A THEORY OF MATHEMATICAL OBJECTS AS A PROTOTYPE OF SET THEORY

KATUZI ONO

Introduction

The theory of mathematical objects, developed in this work, is a trial system intended to be a prototype of set theory. It concerns, with respect to the only one primitive notion "proto-membership", with a field of mathematical objects which we shall hereafter simply call objects. It is a very simple system, because it assumes only one axiom scheme which is formally similar to the aussonderung axiom of set theory. We shall show that in our object theory we can construct a theory of sets which is stronger than the Zermelo set-theory [1] without the axiom of choice.

We use capital Latin letters as variables for objects, and the symbol "ϵ" for proto-membership, the only one primitive notion of our system. Proto-membership is naturally a binary relation, and any object X satisfying $X \epsilon Y$ is called a "proto-member" of Y. Objects and proto-membership can be regarded as prototypes of sets and membership respectively of the ordinary set-theoretical systems such as the systems of Zermelo, Fraenkel [2], etc. Our assumption for proto-membership is much weaker than the assumptions of these systems, especially in the following two points:

1) We do not assume that for any pair of objects there is an object containing them as its proto-members.

2) We do not assume that every object is completely determined by its proto-members. In fact, there can be many individual objects having no proto-members at all.

It is true that the pair-set axiom and the extensionality axiom are indispensable for set theory. We believe, however, it is an important question worth discussing in detail whether we can establish the notions of sets, membership, and equality in such a way that these axioms hold together with other

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set-theoretical axioms such as the axioms of sum sets, of power sets, of infinity, or of fundierung. von Neumann [3] proved consistency of the fundierung axiom with respect to his system, and Gandy [4] proved consistency of the extensionality axiom with respect to a modified system of the Bernays-Gödel set-theory [5], [6]. We show in our system that the notions of sets, membership, and equality can be defined in such a way that these axioms are altogether provable with respect to these notions.

Our object theory is founded on a unique principle (1.1.4)[1] which can be taken as a generalization of the aussonderung axiom. The logic in which we describe our system is the usual predicate logic. In describing axioms and theorems we usually omit universal quantifiers of a formula standing at its top and having the whole formula as their scope.

For any binary relations λ and φ we employ the relation product "$\lambda\varphi$" (See (1.1.1).). Binary relations often employed in this work are, for instance, the primitive notion "ϵ" itself, the "identity $\equiv$", defined in (1.2.3), and the "proto-inclusion $\subseteq$", defined in (1.1.2) as $X \subseteq Y \overset{df}{\equiv} \forall S(S \epsilon X \rightarrow S \epsilon Y)$, the last two of which can be regarded as prototypes of equality and inclusion respectively. Any object X satisfying $X \subseteq Y$ is called a "subobject" of the object Y.

Our system of object theory is exactly and formally introduced in Chapter (1). Here we give a sketch of the outline of our object theory informally.

In set theory, the device of aussonderung of the form $\exists p \forall x(x \in p \equiv \cdot x \in m \wedge \mathfrak{A}(x))$ is powerful enough to avoid well-known contradictions caused by the abstraction of the form $\exists p \forall x(x \in p \equiv \mathfrak{A}(x))$. Regarding it as a sole generating principle, however, the controlling power of the aussonderung axiom is too weak, since in generating a new set p the restriction on its members to those of a given set m seems too strict. Restriction to members of a given set could be modified safely by finding out a suitable weaker substitute for membership. In our axiom scheme, namely, *restriction to members* of a given set is replaced by *restriction to "satellites"* of a given object. Naturally, we have to replace sets by objects, and membership by proto-membership, so the only assumption of our system can be expressed as

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[1] In the numbering of the forms (a), (a, b), and (a, b, c) in this work, a, b and c denote the numbers of chapters, sections, and paragraphs respectively. Each paragraph is usually a theorem or a definition.

$$\exists P \forall X(X \in P \equiv \cdot X\sigma M \wedge \mathfrak{A}(X)),$$

where $X\sigma M$ denotes that X is a satellite of M. (See (1.1.4).)

We explain now the notion of satellites. Formally, $X\sigma Y$ is defined in (1.1.3) as $X\sigma Y \overset{df}{\equiv} \forall P(\sigma(P) \wedge Y \subseteq \in P \cdot \rightarrow X \subseteq \in P)$, where $\sigma(P)$ denotes $\forall S(S \in \in P \equiv S \subseteq \in P)$. To see more closely what is really a satellite of an object M, we need the notion of "unit objects" of an object M (formally defined in (1.4.3)), the notion corresponding to the notion of the unit set of a set in set theory. Since we assume nothing corresponding to the extensionality axiom, we have to say like "a unit object of a given object" instead of saying like "the unit object of a given object".

Any subobject of an object M, any subobject of a proto-member of M, any subobject of a proto-member of a proto-member of M, and so on, are called "constituents" of M (formally defined in (1.5.1).). Also any subobject of an object M, any unit object of a subobject of M, any unit object of a unit object of a subobject of M, and so on, are called "ancestors" of M (formally defined in (1.5.8).). Any ancestor of a constituent of an object M is a satellite of M. (See (1.7.1), (1.7.3), and (1.7.5).) When we replace objects and proto-membership by their corresponding notions, sets and membership in the Fraenkel set-theory, respectively, any set x is a satellite (in the interpreted sense) of m if and only if x and m satisfy the formal definition of $x\hat{\sigma}m$. ($\hat{\sigma}$ be the interpreted relation of σ.) Moreover, in the Fraenkel set-theory, every proposition of the form $\exists p \forall x(x \in p \equiv \cdot x\hat{\sigma}m \wedge \mathfrak{A}(x))$ holds, so our system can be imbedded in the Fraenkel set-theory. In other words, our system is consistent if the Fraenkel system is so. An outline of this proof of consistency relative to the Fraenkel set-theory is described in the last Chapter (7) as a supplementary remark.

In our object theory, namely, we assume that all the satellites of a given object satisfying an arbitrary given condition form an object. The notion of satellites is so defined as to be able to construct a set theory as far as possible only on the basis of this simple assumption. In our object theory, we construct a set theory stronger than the Zermelo system without the axiom of choice, but it has not been decided whether we can construct a system including the axiom of replacement in our object theory. Throughout this work, we do not discuss anything concerning the axiom of choice. However, it will be possible to prove consistency of the axiom of choice, together with the axiom of replacement, perhaps by following Gödel's consistency proof [6].

In Chapter (1), we show some elementary properties of satellites, constituents, and ancestors. In this Chapter we show also existence of "null objects" (corresponding to the null set) and existence of unit objects, "sum objects", and "power objects" (the last two correspond to the sum set and the power set respectively) of a given object in our system, even though uniqueness can never be proved. Also here, accordingly, we have to say such as "a null object", "a sum object of X", or "a power object of X" instead of saying like "the null object", "the sum object of X", or "the power object of X", respectively. In this Chapter we prove further a proposition corresponding to the aussonderung axiom and, even more, a proposition corresponding to the axiom of infinity with respect to identity in our object theory. (See (1.3.1), (1.8.2), (1.8.4), and (1.8.5).)

