{S}}$ has an automorphic $\overline{\mathbb{Q}}_p$-point. We claim that this can be done by applying [BGG13, Theorem A.4.1] to $\overline{\rho}$. The only hypotheses of [BGG13, Theorem A.4.1] that are not obviously satisfied are those pertaining to potential diagonalisability. By [BGG13, Lemma 3.1.1], we may choose a finite solvable extension F'/F of CM fields which is linearly disjoint from $\overline{F}^{\ker\overline{\rho}}(\zeta_p)$, such that $\overline{\rho}|_{G_{F'}}$ has an automorphic lift which is potentially diagonalisable at all places dividing p.

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All $\overline{\mathbb{Q}}_p$-points of each $R^{\chi,\gamma}_{G_{F_v}}$ are potentially diagonalisable by [BGGT, Lemma 1.4.3], so [BGG13, Theorem A.4.1] produces a $\overline{\mathbb{Q}}_p$-point of $R_{\mathcal{S}}$ which is automorphic upon restriction to $G_{F'}$. Since F'/F is solvable, the result follows by solvable base change (see [BGHT11, Lemma 1.4]). $\square$

3. Weight-one forms

3.1 Let $p>2$ be a prime number. Let $F/\mathbb{Q}$ be a totally real field of degree $d>1$, $\mathcal{O}_F$ its ring of integers, and $\mathfrak{d}_F$ its different ideal. Let $\mathbb{S}=\{v\mid p\}$ be the set of all primes lying above p. We assume that p is unramified in F. Let $N>3$ be an integer coprime to p.

In this section, we make our geometric arguments. We begin by recalling some standard definitions. Let K be a finite extension of $\mathbb{Q}_p$ (which we will assume to be sufficiently large without further comment), and let $\mathcal{O}_K$, $\mathfrak{m}_K$ and κ denote, respectively, its ring of integers, its maximal ideal and its residue field. Let $X/\mathcal{O}_K$ be the Hilbert modular scheme representing the functor which associates to an $\mathcal{O}_K$-scheme S the set of all polarised abelian schemes with real multiplication and $\Gamma_{00}(N)$-structure $\underline{A}/S=(A/S,\iota,\lambda,\alpha)$ as follows:

- A is an abelian scheme of relative dimension d over S;
- the real multiplication $\iota\colon \mathcal{O}_F\hookrightarrow \mathrm{End}_S(A)$ is a ring homomorphism endowing A with an action of $\mathcal{O}_F$;
- the map λ is a polarisation as in [DP94];
- α is a rigid $\Gamma_1(N)$-level structure, that is, $\alpha\colon \mu_N\otimes_{\mathbb{Z}}\mathfrak{d}_F^{-1}\to A$, an $\mathcal{O}_F$-equivariant closed immersion of group schemes.

Let X_K/K and $\overline{X}/\kappa$ denote, respectively, the generic and special fibres of X. Let $\widetilde{X}$ denote a toroidal compactification of X. We define $\widetilde{X}_K$ and $\overline{\widetilde{X}}$ similarly.

Let Y be the scheme representing the functor which associates to an $\mathcal{O}_K$-scheme S the set of all $(\underline{A}/S,C)=(A/S,\iota,\lambda,\alpha,C)$, where $(A/S,\iota,\lambda,\alpha)$ is as above and C is an $\mathcal{O}_F$-invariant isotropic finite flat subgroup scheme of $A[p]$ of order p^g. Let $\widetilde{Y}$ denote a toroidal compactification of Y obtained using the same choices of polyhedral decompositions as for $\widetilde{X}$. We introduce the notation $Y_K,\overline{Y},\widetilde{Y}_K,\overline{\widetilde{Y}}$ in the same way as we did for X. The ordinary locus in $\overline{X}$ is denoted by $\overline{X}^{\mathrm{ord}}$. It is Zariski dense in $\overline{X}$.

