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ON CONVEX AND STARLIKE FUNCTIONS IN A SECTOR

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Abstract

Let f be analytic in $D = \{z : |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$. For $\gamma > 0$, the largest $\alpha(\gamma)$ and $\beta(\gamma)$ are found such that

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \Longrightarrow \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta} \Longrightarrow \frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}.$$

The results solve the inclusion problem for convex and starlike functions defined in a sector.

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Introduction

For $0 \le \alpha < 1$, let $K(\alpha)$ and $St(\alpha)$ denote the classes of functions f analytic in $D = \{z : |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$ which are convex and starlike of order α; that is, which satisfy

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$$

and

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha$$

respectively.

In [2] and [7] MacGregor, Wilken and Feng determined the exact $\beta(\alpha)$ such that $K(\alpha) \subset St(\beta)$ and in [1], Brickman et al found the exact $\gamma(\alpha)$ such that $f \in K(\alpha)$ implies $\operatorname{Re} f(z)/z > \gamma(\alpha)$, thus generalising Marx [3] and Strohhäcker's [6] classical results in the case $\alpha = 0$. By considering the extreme points of the closed convex hull of $St(\alpha)$, it follows easily from the results in [1] that if $f \in St(\alpha)$, then

$$\operatorname{Re}\frac{f(z)}{z} > 2^{-2(1-\alpha)} \qquad \text{if} \qquad \frac{1}{2} \le \alpha < 1,$$

but that if $0 \le \alpha < 1/2$, then $\operatorname{Re} f(z)/z$ can become minus infinity.

In this paper, we solve the corresponding problems for the classes $C(\alpha)$ and $S^*(\alpha)$ of convex and starlike functions in a sector defined as follows.

Let f be analytic in D and $f(0) = f'(0) - 1 = 0$. Then for $0 < \alpha \le 1$ the classes $C(\alpha)$ and $S^*(\alpha)$ are defined by

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2},$$

and

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}$$

respectively. Our results are an immediate consequence of the following subordination theorem.

Results

THEOREM 1. *Let f be analytic in D with $f(0) = f'(0) - 1 = 0$ and $0 < \beta \le 1$. Then for $z \in D$,*

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta)}$$

implies

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta}, \tag{1}$$

where

$$\alpha(\beta) = \frac{2}{\pi}\arctan\left[\tan\frac{\beta\pi}{2} + \frac{\beta}{(1+\beta)^{(1+\beta)/2}(1-\beta)^{(1-\beta)/2}\cos(\beta\pi/2)}\right] \tag{2}$$

Also for $\gamma > 0$,

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta(\gamma)} \tag{3}$$

implies

$$\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}, \tag{4}$$

where

$$\beta(\gamma) = \frac{2}{\pi} \arctan \gamma. \tag{5}$$

Furthermore $\alpha(\beta)$ *is the largest number such that* (1) *holds and* $\beta(\gamma)$ *is the largest number such that* (4) *holds.*

We shall use the following lemma (see for example [4]).

LEMMA. *Let* F *be analytic in* D *and* G *be analytic and univalent in* $\bar{D}$, *with* $F(0) = G(0)$. *If* $F \not\prec G$, *then there is a point* $z_0 \in D$ *and* $\zeta_0 \in \partial D$ *such that* $F(|z| < |z_0|) \subset G(D)$, $F(z_0) = G(\zeta_0)$ *and* $z_0 F'(z_0) = m\zeta_0 G'(\zeta_0)$ *for* $m \geq 1$.

PROOF OF THEOREM 1. Write

$$p(z) = \frac{zf'(z)}{f(z)},$$

so that p is analytic in D and $p(0) = 1$. Thus we need to show that

$$p(z) + \frac{zp'(z)}{p(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}$$

implies

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta},$$

whenever $\alpha = \alpha(\beta)$.

Now let $h(z) = [(1+z)/(1-z)]^{\alpha}$ and $q(z) = [(1+z)/(1-z)]^{\beta}$. Then $|\arg h(z)| < \alpha\pi/2$ and $|\arg q(z)| < \beta\pi/2$. Suppose that $p \not\prec q$, then from the Lemma there is a point $z_0 \in D$ and $\zeta_0 \in \partial D$ such that $p(z_0) = q(\zeta_0)$ and $p(|z| < |z_0|) \subset q(D)$.

We next note that $p(z) \neq 0$ for $z \in D$, since otherwise we can write $p(z) = (z - z_1)^k p_1(z)$ for some $k \geq 1$ and p_1 analytic in D, such that $p_1(z_1) \neq 0$. Then

$$p(z) + \frac{zp'(z)}{p(z)} = \frac{zp_1'(z)}{p_1(z)} + \frac{kz}{z - z_1} + (z - z_1)^k p_1(z). \tag{6}$$

Thus choosing $z \to z_1$, suitably the argument of the right-hand side of (6) can take any value between 0 and 2π, which contradicts the hypotheses of the Theorem.