It seems a big merit of the notion of satellites that the natural numbers can be introduced quite naturally by it. Namely, any object which is a satellite of every object can be taken as a representative of a natual number and is called a "proto-number". We prove that every null object is a proto-number, that every unit object of a proto-number is also a proto-number, and that only those objects which can be shown to be proto-numbers by the above two principles are proto-numbers. (See (1.8).)

Moreover, we see that any null object is not a unit object of any object, and that X and Y are identical if any unit obiect of X is identical with a unit object of Y. In short, proto-numbers satisfy the Peano axioms by suitable interpretation except for that there can be many null objects and that for each object there can be many unit objects of it. To establish a full theory of natural numbers, we have to identify all the proto-numbers which represent the same natural number. This can be done only in Chapter [6].

Any proposition corresponding to the fundierung axiom seems unprovable in our system. In Chapter (2), we introduce the notions of "semi-regularity" and "regularity". Any object is called semi-regular if and only if it has no such constituent S that S is a constituent of a proto-member of S itself. The notion of regularity is a notion more complicated and stronger than semi-regularity. (See (2.2.1), (2.2.2), and (2.2.4).) It will not be proved that all the objects are regular, nor even that all the objects are semi-regular. However,

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we prove in (3.2.7) that all the "producible" objects (defined in (3.2.1)) are regular. More in detail, we prove in (2.2.3) that any null object is regular, in (3.2.3) that any satellite of a regular object is regular, and further in (3.2.4) that any object formed by all the satellites of a regular object is also regular.

Chapter (3) is devoted to giving an exact description of producible objects i.e. of those objects whose existence can be really confirmed by our axioms only. Starting from any proto-number X_1, we can construct an infinite sequence of objects X_1, X_2, X_3, $\cdots$ successively by the rule that X_{n+1} is an object formed by all the satellites of X_n. This sequence is monotone increasing in the sense that X_n is a subobject of X_{n+1}. Any object formed by the first finite terms of a sequence of this kind is called a "basic" object. Any satellite of a proto-member of a basic object is a producible object. By technical reason, however, the notion of basic objects is formally defined in a slightly modified way in (3.1.1).

We prove that there is at least one producible object, that every satellite of any producible object is also producible, and that every object formed by all the satellites of any producible object is also producible. (See (3.2.2), (3.2.3), and (3.2.4).) In short, the field of producible objects is closed with respect to generation of new objects by our axiom scheme.

The field of producible objects is very important, firstly because every producible object is regular (See (3.2.7).), secondly because for any two producible objects there is a producible "pair object" of them (See (3.2.10). Pair objects are defined in (3.2.9).), and thirdly because we can define membership and equality in such a way that the extensionality axiom together with other equality axioms holds for nroducible objects with respect to these two notions.

In Chapter (4) we introduce the notions of "membership" and "equallity" so that all the axioms concerning equality (in the ordinary sense) are provable for producible objects with respect to these notions (See (4.10) and (4.12).). Membership and equality are weaker than proto-membership and identity respectively. They are defined simultaneously keeping the relation in mind that an object X is a member of an object Y if and only if X is equal to a proto-member of Y. The process of defining equality is not so simple. Namely, after introducing the notion of "ε-objects" (defined in (4.6.1)), X is called to be equal to Y if and only if either X and Y are identical or there is a pair object of X and Y which is a proto-member of an ε-object. Roughly speaking, ε-objects can be taken as those formed by some pair objects of mutually equal objects, but the definition turns out to be complicated because we have, in advance, to give a condition for an object to be an ε-object without employing the notion of equality.

Broadly speaking, producible objects can be regarded as sets with respect to membership and equality. Namely, if we restrict ourselves to consider only "properties modulo equality" i.e. properties common to all the objects mutually equal to each other, as is defined in (5.1.1), we can establish a theory of sets. An example of properties modulo equality is the property defined by the "set-theoretical image $|\mathfrak{A}(X)|$" of any proposition $\mathfrak{A}(X)$ that is obtained by replacing proto-membership in $\mathfrak{A}(X)$ by membership and restricting the ranges of all the quantifiers in $\mathfrak{A}(X)$ to producible objects. (See (5.3.1), (5.3.2), (5.3.5), and (6.1.4).) In Chapter (5), we study properties modulo equality, especially in connection with the set-theoretical images of propositions.

In Chapter (6), we show that a theory of sets can be established with respect to membership and equality. In our theory of sets, all the axioms of the Zermelo system are provable except for the axiom of choice. Also the fundierung axiom is provable in our set theory. Moreover, the set-theoretical images of all the axioms of our object theory are provable, which are surely provable in the Fraenkel system but it seems that some of them may be unprovable in the Zermelo system.

(1) Theory of objects

Our system of object theory is founded on a unique axiom scheme taking proto-membership "ϵ" as the only primitive notion. "$X \epsilon Y$" is read "X is a *proto-member* of Y".

(1.1) Axioms. Before describing the axiom scheme, we define "subobjects" and "satellites".

(1.1.1) The relation product $\lambda\varphi$ of two binary relations λ and φ is defined by $X\lambda\varphi Y \overset{df}{\equiv} \exists S(X\lambda S \wedge S\varphi Y)$.

(1.1.2) Definition: $X \subseteq Y \overset{df}{\equiv} \forall S(S \epsilon X \rightarrow S \epsilon Y)$. The binary relation "$\subseteq$" is

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called *proto-inclusion*, and any object X satisfying $X \subseteq Y$ is called a *subobject* of Y.

(1.1.3) **Definition**: $\sigma(P) \overset{df}{\equiv} \forall S(S \in \in P \equiv S \subseteq \in P)$ and

$$X \sigma Y \overset{df}{\equiv} \forall P(\sigma(P) \wedge Y \subseteq \in P \cdot \rightarrow X \subseteq \in P).$$

Any object P satisfying $\sigma(P)$ is called a *σ-object* and any object X satisfying $X\sigma Y$ is called a *satellite* of Y.

(1.1.4) **Axiom scheme**: All the formulas of the form

$$\exists P \forall X(X \in P \equiv \cdot X \sigma M \wedge \mathfrak{A}(X))$$

and formulas of this form only are axioms of our object theory, where in $\mathfrak{A}(X)$ any number of free variables other than P may occur.

Namely, we assume that there is an object formed by all the satellites of an object satisfying an arbitrary given condition.

(1.2) **Some elementary properties.**

(1.2.1) $X \subseteq X$ and $X \subseteq \subseteq Y \rightarrow X \subseteq Y$. (Reflexivity and transitivity of proto-inclusion.)

(1.2.2) $X \in \subseteq Y \rightarrow X \in Y$.

(1.2.3) **Definition**: $X = Y \overset{df}{\equiv} \forall P(X \in P \equiv Y \in P)$. The binary relation "$=$" is called *identity*.

(1.2.4) $X = X$, $X = Y \rightarrow Y = X$, and $X = = Y \rightarrow X = Y$. (Reflexivity, symmetricity, and transitivity of identity.)

(1.2.5) **Definition**: $X \simeq Y \overset{df}{\equiv} \forall S(S \in X \equiv S \in Y)$. The binary relation "$\simeq$" is called *proto-equality* (prototype of equality).

(1.2.6) $X \simeq Y \equiv \cdot X \subseteq Y \wedge Y \subseteq X$.