There are two finite flat maps

$$\pi_1,\pi_2:\widetilde{Y}\to\widetilde{X},$$

where π_1 forgets the subgroup C and π_2 quotients out by C. We define the Atkin–Lehner involution $w:\widetilde{Y}\to\widetilde{Y}$ to be the map which, on the non-cuspidal locus, sends $(\underline{A},C)$ to $(\underline{A}/C,A[p]/C)$; it is an automorphism of $\widetilde{Y}$. We have $\pi_2=\pi_1\circ w$. We also define the Frobenius section $s:\overline{\widetilde{X}}\to\overline{\widetilde{Y}}$ which, on the non-cuspidal locus, sends $\underline{A}$ to $(\underline{A},\mathrm{Ker}(\mathrm{Frob}_A))$. Our convention is to use the same notation to denote maps between the various versions of X and Y.

Let $\epsilon:\underline{\mathcal{A}}^{\mathrm{univ}}\to X$ be the universal abelian scheme. Let

$$\underline{\Omega}=\epsilon_*\Omega^1_{\underline{\mathcal{A}}^{\mathrm{univ}}/X}$$

be the Hodge bundle on X. Since p is assumed to be unramified in F, $\underline{\Omega}$ is a locally free $(\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{O}_X)$-module of rank one. We define $\underline{\omega}=\wedge^d\underline{\Omega}$. The sheaf $\underline{\omega}$ naturally extends to $\widetilde{X}$ as an invertible sheaf. Let $\epsilon':\underline{\mathcal{B}}^{\mathrm{univ}}\to Y$ be the universal abelian scheme over Y with the designated subgroup $\mathcal{C}$. We have

$$\pi_1^*\underline{\omega} = \wedge^d \epsilon_* \Omega^1_{\mathcal{B}^{\mathrm{univ}}/Y},$$
$$\pi_2^*\underline{\omega} = \wedge^d \epsilon_* \Omega^1_{(\mathcal{B}^{\mathrm{univ}}/\mathcal{C})/Y}.$$

Let

$$\mathrm{pr}^* : \pi_1^*\underline{\omega} \to \pi_2^*\underline{\omega}$$

denote the pullback under the natural projection $\mathrm{pr} : \underline{\mathcal{B}}^{\mathrm{univ}} \to \underline{\mathcal{B}}^{\mathrm{univ}}/\mathcal{C}$. We will often write $\pi_1^*\underline{\omega}$ as simply $\underline{\omega}$.

Let R be an $\mathcal{O}_K$-algebra. A (geometric) Hilbert modular form of parallel weight $k \in \mathbb{Z}$ and level $\Gamma_1(N)$ defined over R is a section of $\underline{\omega}^k$ over $X \otimes_{\mathcal{O}_K} R$. Every such section extends to $\widetilde{X} \otimes_{\mathcal{O}_K} R$ by the Koecher principle. We denote the space of such forms by $M_k(\Gamma_1(N), R)$ and the subspace of cuspforms (those sections vanishing on the cuspidal locus) by $S_k(\Gamma_1(N), R)$. If R is a κ-algebra, then elements of $M_k(\Gamma_1(N), R)$ are referred to as *mod p Hilbert modular forms.* Every such form is a section of $\underline{\omega}^k$ over $\overline{X} \otimes_\kappa R$ and extends automatically to $\widetilde{\overline{X}} \otimes_\kappa R$.

Given any fractional ideal $\mathfrak{a}$ of $\mathcal{O}_F$, let $X_\mathfrak{a}$ denote the subscheme of X where the polarisation module of the abelian scheme is isomorphic to $(\mathfrak{a}, \mathfrak{a}^+)$ as a module with notion of positivity. Then $X_\mathfrak{a}$ is a connected component of X, and every connected component of X is of the form $X_\mathfrak{a}$ for some $\mathfrak{a}$. The same is true for $Y_\mathfrak{a}$ and Y.

In our arguments, we will use Tate objects over Hilbert modular varieties and q-expansions for Hilbert modular forms. For more details on these notions, we refer the reader to [AG05, § 6], although our notation will be slightly different. For example, given a fractional ideal $\mathfrak{a}$, one can define a map $\underline{q} : \mathfrak{a}^{-1} \to \mathbb{G}_m \otimes \mathfrak{d}_F^{-1}$ by sending an element $\alpha \in \mathfrak{a}^{-1}$ to the point of the torus $\mathbb{G}_m \otimes \mathfrak{d}_F^{-1} = \mathbb{G}_m \otimes \mathcal{O}_F^*$ whose value at the parameter X^ξ (for $\xi \in \mathcal{O}_F$) is $q^{\xi\alpha}$. We will replace the notation $\underline{q}(\mathfrak{a}^{-1})$ employed by [AG05] with $q^{\mathfrak{a}^{-1}}$. Let