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Since $p(z_0) = q(\zeta_0) \neq 0$, it follows that $\zeta_0 \neq \pm 1$. Thus we can write $ri = (1+\zeta_0)/(1-\zeta_0)$ for $r \neq 0$. Then the Lemma gives

$$\begin{aligned} p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} &= q(\zeta_0) + \frac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)} \\ &= (ri)^\beta + \frac{m\beta(1+r^2)i}{2r}. \end{aligned}$$

Next assume that $r > 0$. (If $r < 0$, the proof is similar.) Since $m \geq 1$, an elementary argument shows that

$$\begin{aligned} \arg\left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right) &= \arctan\left[\tan\frac{\beta\pi}{2} + \frac{m\beta(1+r^2)}{2r^{\beta+1}\cos(\beta\pi/2)}\right] \\ &\geq \arctan\left[\tan\frac{\beta\pi}{2} + \frac{\beta(1+r^2)}{2r^{\beta+1}\cos(\beta\pi/2)}\right] \\ &\geq \arctan\left[\tan\frac{\beta\pi}{2} + \frac{\beta}{(1+\beta)^{(1+\beta)/2}(1-\beta)^{(1-\beta)/2}\cos(\beta\pi/2)}\right] \\ &= \alpha(\beta)\pi/2. \end{aligned}$$

Hence

$$\frac{\alpha(\beta)\pi}{2} \leq \arg\left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right) \leq \frac{\pi}{2},$$

which contradicts the fact that $|\arg h(z)| < \alpha(\beta)\pi/2$ provided that (2) holds.

To show that (2) is exact, let

$$p(z) = \left(\frac{1+z}{1-z}\right)^\beta.$$

Then from the minimum principle for harmonic functions, it follows that

$$\inf_{|z|<1} \arg\left(p(z) + \frac{zp'(z)}{p(z)}\right)$$

is attained at some point $z = e^{i\theta}$ for $0 < \theta < 2\pi$. Thus

$$p(z) + \frac{zp'(z)}{p(z)} = \left(\frac{\sin\theta}{1-\cos\theta}\right)^\beta e^{\beta\pi i/2} + \frac{i\beta}{\sin\theta},$$

and so writing $t = \cos\theta$, we obtain

$$\arg\left(p(z) + \frac{zp'(z)}{p(z)}\right) = \arctan\left[\tan\frac{\beta\pi}{2} + \frac{\beta}{(1+t)^{(1+\beta)/2}(1-t)^{(1-\beta)/2}\cos(\beta\pi/2)}\right]$$

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and an elementary calculation shows that the minimum of this expression is attained when $t = \beta$.

We next show that (3) implies (4). Write $p(z) = f(z)/z$, so that again p is analytic in D and $p(0) = 1$. Thus this time we need to show that

$$1 + \frac{zp'(z)}{p(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta(\gamma)}$$

implies

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\gamma},$$

whenever (5) holds.

As before, let $h(z) = [(1+z)/(1-z)]^{\beta}$ and $q(z) = [(1+z)/(1-z)]^{\gamma}$ and assume that $p \not\prec q$. Then there exists $z_0 \in D$ and $\zeta_0 \in \partial D$ such that $p(z_0) = q(\zeta_0)$ and $p(|z| < |z_0|) \subset q(D)$. Since $p(z_0) = q(\zeta_0) \neq 0$, the Lemma gives

$$1 + \frac{z_0 p'(z_0)}{p(z_0)} = 1 + \frac{m\gamma(1+r^2)i}{2r}.$$

The result now follows using the same arguments as before.

To show that $\beta(\gamma)$ is the largest number such that (4) holds we again let $p(z) = [(1+z)/(1-z)]^{\beta}$ and $z = e^{i\theta}$. Then

$$\arg\left(1 + \frac{zp'(z)}{p(z)}\right) = \arctan\left[\frac{\beta}{\sin\theta}\right],$$

which has a minimum when $\theta = \pi/2$.

The following is an immediate consequence of Theorem 1.

COROLLARY. *For* $\gamma > 0$,

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\gamma)}$$

implies

$$\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}, \tag{7}$$

where

$$\alpha(\gamma) = \frac{2}{\pi}\arctan\left[\tan\frac{\beta(\gamma)\pi}{2} + \frac{\beta(\gamma)}{(1+\beta(\gamma))^{(1+\beta(\gamma))/2}(1-\beta(\gamma))^{(1-\beta(\gamma))/2}\cos(\beta(\gamma)\pi/2)}\right]$$

and $\beta(\gamma)$ *is given by* (5). *Also* $\alpha(\gamma)$ *is the largest number such that* (7) *holds.*

For $\alpha > 0$, denote by $R(\alpha)$ the class of functions f analytic in D with $f(0) = f'(0) - 1 = 0$, satisfying

$$\left|\arg \frac{f(z)}{z}\right| < \frac{\alpha\pi}{2},$$

for $z \in D$. We now restate the results in terms of $C(\alpha)$, $S^*(\alpha)$ and $R(\alpha)$ as follows.

THEOREM 2. *For $\gamma > 0$,*

$$C(\alpha(\beta(\gamma))) \subset S^*(\beta(\gamma)) \subset R(\gamma),$$

where $\alpha(\beta)$ and $\beta(\gamma)$ are given by (2) *and* (5) *respectively. Furthermore, $\alpha(\beta(\gamma))$ and $\beta(\gamma)$ are the largest numbers such that the inclusion holds.*

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