(1.2.7) $X \simeq X$, $X \simeq Y \rightarrow Y \simeq X$, and $X \simeq \simeq Y \rightarrow X \simeq Y$. (Reflexivity, symmetricity, and transitivity of proto-equality.)

(1.2.8) $X = \in Y \equiv X \in Y$ and $X \in \simeq Y \equiv X \in Y$.

(1.2.9) *Remark.* We can neither introduce here term-symbols of the form $\{X;\ \mathfrak{A}(X)\}$, nor adopt the way of talking such as "the object formed by all those objects X which satisfy the condition $\mathfrak{A}(X)$", even when it is certain that there

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exists such an object. For, there can be possibly more than two (in the sense of identity) objects P satisfying $\forall X(X \in P \equiv \mathfrak{A}(X))$, since the axiom of extensionality with respect to identity is not assumed in our system. However, we introduce a proposition-symbol $P\{X;\ \mathfrak{A}(X)\}$ for any $\mathfrak{A}(X)$ which is defined by

Definition: $P\{X;\ \mathfrak{A}(X)\} \overset{df}{\equiv} \forall X(X \in P \equiv \mathfrak{A}(X))$.

By this definition the axiom scheme of our object theory can be expressed as $\exists P \cdot P\{X;\ X\sigma M \wedge \mathfrak{A}(X)\}$.

(**1.2.10**) *Remark.* Concerning uniqueness,
we can not assert $P\{X;\ \mathfrak{A}(X)\} \wedge Q\{X;\ \mathfrak{A}(X)\} \cdot \rightarrow P \equiv Q$,
but we can assert $P\{X;\ \mathfrak{A}(X)\} \wedge Q\{X;\ \mathfrak{A}(X)\} \cdot \rightarrow P \simeq Q$.

(**1.2.11**) $X \in Y \rightarrow X\sigma Y$. (Proto-membership implies σ-relation.)

Proof. Let X be any proto-member of an object Y, and P be any σ-object. Then, $Y \subseteq \in P$ implies $X \in \in P$ by (1.2.2), which implies $X \subseteq \in P$ because P is a σ-object.

(**1.2.12**) $X \subseteq Y \rightarrow X\sigma Y$. (Proto-inclusion implies σ-relation.)

Proof. Let X be a subobject of an object Y. Then, by transitivity of proto-inclusion, $Y \subseteq \in P$ implies $X \subseteq \in P$ for any P, especially for any σ-object P.

(**1.2.13**) $X\sigma X$. (Reflexivity of σ.)

Proof. By (1.2.12) and reflexivity of proto-inclusion.

(**1.2.14**) $X\sigma\sigma Y \rightarrow X\sigma Y$. (Transitivity of σ.)

Proof. Let X be a satellite of a satellite Z of Y. For any σ-object P, $Y \subseteq \in P$ implies $Z \subseteq \in P$ by the assumption $Z\sigma Y$, and $Z \subseteq \in P$ implies $X \subseteq \in P$ by the assumption $X\sigma Z$, so $Y \subseteq \in P$ implies $X \subseteq \in P$.

(**1.3**) **Aussonderung.**

(1.3.1) $\exists P \cdot P\{X;\ X \in M \wedge \mathfrak{A}(X)\}$ and $\exists P \cdot P\{X;\ X \subseteq M \wedge \mathfrak{A}(X)\}$, where in $\mathfrak{A}(X)$ any number of free variables other than P may occur.

Proof. By (1.2.11) and (1.2.12), $X \in M \wedge \mathfrak{A}(X)$ and $X \subseteq M \wedge \mathfrak{A}(X)$ can be expressed as $X\sigma M \wedge X \in M \wedge \mathfrak{A}(X)$ and $X\sigma M \wedge X \subseteq M \wedge \mathfrak{A}(X)$ respectiviely, so we can get the theorem directly from our axiom scheme.

(1.3.2) $\exists P \cdot P\{X;\ X\sigma M\}$. (For any object M there is an object formed by all the satellites of M.)

Proof. Because $X\sigma M$ can be expressed as $X\sigma M \wedge X\sigma M$, we can get the theorem directly from the axiom scheme.

(1.3.3) $\exists X \cdot X\{T;\ T \simeq \in Y\}$.

Proof. By (1.2.6), (1.2.11), (1.2.12), and transitivity of σ.

(1.3.4) $X \equiv Y \to X \simeq Y$. (Identity implies proto-equality.)

Proof. Let X and Y be any two mutually identical objects. By (1.3.1) take any P satisfying $P\{T;\ T \subseteq X \wedge X \simeq T\}$. By reflexivity of proto-inclusion and proto-equality, $X \in P$, so $Y \in P$. Namely $X \simeq Y$.

(1.3.5) $X \equiv Y \to \cdot\, \mathfrak{A}(X) \equiv \mathfrak{A}(Y)$. (The proposition corresponding to the second axiom of equality.)

Proof. By (1.3.4).

(1.4) Null objects, unit objects, sum objects, and power objects.

(1.4.1) **Definition**: $\emptyset(P) \overset{df}{\equiv} : \neg \exists S \cdot S \in P$.

Any object P satisfying $\emptyset(P)$ is called a ***null object***.

(1.4.2) $\exists X \cdot \emptyset(X)$. (Existence of null objects.)

Proof.[2] Because $P\{T;\ T\sigma M \wedge \neg T\sigma M\}$ contradicts $S \in P$, the former also contradicts $\exists U \cdot U \in P$ since the free variable S does not occur in it. Accordingly, $P\{T;\ T\sigma M \wedge \neg T\sigma M\}$ implies $\emptyset(P)$, so also $\exists X \cdot \emptyset(X)$. Since the free variable P does not occur in $\exists X \cdot \emptyset(X)$, $\exists P \cdot P\{T;\ T\sigma M \wedge \neg T\sigma M\}$ implies $\exists X \cdot \emptyset(X)$. Hence the axiom $\forall M \exists P \cdot P\{T;\ T\sigma M \wedge \neg T\sigma M\}$ implies $\exists X \cdot \emptyset(X)$.

(1.4.3) **Definition**: $U\{X\} \overset{df}{\equiv} U\{T;\ T \equiv X\}$.

Any object U satisfying $U\{X\}$ is called a ***unit object*** of X.

(1.4.4) $\exists U \cdot U\{X\}$. (Existence of unit objects of X.)

Proof. By (1.3.1), take any object U satisfying $U\{T;\ T \subseteq X \wedge T \equiv X\}$. Then,

[2] We prove the theorem particularly in detail, as we wish to show that our axiom scheme logically implies absolute existence of an object. It should be also remarked here that in our object theory we may consider any object X satisfying any condition $\mathfrak{A}(X)$ but its existence is negated whenever $\mathfrak{A}(X)$ contradicts the axioms.

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U is clearly a unit object of X by (1.2.6) and (1.3.4).

(1.4.5) $\exists P \cdot P\{T;\ T \in \in A\}$. (Existence of sum objects of A. Here we call any object P satisfying $P\{T;\ T \in \in A\}$ a ***sum object*** of A.)

Proof. Because $T \in \in A$ can be expressed as $T\sigma A \wedge T \in \in A$ by (1.2.11) and transitivity of σ.

(1.4.6) $\exists P \cdot P\{T;\ T \subseteq A\}$. (Existence of power objects of A. Here we call any object P satisfying $P\{T;\ T \subseteq A\}$ a ***power object*** of A.)