$$Ta_\mathfrak{a} = \underline{(\mathbb{G}_m \otimes \mathfrak{d}_F^{-1})/q^{\mathfrak{a}^{-1}}}$$

denote a cusp on $X_\mathfrak{a}$, where the underline indicates the inclusion of standard PEL structure. Pick $c \in p^{-1}\mathfrak{a}^{-1} - \mathfrak{a}^{-1}$. Then

$$Ta_{\mathfrak{a},c} = (Ta_\mathfrak{a}, \langle q^c \rangle)$$

is a cusp on $Y_\mathfrak{a}$, where $\langle q^c \rangle$ denotes the $\mathcal{O}_F$-submodule of $Ta_\mathfrak{a}[p]$ generated by q^c.

We now prove our main result, the following theorem.

Theorem 3.1.1. *Let $p > 2$ be a prime which is unramified in F, a totally real field. Let $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible modular representation such that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. If $p = 3$ (respectively, $p = 5$), assume further that the projective image of $\overline{\rho}(G_{F(\zeta_p)})$ is not conjugate to $\mathrm{PSL}_2(\mathbb{F}_3)$ (respectively, $\mathrm{PSL}_2(\mathbb{F}_5)$).*

Suppose that for each prime $v \mid p$, $\overline{\rho}|_{G_{F_v}}$ is unramified and that the eigenvalues of $\overline{\rho}(\mathrm{Frob}_v)$ are distinct.

Then there is a mod p Hilbert modular form h of parallel weight 1 and level coprime to p such that $\overline{\rho}_h \cong \overline{\rho}$. Furthermore, h can be chosen to have level bounded in terms of the Artin conductor of $\overline{\rho}$.

Proof. For each $v \in \mathbb{S}$, let the eigenvalues of $\overline{\rho}(\mathrm{Frob}_v)$ be $\gamma_{v,1} \neq \gamma_{v,2}$. Let $\mathfrak{N}$ denote the Artin conductor of $\overline{\rho}$, and let $N > 3$ be an integer coprime to p and divisible by $\mathfrak{N}$. By Theorem 2.1.1, we see that for each subset $I \subset S$ there is a Hilbert modular eigenform f_I of weight p and level $\Gamma_1(N)$ (defined over K, a finite extension of $\mathbb{Q}_p$) such that $\overline{\rho}_{f_I} \cong \overline{\rho}$, and for each prime $v \in \mathbb{S}$

we have $T_v f = \widetilde{\gamma}_{I,v} f$, where $\widetilde{\gamma}_{I,v}$ lifts $\gamma_{v,1}$ if $v \in I$ and $\gamma_{v,2}$ if $v \notin I$. Since $\overline{\rho}_{f_I} \cong \overline{\rho}$, we see that for each prime $\mathfrak{l} \nmid Np$ of F, the f_I are eigenvectors for $T_\mathfrak{l}$ with eigenvalues $\lambda_{I,\mathfrak{l}}$ whose reductions mod $\mathfrak{m}_K$ are independent of I. By a standard argument, using [Shi78, Proposition 2.3], we can furthermore assume (at the possible cost of passing to forms of level N^2) that for each prime $\mathfrak{l} \mid N$, we have $T_\mathfrak{l} f_I = 0$ for all I. See [Hid88, § 2] for definition of the Hecke operators for Hilbert modular forms.

We can and do assume that each f_I is normalised, in the sense that (in the notation of [Shi78]) $c(\mathcal{O}_F, f_I) = 1$. For any $I \subset \mathbb{S}$, let $\widetilde{\gamma}_I = \Pi_{v \mid p} \widetilde{\gamma}_{I,v}$; this is the T_p-eigenvalue of f_I. Set

$$f = \sum_{I \subset \mathbb{S}} (-1)^{|I|} \widetilde{\gamma}_I f_I,$$
$$g = \sum_{I \subset \mathbb{S}} (-1)^{|I|} f_I.$$

We begin with a lemma.

Lemma 3.1.2. *The section $\pi_1^* f - \mathrm{pr}^* \pi_2^* g$ of $\underline{\omega}^p$ on $\widetilde{Y}$ has q-expansion with coefficients in $\mathfrak{m}_K$ at every cusp of the form $Ta_{\mathfrak{a},c}$.*

Proof. We let η denote the canonical generator of the sheaf $\underline{\omega}$ on the base of $Ta_\mathfrak{a}$ or $Ta_{\mathfrak{a},c}$. We first remark that by [Shi78, (2.23)], if h is a normalised Hilbert modular eigenform of parallel weight k and $h(Ta_\mathfrak{a}) = \sum_{\xi \in (\mathfrak{a}^{-1})^+} c_\xi q^\xi \eta^k$, then $c_\xi = c(\xi\mathfrak{a}, h)$ is the eigenvalue of the $T_{\xi\mathfrak{a}}$ operator on h, for all $\xi \in (\mathfrak{a}^{-1})^+$.

Write $f(Ta_\mathfrak{a}) = \sum_{\xi \in (\mathfrak{a}^{-1})^+} a_\xi(\mathfrak{a}) q^\xi \eta^p$ and $g(Ta_\mathfrak{a}) = \sum_{\xi \in (\mathfrak{a}^{-1})^+} b_\xi(\mathfrak{a}) q^\xi \eta^p$. We have

$$\pi_1^* f(Ta_{\mathfrak{a},c}) = f(Ta_\mathfrak{a}) = \sum_{\xi \in (\mathfrak{a}^{-1})^+} a_\xi(\mathfrak{a}) q^\xi \eta^p,$$
$$\mathrm{pr}^* \pi_2^* g(Ta_{\mathfrak{a},c}) = \mathrm{pr}^* g(Ta_{p\mathfrak{a}}) = \sum_{\xi \in p^{-1}(\mathfrak{a}^{-1})^+} b_\xi(p\mathfrak{a}) q^\xi \eta^p.$$

It is therefore enough to show that $b_\xi(p\mathfrak{a}) \equiv_{\mathfrak{m}_K} 0$ if $\xi \in p^{-1}(\mathfrak{a}^{-1})^+ - (\mathfrak{a}^{-1})^+$ and that $a_\xi(\mathfrak{a}) \equiv_{\mathfrak{m}_K} b_\xi(p\mathfrak{a})$ for $\xi \in (\mathfrak{a}^{-1})^+$.

For the first statement, let $v \in \mathbb{S}$ be such that v does not divide $\xi p\mathfrak{a}$. Then we can write

$$b_\xi(p\mathfrak{a}) = \sum_{I \subset \mathbb{S}} (-1)^{|I|} c(\xi p\mathfrak{a}, f_I) = \sum_{v \notin I} (-1)^{|I|} c(\xi p\mathfrak{a}, f_I) - \sum_{v \in I} (-1)^{|I|} c(\xi p\mathfrak{a}, f_I) \equiv_{\mathfrak{m}_K} 0,$$

since for any I not containing v, we have $c(\xi \mathfrak{p}\mathfrak{a}, f_I) \equiv_{\mathfrak{m}_K} c(\xi p\mathfrak{a}, f_{I \cup \{v\}})$, as v does not divide $\xi p\mathfrak{a}$.

For the second statement, note, firstly, that for $\xi \in (\mathfrak{a}^{-1})^+$ we can write

$$b_\xi(p\mathfrak{a}) = \sum_{I \subset \mathbb{S}} (-1)^{|I|} c(\xi p\mathfrak{a}, f_I),$$

$$a_\xi(\mathfrak{a}) = \sum_{I \subset \mathbb{S}} (-1)^{|I|} \widetilde{\gamma}_I c(\xi\mathfrak{a}, f_I).$$

Now, if h is a Hilbert modular eigenform, then for any integral ideal $\mathfrak{m}$ of $\mathcal{O}_F$, we have $c(p\mathfrak{m}, h) \equiv c((p), h) c(\mathfrak{m}, h) \bmod p$. Since for any $I \subset \mathbb{S}$ we have $c((p), f_I) = \widetilde{\gamma}_I$, the result follows. $\square$

For any section $h \in H^0(\widetilde{X}, \underline{\omega}^k)$, we denote its image in $H^0(\widetilde{\overline{X}}, \underline{\omega}^k)$ by $\bar{h}$.