Proof. Because $T \subseteq A$ can be expressed as $T\sigma A \wedge T \subseteq A$ by (1.2.12).

(1.4.7) $\emptyset(X) \to X \subseteq Y$. (Any null object is a subobject of every object.)

(1.4.8) $\emptyset(X) \to X\sigma Y$. (Any null object is a satellite of every object.)

Proof. By (1.2.12) and (1.4.7).

(1.4.9) $X\{Y\} \to X\sigma Y$. (Unit-object relation implies σ-relation.)

Proof. Let X be a unit object of Y. For any σ-object P satisfying $Y \subseteq \in P$, holds $Y \in \in P$ i.e. Y is a proto-member of a proto-member Z of P. Accordingly, by (1.3.5), the unit object X of Y is a subobject of the proto-member Z of P. Hence $X\sigma Y$.

(1.4.10) $X \subseteq Y \wedge Y\{Z\} \cdot \to \cdot \emptyset(X) \vee X\{Z\}$. (Any subobject of a unit object of an object Z is either a null object or a unit object of Z.)

Proof. Let X be a subobject of a unit object Y of an object Z. Assume further that X is not a null object.

To show that X is a unit object of Z, take any proto-member T of X. Then, by (1.2.2), T is also a proto-member of the unit object Y of Z, so $T \equiv Z$. Conversely, take any object T which is identical with Z. Since X is not a null object, there is a proto-member S of X, which is also a proto-member of Y by (1.2.2). Hence $S \equiv Z$. By symmetricity and transitivity of identity, T is identical with the proto-member S of X, so $T \in X$ by (1.3.5).

(1.5) Constituents and ancestors.

(1.5.1) Definition: $\kappa(P) \overset{df}{\equiv} \forall S(S \in \in P \to S \subseteq \in P)$ and

$$X\kappa Y \overset{df}{\equiv} \forall P(\kappa(P) \wedge Y \subseteq \in P \cdot \to X \subseteq \in P).$$

Any object P satisfying $\kappa(P)$ is called a ***κ-object***. Any object X satisfying

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$X\kappa Y$ is called a *constituent* of Y.

(1.5.2) $\sigma(P)\to\kappa(P)$. (Any σ-object is a κ-object.)

(1.5.3) $X\kappa Y\to X\sigma Y$. (κ-relation implies σ-relation.)

(1.5.4) $X\in Y\to X\kappa Y$. (Proto-membership implies κ-relation.)

Proof. Similar to the proof of (1.2.11) by making use of (1.2.2).

(1.5.5) $X\subseteq Y\to X\kappa Y$. (Proto-inclusion implies κ-relation.)

Proof. Similar to the proof of (1.2.12), by making use of transitivity of proto-inclusion.

(1.5.6) $X\kappa X$. (Reflexivity of κ.)

(1.5.7) $X\kappa\kappa Y\to X\kappa Y$. (Transitivity of κ.)

Proof. Similar to the proof of transitivity of σ.

(1.5.8) Definition: $\alpha(P)\overset{df}{\equiv}\forall S(S\subseteq\in P\to S\in\in P)$ and

$$X\alpha Y\overset{df}{\equiv}\forall P(\alpha(P)\wedge Y\subseteq\in P\cdot\to X\subseteq\in P).$$

Any object P satisfying $\alpha(P)$ is called an α-*object*. Any object X satisfying $X\alpha Y$ is called an *ancestor* of Y.

(1.5.9) $\sigma(P)\to\alpha(P)$. (Any σ-object is an α-object.)

(1.5.10) $X\alpha Y\to X\sigma Y$. (α-relation implies σ-relation.)

(1.5.11) $X\subseteq Y\to X\alpha Y$. (Proto-inclusion implies α-relation.)

Proof. Similar to the proof of (1.2.12), by making use of transitivity of proto-inclusion.

(1.5.12) $X\{Y\}\to X\alpha Y$. (Unit-object relation implies α-relation.)

Proof. Similar to the proof of (1.4.9), by making use of (1.3.5).

(1.5.13) $X\alpha X$. (Reflexivity of α.)

(1.5.14) $X\alpha\alpha Y\to X\alpha Y$. (Transitivity of α.)

Proof. Similar to the proof of transitivity of σ.

(1.5.15) $\emptyset(X)\to X\kappa Y$ and $\emptyset(X)\to X\alpha Y$. (Any null object is a constituent as well as an ancestor of every object.)

Proof. By (1.4.7), (1.5.5), and (1.5.11).

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(1.6) Minimum properties.

(1.6.1) $\forall XY$[3]$(X\in Y\vee X\subseteq Y\vee X\{Y\}\cdot\rightarrow\mathfrak{A}(X, Y))\wedge\forall XYZ(\mathfrak{A}(X, Y)\wedge\mathfrak{A}(Y, Z)\cdot\rightarrow\mathfrak{A}(X, Z)):\rightarrow\cdot U\sigma V\rightarrow\mathfrak{A}(U, V)$. (Minimum property of σ.)

Proof. Let $\mathfrak{A}(X, Y)$ be any transitive relation which satisfies $\forall XY(X\in Y\vee X\subseteq Y\vee X\{Y\}\cdot\rightarrow\mathfrak{A}(X, Y))$. Take an object P which satisfies $P\{T;\ T\sigma V\wedge\mathfrak{A}(T, V)\}$, V being an arbitrary object.

We prove first $\sigma(P)$: Take any subobject S of a proto-member T of P. Then, $\mathfrak{A}(S, T)$ and $T\sigma V\wedge\mathfrak{A}(T, V)$ hold by assumption. Moreover $S\sigma T$ by (1.2.12). Take now a unit object W of S by (1.4.4). Then, $W\sigma S$ by (1.4.9), and $\mathfrak{A}(W, S)$ holds by assumption. By transitivity of σ and of the relation $\mathfrak{A}(X, Y)$, holds $W\sigma V\wedge\mathfrak{A}(W, V)$. Accordingly $W\in P$; so, by reflexivity of identity, S is a proto-member of the proto-member W of P. Conversely, let S be any proto-member of a proto-member T of P. Then, $T\sigma V\wedge\mathfrak{A}(T, V)$ holds. Moreover, $S\sigma T$ holds by (1.2.11), and $\mathfrak{A}(S, T)$ holds by assumption. By transitivity of σ and of the relation $\mathfrak{A}(X, Y)$, holds $S\sigma V\wedge\mathfrak{A}(S, V)$ i.e. $S\in P$. Hence by reflexivity of proto-inclusion, S is a subobject of the proto-member S of P.

Next we prove $V\in P$: By reflexivity of σ and proto-inclusion, hold $V\sigma V$ and $V\subseteq V$, and the latter implies $\mathfrak{A}(V, V)$ by assumption. Hence $V\in P$.

Now we prove that $U\sigma V$ implies $\mathfrak{A}(U, V)$: If U is a satellite of V, U is a subobject of a proto-member W of P, since P is a σ-object and V is, by reflexivity of proto-inclusion, a subobject of the proto-member V of P. Any proto-member W of P satisfies clearly $\mathfrak{A}(W, V)$, and any subobject U of W satisfies $\mathfrak{A}(U, W)$ by assumption. Accordingly, $\mathfrak{A}(U, V)$ holds by transitivity of the relation $\mathfrak{A}(X, Y)$.