Corollary 3.1.3. *We have the following equality of sections of $\underline{\omega}^p$ on $\tilde{\overline{X}}$:*

$$s^*\pi_2^*\bar{f} = s^*w^*\mathrm{pr}^*\pi_2^*\bar{g}.$$

Proof. By Lemma 3.1.2, the reduction of $\pi_1^*f - \mathrm{pr}^*\pi_2^*g$ modulo $\mathfrak{m}_K$ is a section of $\underline{\omega}^p$ on $\tilde{\overline{Y}}$ which vanishes on every irreducible component of $\tilde{\overline{Y}}$ containing the reduction of a cusp of the form $Ta_{\mathfrak{a},c}$. These components are exactly the irreducible components of $ws(\tilde{\overline{X}})$, since $w(Ta_{\mathfrak{a},c}) = (Ta_{p\mathfrak{a}}, \langle\zeta\rangle) = s(Ta_{p\mathfrak{a}})$ (where $\langle\zeta\rangle$ is the $\mathcal{O}_F$-module generated by a pth root of unity ζ). Pulling back under $w \circ s$, we obtain the desired equality on $\tilde{\overline{X}}$. □

For any scheme Z over κ, let $\mathrm{Fr} : Z \to Z^{(p)}$ denote the relative Frobenius morphism. Since $\tilde{\overline{X}}$ has a model over $\mathbb{F}_p$, we have $\tilde{\overline{X}}^{(p)} = \tilde{\overline{X}}$. For any non-negative integer k, we define a morphism

$$V : H^0(\tilde{\overline{X}}, \underline{\omega}^k) \to H^0(\tilde{\overline{X}}, \underline{\omega}^{kp})$$

as follows: choose a trivialisation $\{(U_i, \eta_i)\}$ for $\underline{\omega}$ on $\tilde{\overline{X}}$, and let $f \in H^0(\tilde{\overline{X}}, \underline{\omega}^k)$ be given by $f_i\eta_i^k$ on U_i; then, there is a unique section $V(f)$ in $H^0(\tilde{\overline{X}}, \underline{\omega}^{kp})$ whose restriction to U_i is $\mathrm{Fr}^*(f_i)\eta_i^{kp}$.

Calculating on points, we see easily that for $\tilde{\overline{X}}$, we have $\pi_2 \circ s = \mathrm{Fr}$. Let $\mathbf{h}$ denote the Hasse invariant of parallel weight $p-1$; it can be defined as follows. Let U be an open subset of $\overline{X}$ over which $\underline{\omega}$ is trivial, and let A_U denote the universal abelian scheme over U. Let η be a non-vanishing section of $\underline{\omega}$ on U; it can be thought of as a section of $\Omega_{A_U/U}$. We let $\eta^{(p)}$ denote the induced section of $\Omega_{A_U^{(p)}/U}$. Let $\mathrm{Ver} : A_U^{(p)} \to A_U$ be the Verschiebung morphism. Then, there is a unique $\lambda \in \mathcal{O}_{\overline{X}}(U)$ such that $\mathrm{Ver}^*\eta = \lambda\eta^{(p)}$. We define a section of ω^{p-1} on U via

$$\mathbf{h}_{U,\eta} := \lambda\eta^{p-1}.$$

It is easy to see that there is a unique section of ω^{p-1} on $\overline{X}$, denoted by $\mathbf{h}$, such that $\mathbf{h}_{|_U} = \mathbf{h}_{U,\eta}$ for any choice of (U, η) as above. See [AG05, § 7.11] for an equivalent construction.

Proposition 3.1.4. *We have $V(\bar{f}) = V(\mathbf{h})\bar{g}$ as sections of $\underline{\omega}^{p^2}$ on $\overline{X}$. Furthermore, $\bar{f}$ is divisible by $\mathbf{h}$ and $\bar{f}/\mathbf{h}$ is a mod p Hilbert modular form of parallel weight one defined over κ.*

Proof. Let U be an open subset of $\overline{X}$ over which $\underline{\omega}$ is trivialisable, and let η be a non-vanishing section of $\underline{\omega}$ over U. We claim that if $\mathbf{h} = \lambda\eta^{p-1}$ on U, then

$$\lambda s^*\pi_2^*\eta = s^*w^*\mathrm{pr}^*\pi_2^*\eta.$$

Evaluating both sides at a point corresponding to $\underline{A}$, we need to show that for the natural projection

$$\mathrm{pr} : A/\mathrm{Ker}(\mathrm{Frob}_A) \to A/A[p] \cong A \tag{1}$$

we have $\mathrm{pr}^*\eta = \lambda\eta^{(p)}$, which follows from the definition of the Hasse invariant, since pr is the Verschiebung morphism of A.

Now, writing $\bar{f} = F\eta^p$ and $\bar{g} = G\eta^p$ on U, and using the above claim, over U we can write

$$\lambda^p s^*\pi_2^*\bar{f} = (s^*\pi_2^*F)(\lambda^p s^*\pi_2^*\eta^p) = \mathrm{Fr}^*(F)(s^*w^*\mathrm{pr}^*\pi_2^*\eta^p).$$

On the other hand, we have

$$\lambda^p s^*w^*\mathrm{pr}^*\pi_2^*\bar{g} = \lambda^p(s^*w^*\pi_2^*G)(s^*w^*\mathrm{pr}^*\pi_2^*\eta^p) = \lambda^p G(s^*w^*\mathrm{pr}^*\pi_2^*\eta^p).$$

At a point corresponding to an *ordinary* abelian variety A, the section $s^*w^*\mathrm{pr}^*\pi_2^*\eta$ specialises to $\mathrm{pr}^*\eta$, where pr is as in (1) and, hence, non-vanishing. Corollary 3.1.3 now implies that

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over $\overline{X}^{\mathrm{ord}} \cap U$ we have

$$\mathrm{Fr}^*(F) = \lambda^p G = \mathrm{Fr}^*(\lambda) G,$$

with the last equality holding because $\mathbf{h}$ is defined on a model of $\overline{X}$ over $\mathbb{F}_p$. Running over a trivialising open covering of $\overline{X}$ for ω, we conclude that $V(\bar{f}) = V(\mathbf{h})\bar{g}$ on $\overline{X}^{\mathrm{ord}}$. Since $\overline{X}^{\mathrm{ord}}$ is Zariski dense in $\overline{X}$, it follows that

$$V(\bar{f}) = V(\mathbf{h})\bar{g}$$

as sections of ω^{p^2} over $\overline{X}$.

For the second statement, we view F/λ as a function on the ordinary part of U and show that it extends to all of U. Since U is smooth over κ, it is enough to show that the Weil divisor of F/λ is effective. But the coefficients appearing in that divisor are the coefficients of the Weil divisor of G multiplied by p. Since G has an effective Weil divisor, so does F/λ, and hence F is divisible by λ on U. Repeating this argument over an open covering of $\widetilde{\overline{X}}$, we obtain that $\bar{f}$ is divisible by $\mathbf{h}$ and that $\bar{f}/\mathbf{h}$ is a mod p Hilbert modular form of parallel weight one defined over κ, as claimed. □

We can now finish the proof of Theorem 3.1.1. The desired mod p Hilbert modular form h of parallel weight one is $\bar{f}/\mathbf{h}$. Since $\mathbf{h}$ has q-expansion 1 at all unramified cusps, it follows that h satisfies the desired assumptions. □

4. Serre's conjecture implies Artin's conjecture

4.1 In this final section we generalise the arguments of [Kha97] to show that for a fixed totally real field F, the weak form of Serre's conjecture for F implies the strong form of Artin's conjecture for two-dimensional totally odd representations over F. To be precise, the weak version of Serre's conjecture that we have in mind is the following (cf. [BDJ10, Conjecture 1.1], where it is described as a folklore conjecture).

Conjecture 4.1.1. Suppose that $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is continuous, irreducible and totally odd. Then $\overline{\rho} \cong \overline{\rho}_f$ for some Hilbert modular eigenform f.

Meanwhile, we have the following strong form of Artin's conjecture.

Conjecture 4.1.2. Suppose that $\rho : G_F \to \mathrm{GL}_2(\mathbb{C})$ is continuous, irreducible and totally odd. Then $\rho \cong \rho_f$ for the some Hilbert modular eigenform f (necessarily of parallel weight one).