We can prove similarly the following two theorems, minimum property of κ and that of α, by making use of (1.2.11); (1.2.12); (1.4.4); (1.4.9); transitivity of σ; and reflexivity of proto-inclusion, identity, and σ.

(1.6.2) $\forall XY(X\in Y\vee X\subseteq Y\cdot\rightarrow\mathfrak{A}(X, Y))$
$\wedge\forall XYZ(\mathfrak{A}(X, Y)\wedge\mathfrak{A}(Y, Z)\cdot\rightarrow\mathfrak{A}(X, Z)):\rightarrow\cdot U\kappa V\rightarrow\mathfrak{A}(U, V)$.

(1.6.3) $\forall XY(X\subseteq Y\vee X\{Y\}\cdot\rightarrow\mathfrak{A}(X, Y))$
$\wedge\forall XYZ(\mathfrak{A}(X, Y)\wedge\mathfrak{A}(Y, Z)\cdot\rightarrow\mathfrak{A}(X, Z)):\rightarrow\cdot U\alpha V\rightarrow\mathfrak{A}(U, V)$.

(1.6.4) $\forall XY(X\in Y\vee X\subseteq Y\vee X\{Y\}\cdot\rightarrow\cdot\mathfrak{A}(X)\rightarrow\mathfrak{A}(Y))$ and $U\sigma V$ imply

[3] In this work, quantifiers of the forms $\forall X\cdots Z$ and $\exists X\cdots Z$ stand for $\forall X\cdots\forall Z$ and $\exists X\cdots\exists Z$ respectively.

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$\mathfrak{A}(U) \to \mathfrak{A}(V)$.

Proof. Because the relation $\mathfrak{A}(X) \to \mathfrak{A}(Y)$ is transitive, this is a special case of (1.6.1).

Similarly, as special cases of (1.6.2) and (1.6.3), we have

(1.6.5) $\forall XY(X \in Y \vee X \subseteq Y \cdot \to \cdot \mathfrak{A}(X) \to \mathfrak{A}(Y))$ and $U \kappa V$ imply $\mathfrak{A}(U) \to \mathfrak{A}(V)$.

(1.6.6) $\forall XY(X \subseteq Y \vee X\{Y\} \cdot \to \cdot \mathfrak{A}(X) \to \mathfrak{A}(Y))$ and $U \alpha V$ imply $\mathfrak{A}(\quad) \to \mathfrak{A}(V)$.

(1.7.1) $X \kappa Y \equiv \cdot X \subseteq Y \vee X \kappa \in Y$. (Any object X is a constituent of an object Y if and only if X is either a subobject of Y or a constituent of a proto-member of Y.)

Proof. Take an object P satisfying $P\{T;\ T \sigma Y \wedge (T \subseteq Y \vee T \kappa \in Y)\}$. Then, $P\{T;\ T \subseteq Y \vee T \kappa \in Y\}$ holds by (1.2.11), (1.2.12), (1.5.3), and transitivity of σ.

We prove first $\kappa(P)$: Take any proto-member S of a proto-member T of P. Then, $T \subseteq Y$ or $T \kappa \in Y$. $S \kappa \in Y$ is provable, by making use of (1.2.2) and reflexivity of κ in the case $T \subseteq Y$, and by making use of (1.5.4) and transitivity of κ in the case $T \kappa \in Y$. Hence $S \in P$, so, by reflexivity of proto-inclusion, S is a subobject of the proto-member S of P.

Moreover $Y \in P$ by reflexivity of proto-inclusion, so Y is a subobject of the proto-member Y of P again by reflexivity of proto-inclusion. Hence any constituent X of Y is also a subobject of a proto-member W of P. By (1.5.5) and transitivity of proto-inclusion and of κ holds $X \subseteq Y \vee X \kappa \in Y$, because either $W \subseteq Y$ or $W \kappa \in Y$.

Conversely, $X \subseteq Y \vee X \kappa \in Y$ implies $X \kappa Y$ by (1.5.4), (1.5.5), and transitivity of κ.

(1.7.2) $X \kappa Y \equiv \cdot X \subseteq Y \vee X \subseteq \in \kappa Y$.

Proof. The proof is simiar to that of (1.7.1). Namely, by (1.2.11), (1.2.12), (1.5.3), and transitivity of σ, we can take an object P satisfying $P\{T;\ T \subseteq Y \vee T \subseteq \in \kappa Y\}$. By (1.2.2), (1.5.4), transitivity of κ, and reflexivity of proto-inclusion and κ, we can show for this P that P is a κ-object and that Y is a subobject of the proto-member Y of P. Consequently, any constituent X of Y is a subobject of a proto-member W of P, i.e. there is an object W satisfying $X \subseteq W$ and $W \subseteq Y \vee W \subseteq \in \kappa Y$, which imply $X \subseteq Y \vee X \subseteq \in \kappa Y$ by transitivity of proto-inclusion. On the other hand, $X \subseteq Y$ as well as $X \subseteq \in \kappa Y$ implies $X \kappa Y$

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by (1.5.4), (1.5.5), and transitivity of κ.

(1.7.3) $X\alpha Y \equiv \cdot X \subseteq Y \vee \exists Z(X\{Z\} \wedge Z\alpha Y)$. (Any object X is an ancestor of an object Y if and only if X is either a subobject of Y or a unit object of an ancestor of Y.)

Proof. The proof is similar to that of (1.7.1).

Namely, by (1.2.12), (1.4.9), (1.5.10), and transitivity of σ, we can take an object P satisfying $P\{T;\ T \subseteq Y \vee \exists Z(T\{Z\} \wedge Z\alpha Y)\}$. For this P we can show $\alpha(P)$: Namely, take any subobject S of a proto-member T of P, and also take any unit object U of S by (1.4.4). Then, $S \in U$ by reflexivity of identity, and moreover, T is either a subobject of Y or a unit object of an ancestor Z of Y. In the case $T \subseteq Y$, holds $S \subseteq Y$ by transitivity of proto-inclusion. Hence $S\alpha Y$ by (1.5.11). Accordingly $U \in P$, so S is a proto-member of the proto-member U of P. In the case $T\{Z\} \wedge Z\alpha Y$, $S\alpha Y$ by (1.5.11), (1.5.12), and transitivity of α, so the unit object U of S is a proto-member of P. Accordingly, S is a proto-member of the proto-member U of P.

Moreover, by reflexivity of proto-inclusion, we can show that Y is a subobject of the proto-member Y of P. Consequently, any ancestor X of Y is a subobject of a proto-member W of P, namely, there is an object W satisfying $W \subseteq Y \vee \exists Z(W\{Z\} \wedge Z\alpha Y)$. From this we can easily deduce $X \subseteq Y \vee \exists Z(X\{Z\} \wedge Z\alpha Y)$ by making use of (1.4.7), (1.4.10), and transitivity of proto-inclusion.

Conversely, $X \subseteq Y$ as well as $\exists Z(X\{Z\} \wedge Z\alpha Y)$ implies $X\alpha Y$ by (1.5.11), (1.5.12), and transitivity of α.

(1.7.4) $X \in \alpha Y \equiv \cdot X \in Y \vee X\alpha Y$. (Any object X is a proto-member of an ancestor of Y if and only if X is either a proto-member of Y or an ancestor of Y.)