In order to show that Conjecture 4.1.1 implies Conjecture 4.1.2, we follow [Kha97, proof of Proposition 1], using Theorem 3.1.1 in place of the results of Gross, Coleman and Voloch used in [Kha97]. The argument is slightly more involved than in [Kha97], because we have to be careful to show that the p-distinguishedness hypothesis in Theorem 3.1.1 is satisfied.

Theorem 4.1.3. *Fix a totally real field F. Then Conjecture 4.1.1 implies Conjecture 4.1.2.*

Proof. Suppose that $\rho : G_F \to \mathrm{GL}_2(\mathbb{C})$ is continuous, irreducible and totally odd. Then $\rho(G_F)$ is finite, so after conjugation we may assume that $\rho : G_F \to \mathrm{GL}_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers in some number field K. We will show that there are a fixed integer N and infinitely many rational primes p such that for each such p, if $\overline{\rho}_p$ denotes the reduction of ρ mod p (or, rather, modulo a prime of $\mathcal{O}_K$ above p), then $\overline{\rho}_p$ arises from the reduction mod p of the Galois representation associated to an eigenform in $S_1(\Gamma_1(N), \mathbb{C})$, the space of cuspidal Hilbert modular forms of parallel weight one and level $\Gamma_1(N)$ over $\mathbb{C}$. Since $S_1(\Gamma_1(N), \mathbb{C})$ is finite-dimensional, there are only finitely many such eigenforms, so we see that one eigenform f must work for infinitely many p; but then it is easy to see that $\rho \cong \rho_f$, as required.

Firstly, we claim that it suffices to prove that there exist a fixed N and infinitely many primes p such that $\overline{\rho} \cong \overline{\rho}_f$ for some eigenform $f \in S_1(\Gamma_1(N), \overline{\mathbb{F}}_p)$ (using the notation of § 3). To see this, note that for all but finitely many primes p, the finitely generated $\mathbb{Z}$-module $H^1(X, \underline{\omega})$ is p-torsion free, so that for all but finitely many p the reduction map $H^0(X, \underline{\omega}) \to H^0(\overline{X}, \underline{\omega})$ is surjective, and the Deligne–Serre lemma [DS74, Lemma 6.11] allows us to lift from $S_1(\Gamma_1(N), \overline{\mathbb{F}}_p)$ to $S_1(\Gamma_1(N), \mathbb{C})$.

We are thus reduced to showing that there are infinitely many primes p for which $\overline{\rho}_p$ satisfies the hypotheses of Theorem 3.1.1. First, note that there is at most one prime p for which $\rho|_{G_{F(\zeta_p)}}$ is reducible, so if we exclude any such prime, as well as the (finitely many) primes dividing $\#\rho(G_F)$, the primes which ramify in F, and the primes less than 7, then $\overline{\rho}_p$ will satisfy the requirements of the first paragraph of Theorem 3.1.1.

If we also exclude the finitely many primes p for which $\rho|_{G_{F_v}}$ is ramified for some $v \mid p$, we see that it is enough to show that there are infinitely many p such that for all $v \mid p$, $\overline{\rho}_p(\mathrm{Frob}_v)$ has distinct eigenvalues.

In fact, we claim that it is enough to show that there are infinitely many p such that for all $v \mid p$, $\rho(\mathrm{Frob}_v)$ is not scalar. To see this, suppose that $\rho(\mathrm{Frob}_v)$ is not scalar but $\overline{\rho}_p(\mathrm{Frob}_v)$ is scalar. Then it must be the case that the difference of the eigenvalues of $\rho(\mathrm{Frob}_v)$ is divisible by some prime above p. Now, there are only finitely many non-scalar elements in $\rho(G_F)$, and for each of these elements there are only finitely many primes dividing the difference of their eigenvalues, so excluding this finite set of primes gives the claim.