Proof. Assume first that X is a proto-member of an ancestor Z of Y. By (1.7.3), Z is either a subobject of Y or a unit object of an ancestor T of Y. In the case $Z \subseteq Y$, X is surely a proto-member of Y by (1.2.2). In the case $Z\{T\} \wedge T\alpha Y$, holds $X = T$, so $X\alpha Y$ by (1.3.5).

Conversely, any proto-member X of Y is a proto-member of the ancestor Y of Y by reflexivity of α. On the other hand, we can take for any ancestor X of Y, a unit object Z of X by (1.4.4). Then, by reflexivity of identity, X is a proto-member of Z, which is an ancestor of Y by (1.5.12) and transitivity of α.

(**1.7.5**) $X\sigma Y \equiv X\alpha\kappa Y$. ($\sigma$-relation is equivalent to the relation product $\alpha\kappa$.)

Proof. We can take an object P satisfying $P\{T;\ T\alpha\kappa Y\}$ by (1.5.3), (1.5.10), and transitivity of σ.

At first, we prove $\sigma(P)$: Take any subobject S of a proto-momber T of P, and then, take a unit object U of S by (1.4.4). By (1.5.11), (1.5.12), and transitivity of α, $T\alpha\kappa Y$ implies $U\alpha\kappa Y$, so holds $U\in P$. Consequently, S is a proto-member of the proto-member U of P by reflexivity of identity. Conversely, take any proto-member S of a proto-member T of P. Then, T is an ancestor of a constituent Z of Y. By (1.7.4), S is either a proto-member of Z or an ancestor of Z. In the case $S\in Z$, holds $S\kappa Y$ by (1.5.4) and transitivity of κ, so S is an ancestor of the constituent S of Y by reflexivity of α. In the case $S\alpha Z$, $S\alpha\kappa Y$ holds too. Accordingly, $S\in P$ anyway, so S is a subobject of the proto-member S of P by reflexivity of proto-inclusion.

Moreover, $Y\subseteq\in P$ by reflexivity of proto-inclusion, α, and κ. Consequently, any satellite X of Y is a subobject of a proto-member U of P. Since the proto-member U of P is an ancestor of a constituent of Y, X itself is an ancestor of a constituent of Y by (1.5.11) and transitivity of α.

Conversely, $X\alpha\kappa Y$ implies $X\sigma Y$ by (1.5.3), (1.5.10), and transitivity of σ.

(1.8) Proto-numbers.

(**1.8.1**) **Definition:** $\nu(X) \overset{df}{\equiv} \forall P \cdot X\sigma P$.

Any object is called a *proto-number* if and only if it is a satellite of every object.

(**1.8.2**) $\emptyset(X) \to \nu(X)$. (Any null object is a proto-number.)

Proof. By (1.4.8).

(**1.8.3**) $\nu(X) \to \cdot\, Y\sigma X \equiv \nu(Y)$. (Any object is a proto-number if and only if it is a satellite of a proto-number.)

Proof. By transitivity of σ.

(**1.8.4**) $\nu(X) \wedge Y\in X \cdot \to \nu(Y)$, $\nu(X) \wedge Y\subseteq X \cdot \to \nu(Y)$, and $\nu(X) \wedge Y\{X\} \cdot \to \nu(Y)$. (Any proto-member as well as any subobject as well as any unit object of a proto-number is a proto-number.)

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Proof. By (1.2.11), (1.2.12), (1.4.9), and (1.8.3).

(**1.8.5**) $\exists P \cdot P\{X;\ \nu(X)\}$. (There is an object formed by all the proto-numbers.)

Proof. Since $\nu(X)$ implies $X\sigma M$, $P\{X;\ X\sigma M \wedge \nu(X)\}$ implies $P\{X;\ \nu(X)\}$. Hence the axiom $\forall M \exists P \cdot P\{X;\ X\sigma M \wedge \nu(X)\}$ implies $\exists P \cdot P\{X;\ \nu(X)\}$.

(**1.8.6**) $\emptyset(X) \equiv \forall Y \cdot X\kappa Y$. (Any object is a null object if and only if it is a constituent of every object.)

Proof. Any null object is a constituent of every object by (1.4.7) and (1.5.5).

Conversely, let X be a constituent of every object. Take now a null object Z by (1.4.2), and a unit obiect U of Z by (1.4.4). Then, U is a κ-object, because there is no proto-member of any proto-member of U by (1.3.5). Moreover, by reflexivity of proto-inclusion and identity, $Z \subseteq Z$ and $Z \in U$, so X is a subobject of a proto-member T of U, because X is a constituent of every object, especially a consituent of Z. Accordingly, T is identical with the null object Z. Hence, by (1.3.5), T is a null object, so X is also a null object by (1.2.2).

(**1.8.7**) $X\kappa Y \wedge \emptyset(Y) \cdot \rightarrow \emptyset(X)$. (Any constituent of a null object is also a null object.)

Proof. By (1.8.6) and transitivity of κ.

(**1.8.8**) $\nu(X) \equiv \forall Y \cdot X\alpha Y$. (Any object is a proto-number if and only if it is an ancestor of every object.)

Proof. To show that any proto-number X is an ancestor of every object, take a null object Z by (1.4.2). Since the proto-number X is a satellite of Z, X is an ancestor of a constituent T of Z by (1.7.5). However, T is a null object by (1.8.7), so X is an ancestor of every object by (1.5.15) and transitivity of α.

Conversely, if X is an ancestor of every object, then X is, by (1.5.10), a satellite of every object. Hence X is a proto-number.

(**1.8.9**) $\nu(X) \equiv \cdot \emptyset(X) \vee \exists Y(X\{Y\} \wedge \nu(Y))$. (Any object is a proto-number if and only if it is either a null object or a unit object of a proto-number.)

Proof. To show that any proto-number X is either a null object or a unit

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object of a proto-number Y, take a null object Z by (1.4.2). Then, by (1.8.8), $X\alpha Z$. Consequently, by (1.7.3), X is either a subobject of Z or a unit object of an ancestor Y of Z. In the case $X \subseteq Z$, X is a null object by (1.2.2), and in the case $X\{Y\} \wedge Y\alpha Z$, Y is a proto-number by (1.5.15), (1.8.8), and transitivity of α.

Conversely, any null object as well as any unit object of a proto-number is a proto-number by (1.8.2) and (1.8.4).

(1.8.10) $\forall X(\mathfrak{O}(X) \to X \in P) \wedge \forall YZ(Y\{Z\} \wedge Z \in P \cdot \to Y \in P) : \to \cdot \nu(U) \to U \in P.$
(A proposition corresponding to the complete induction of the kernel $X \in P$.)

Proof. Assume $\forall X(\mathfrak{O}(X) \to X \in P)$ and $\forall YZ(Y\{Z\} \wedge Z \in P \cdot \to Y \in P)$. By (1.3.1), we can take an object Q satisfying $Q\{X;\ \forall T(T \subseteq X \to T \in P)\}$, because $\forall T(T \subseteq X \to T \in P)$ implies $X \in P$ by reflexivity of proto-inclusion.

Evidently $Q \subseteq P$ by reflexivity of proto-inclusion.