Let $\mathrm{proj}\,\rho$ be the projective representation $\mathrm{proj}\,\rho : G_F \to \mathrm{PGL}_2(\mathbb{C})$ obtained from ρ. We must show that there are infinitely many primes p such that for each place $v \mid p$ of F, $\mathrm{proj}\,\rho(\mathrm{Frob}_v) \neq 1$. Letting $L = \overline{F}^{\ker \mathrm{proj}\,\rho}$, we must show that there are infinitely many primes p such that no place $v \mid p$ of F splits completely in L. Let M be the normal closure of F over $\mathbb{Q}$, and let N be the normal closure of L over $\mathbb{Q}$. Since ρ is totally odd, we see that M is totally real and N is totally imaginary. Consider a complex conjugation $c \in \mathrm{Gal}\,(N/\mathbb{Q})$. By the Cebotarev density theorem, there are infinitely many primes p such that Frob_p is conjugate to c in $\mathrm{Gal}\,(N/\mathbb{Q})$, and it is easy to see that each such prime splits completely in M and thus in F, and that no place $v \mid p$ of F splits completely in L, as required. □

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References

AG05 F. Andreatta and E. Z. Goren, *Hilbert modular forms: mod p and p-adic aspects*, Mem. Amer. Math. Soc. **173** (819) (2005); MR 2110225(2006f:11049).

BGG12 T. Barnet-Lamb, T. Gee and D. Geraghty, *Congruences between Hilbert modular forms: constructing ordinary lifts*, Duke Math. J. **161** (2012), 1521–1580.

BGG13 T. Barnet-Lamb, T. Gee and D. Geraghty, *Serre weights for rank two unitary groups*, Math. Ann. (2013), doi:10.1007/s00208-012-0893-y.

BGGT T. Barnet-Lamb, T. Gee, D. Geraghty and R. Taylor, *Potential automorphy and change of weight*, Ann. of Math. (2), to appear.

BGHT11 T. Barnet-Lamb, D. Geraghty, M. Harris and R. Taylor, *A family of Calabi-Yau varieties and potential automorphy II*, Publ. Res. Inst. Math. Sci. **47** (2011), 29–98; MR 2827723.

BDJ10 K. Buzzard, F. Diamond and F. Jarvis, *On Serre's conjecture for mod l Galois representations over totally real fields*, Duke Math. J. **155** (2010), 105–161.

CHT08 L. Clozel, M. Harris and R. Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. **108** (2008), 1–181.

DP94 P. Deligne and G. Pappas, *Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant*, Compositio Math. **90** (1994), 59–79; MR 1266495(95a:11041).

DS74 P. Deligne and J.-P. Serre, *Formes modulaires de poids* 1, Ann. Sci. Éc. Norm. Supér. (4) **7** (1974), 507–530; MR 0379379(52#284).

Gee07 T. Gee, *Companion forms over totally real fields. II*, Duke Math. J. **136** (2007), 275–284.

Gee11a T. Gee, *Automorphic lifts of prescribed types*, Math. Ann. **350** (2011), 107–144; MR 2785764(2012c:11118).

Gee11b T. Gee, *On the weights of mod p Hilbert modular forms*, Invent. Math. **184** (2011), 1–46; MR 2782251.

GG12 T. Gee and D. Geraghty, *Companion forms for unitary and symplectic groups*, Duke Math. J. **161** (2012), 247–303; MR 2876931.

Gro90 B. H. Gross, *A tameness criterion for Galois representations associated to modular forms (mod p)*, Duke Math. J. **61** (1990), 445–517; MR 1074305(91i:11060).

Hid88 H. Hida, *On p-adic Hecke algebras for* GL_2 *over totally real fields*, Ann. of Math. (2) **128** (1988), 295–384; MR 960949(89m:11046).

Kat73 N. M. Katz, *p-adic properties of modular schemes and modular forms*, in *Modular functions of one variable, III* (Proceedings International Summer School, University of Antwerp, 1972), Lecture Notes in Mathematics, vol. 350 (Springer, Berlin, 1973), 69–190; MR 0447119(56#5434).

Kha97 C. Khare, *Remarks on mod p forms of weight one*, Int. Math. Res. Not. (1997), 127–133; MR 1434905(97m:11070.

Kis08 M. Kisin, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21** (2008), 513–546; MR 2373358(2009c:11194).

Ser87 J.-P. Serre, *Sur les représentations modulaires de degré 2 de* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Duke Math. J. **54** (1987), 179–230; MR 885783(88g:11022).

Shi78 G. Shimura, *The special values of the zeta functions associated with Hilbert modular forms*, Duke Math. J. **45** (1978), 637–679; MR 507462(80a:10043).

Tho12 J. Thorne, *On the automorphy of l-adic Galois representations with small residual image*, J. Inst. Math. Jussieu **11** (2012), 855–920.

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