We prove now $\forall X(\mathfrak{O}(X) \to X \in Q)$: Take any subobject T of any null object X. Then, by (1.2.2), T is also a null object. So, $T \in P$. Hence $X \in Q$.

Next, we prove $\forall YZ(Y\{Z\} \wedge Z \in Q \cdot \to Y \in Q)$: Take any unit object Y of a proto-member Z of Q. Then, by (1.2.2) holds $Z \in P$. Take now any subobject T of Y. Then, by (1.4.10), either $\mathfrak{O}(T)$ or $T\{Z\}$ hold. Anyway $T \in P$ holds by assumption. Hence $Y \in Q$.

Thirdly, we prove $\forall T(T \subseteq \in Q \to T \in Q)$: Take any subobject T of a proto-member X of Q, and take any subobject S of T. Then, $S \subseteq X$ by transitivity of proto-inclusion, so $S \in P$. Consequently $T \in Q$.

By (1.6.6), $U\alpha V$ implies $V \in Q \to U \in Q$, since $\forall XY(X\{Y\} \vee X \subseteq Y \cdot \to \cdot Y \in Q \to X \in Q)$ holds as shown above. Taking V as a null object, which is surely possible by (1.4.2), we know that any ancestor of a null object is a proto-member of Q, since every null object is a proto-member of Q. Because every proto-number is an ancestor of V by (1.8.8), it is a proto-member of Q. Accordingly, it is a proto-member of P by (1.2.2).

(1.8.11) $\forall X(\mathfrak{O}(X) \to \mathfrak{A}(X)) \wedge \forall YZ(Y\{Z\} \wedge \mathfrak{A}(Z) \cdot \to \mathfrak{A}(Y)) : \to \cdot \nu(U) \to \mathfrak{A}(U).$
(A proposition corresponding to the complete induction.)

Proof. Take an object P satisfying $P\{T;\ T\sigma M \wedge \mathfrak{A}(T)\}$. Then, by (1.4.8), (1.4.9), and transitivity of σ, every null object as well as every unit object of any proto-member of P is also a proto-member of P. Accordingly, by (1.8.10),

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every proto-number U satisfies the condition $\mathfrak{A}(U)$ together with $U\sigma M$.

(1.8.12) *Remark.* We can show $\mathit{0}(X) \to \neg X\{Y\}$ and $X\{Y\} \wedge U\{V\} \wedge X \equiv U \cdot \to Y \equiv V$ by (1.3.5) and reflexivity of identity. Accordingly, if we call every null object "zero" and every unit object of X "a number next to X", the system of the above two propositions, (1.8.2), the last formula of (1.8.4), and (1.8.11) can be regarded as a system of the Peano axioms. However, we can not develop the full theory of natural numbers by this interpretation, because we can prove neither that all the null objects are mutually identical nor that all the unit objects of the same proto-number are mutually identical.

(2) Regularity

The main purpose of this Chapter is to introduce a new notion of "regular" objects and thereafter to prove some fundamental properties of regular obiects.

(2.1) Sub-constituents.

(2.1.1) **Definition:** $X\zeta Y \overset{df}{\equiv} \forall S(S\in X \to S\kappa Y)$. Any object X satisfying $X\zeta Y$ is called a *sub-constituent* of Y. (Illustration: See (2.2.2).)

(2.1.2) $X\kappa Y \to X\zeta Y$, especially $X \in Y \to X\zeta Y$ and $X \subseteq Y \to X\zeta Y$. (Any constituent of an object is a sub-constituent of the object. Especially, any proto-member as well as any subobject of an object is a sub-constituent of the object.)

Proof. Any proto-member S of any constituent X of Y is a constituent of Y by (1.5.4) and transitivity of κ. The other two formulas follow immediately from the first formula, (1.5.4), and (1.5.5).

(2.1.3) $X\zeta Y \equiv X\zeta\kappa Y$. ($\zeta$-relation is equivalent to the relation product $\zeta\kappa$.)

Proof. Let X be a sub-constituent of a constituent U of Y. Then, any proto-member S of X is a constituent of U. So, by transitivity of κ, S is also a constituent of Y. Conversely, let X be a sub-constituent of Y. Then, X is a sub-constituent of the constituent Y of Y by reflexivity of κ.

(2.2) Regularity and semi-regularity.

(2.2.1) **Definition:** $\mu(P) \overset{df}{\equiv} \neg\exists S(S\kappa\in S \wedge S\kappa P)$ and

$$\rho(P) \overset{df}{\equiv} \forall Q(Q\zeta P \wedge \forall X\exists Y(X\in Q \to \cdot Y\kappa\in X \wedge Y\in Q)\cdot \to \emptyset(Q)).$$

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Any object P satisfying $\mu(P)$ is called *semi-regular* and any object P satisfying $\rho(P)$ is called *regular*.

(2.2.2) Illustration. Semi-regularity is so defined that any semi-regular object has no constituent S satisfying $S\kappa\in S$, which means $S\subseteq X_1\in\cdots\in X_n\in S$ for some sequence $X_1,\cdots,X_n$ $(n=1,2,\cdots.)$. Regularity can be comprehended in connection with the proposition corresponding to the fundierung axiom $\forall x\exists y(x\in q\rightarrow\cdot y\in x\wedge y\in q)\rightarrow q=\emptyset$. The condition $\forall X\exists Y(X\in Q\rightarrow\cdot Y\kappa\in X\wedge Y\in Q)\rightarrow\emptyset(Q)$ on Q is a modification of the fundierung axiom regarding it as a condition on q. To define $\rho(P)$ so that our object theory can be developed exclusively in the field of regular objects, we had to replace $y\in x$ of the axiom by $Y\kappa\in X$ instead of $Y\in X$ and to require that the modified condition holds for every object Q in the curious range of sub-constituents of P.

(2.2.3) $\emptyset(P)\rightarrow\rho(P)$. (Any null object is regular.)

Proof. Let P be a null object and Q be any sub-constituent of P. Then, any proto-member X of Q is a constituent of the null object P, so by (1.8.7), X is a null object. Consequently, X can not satisfy $Y\kappa\in X\wedge Y\in Q$ for any Y.

(2.2.4) $\rho(P)\rightarrow\mu(P)$. (Reguralty implies semi-regularity.)

Proof. Assume that any object P is not semi-regular. Then, there is a constituent S of P satisfying $S\kappa\in S$.

Take now by (1.4.4) a unit object Q of S. Firstly, holds $Q\zeta P$; for, any proto-member T of Q is identical with S, so it is a constituent of P by (1.3.5). Secondly, $\forall X\exists Y(X\in Q\rightarrow\cdot Y\kappa\in X\wedge Y\in Q)$ holds by (1.3.5) and reflexivity of identity, because $S\kappa\in S\wedge S\in Q$ holds for the only proto-member S of Q. Thirdly, Q is not a null object by reflexivity of identity. Hence P can not be regular.

(2.2.5) $Q\kappa P\wedge\rho(P)\cdot\rightarrow\rho(Q)$, especially $Q\in P\wedge\rho(P)\cdot\rightarrow\rho(Q)$ and $Q\subseteq P\wedge\rho(P)\cdot\rightarrow\rho(Q)$. (Any constituent of a regular object is regular, especially, any proto-member as well as any subobject of a regular object is regular.)

Proof. Let Q be a constituent of a regular object P. Any sub-constituent R of Q, for which $\forall X\exists Y(X\in R\rightarrow\cdot Y\kappa\in X\wedge Y\in R)$ holds, is a null object, because R is also a sub-constituent of P by (2.1.3).

Other two formulas can be derived from this formula, (1.5.4), and (1.5.5).

(2.2.6) $Q\{P\} \wedge \rho(P) \cdot \rightarrow \rho(Q)$. (Every unit object of a regular object is also regular.)

Proof. Let Q be a unit object of a regular object P, and R be any subconstituent of Q, which satisfies $\forall X \exists Y (X \in R \rightarrow \cdot Y \kappa \in X \wedge Y \in R)$. If R is not a null object, take a proto-member U of R. Then, by assumption, there exists a proto-member V of R, which is also a constituent of a proto-member T of U.

At first, we prove $V \kappa P$: Since $R \zeta Q$, the proto-member U of R is a constituent of Q, so U is either a subobject of Q or a constituent of a proto-member W of Q by (1.7.1). In the case $U \subseteq Q$, the proto-member T of U is also a proto-member of the unit object Q of P by (1.2.2), so $T \equiv P$. Accordingly, the constituent V of T is also a constituent of P by (1.3.5). In the case $U \kappa W \wedge W \in Q$, holds $W \equiv P$ since Q is a unit object of P. So, $U \kappa P$ by (1.3.5). Accordingly, the constituent V of T is a constituent of P by transitivity of κ, because the proto-member T of the constituent U of P is a constituent of P by (1.5.4) and transitivity of κ.

Take now an object H which satisfies $H\{S;\ S \in R \wedge S \kappa V\}$ by (1.3.1). Firstly, $H \zeta P$; because any proto-member S of H is surely a constituent of the constituent V of P, so S is a constituent of P by transitivity of κ. Secondly, $\forall X \exists Y (X \in H \rightarrow \cdot Y \kappa \in X \wedge Y \in H)$. For: Since H is a subobject of R, for any proto-member X of H there is a proto-member Y of R satisfying $Y \kappa \in X$. Because X is a constituent of V, Y is also a constituent of V by (1.5.4) and transitivity of κ. Hence, Y is also a proto-member of H. Thirdly, H is not a null object, because V is surely a proto-member of H by reflexivity of κ.

Accordingly, P can never be regular, if R is not a null object.

(2.2.7) $Q \sigma P \wedge \rho(P) \cdot \rightarrow \rho(Q)$. (Every satellite of a regular object is also regular.)

Proof. By (1.6.4), (2.2.5), and (2.2.6).

(2.3) Objects formed by satellites of an object.

(2.3.1) Definition: $X \theta Y \overset{df}{\equiv} X\{T;\ T \sigma Y\}$. Any object whose proto-members are all the satellites of an object is called an *object formed by satellites of the object*.

(2.3.2) $Q \theta P \wedge \rho(P) \cdot \rightarrow \rho(Q)$. (Any object formed by satellites of a regular object is also regular.)

Proof. Let Q be any object formed by satellites of a regular object P. To show that Q is also regular, take any sub-constituent R of Q satisfying $\forall X \exists Y(X \in R \rightarrow \cdot Y \kappa \in X \wedge Y \in R)$. In the following, we shall show that R is a null object.

Namely, if R is not a null object, take any proto-member U of R. Then, U is a constituent of Q; so, by (1.7.1), U is either a subobject of Q or a constituent of a proto-member S of Q. On the other hand, we can take such a proto-member V of R which is a constituent of a proto-member Z of U. In the case $U \subseteq Q$, Z is a proto-member of Q by (1.2.2), so Z is a satellite of P. Since V is a satellite of Z by (1.5.3), V is a satellite of P by transitivity of σ. In the case $U \kappa S \wedge S \in Q$, S is a satellite of P. By (1.2.11), (1.5.3), and transitivity of σ, V is a satellite of P. Thus V is a satellite of the regular object P anyway, so V is also regular by (2.2.7).

Take now an object W satisfying $W\{T;\ T \in R \wedge T \kappa V\}$ by (1.3.1). Then, firstly $W \subsetneq V$, because every proto-member of W is a constituent of V. Secondly, $\forall X \exists Y(X \in W \rightarrow \cdot Y \kappa \in X \wedge Y \in W)$. To show this, take any proto-member X of W. Then, X is a proto-member of R, so we can find such a proto-member Y of R, which is a constituent of a proto-member of X. Since the proto-member X of W is a constituent of V, so Y is also a constituent of V by (1.5.4) and transitivity of κ. Consequently, the proto-member Y of R is a proto-member of W. Thirdly, W is not a null object, because V is surely a proto-member of W by reflexivity of κ.

Hence V can not be regular if R is not a null object.

(2.3.3) $\neg(P\theta Q \wedge P\sigma Q)$. (Any object formed by satellites of an object can never be a satellite of the object.)

Proof. Let P be an object formed by satellites of Q. By the axiom scheme, take an object R satisfying $R\{X;\ X\sigma Q \wedge \neg X \in X\}$. Then, R can not be a satellite of Q, because $R \in R \equiv \neg R \in R$ can not hold. On the other hand, R is clearly a subobject of P, so R is a satellite of P by (1.2.12). Accordingly, if P were a satellite of Q, so R would be a satellite of Q by transitivity of σ contradictory to the fact above stated.

(2.3.4) $P\{T;\ \nu(T)\} \rightarrow \rho(P)$. (Any object formed by all the proto-numbers is regular.)

Proof. Let P be any object formed by all the proto-numbes. Further, take a null object X by (1.4.2). By (1.8.2) and (1.8.3), P is an object formed by satellites of X, so P must be regular by (2.3.2), since the null object X is regular by (2.2.3).

(2.3.5) $X\theta\sigma Z \wedge Y\theta Z \cdot \rightarrow X \subseteq Y$. (Any object formed by satellites of a satellite of an object Z is a subobject of any object formed by satellites of the object Z.)

Proof. Let X be any object formed by all the satellites of a satellite U of an object Z, and Y be any object formed by satellites of the object Z. Take now any proto-member S of X. Then, S is a satellite of the satellite U of Z; so, by transitivity of σ, holds $S\sigma Z$. Hence $S \in Y$.

(3) Basic objects and producible objects

It is not certain in our object theory that for any pair of objects X and Y there exists a "pair object" of X and Y i.e. an object formed by X and Y, even when X and Y are both regular. However, we can get rid of this difficulty by restricting our object field to "producible" objects, whose notion will be introduced in this Chapter.

(3.1) Basic objects.

(3.1.1) Definition:

$$\beta(P) \overset{df}{\equiv} : \rho(P) \wedge \forall X(X\theta \in P \rightarrow \cdot X \in P \vee P\kappa X) \wedge \forall X(X \in P \rightarrow \cdot \nu(X) \vee X\theta \in P).$$

Any object satisfying $\beta(P)$ is called a ***basic object.***

Illustration. Formal definition of basic objects is really complicated. The condition for P to be a basic object is intended to describe essential property of any object which is formed by some proto-numbers and $X_1, \cdots, X_n$ ($n=0, 1, 2, \cdots$.) in an infinite sequence $X_1, X_2, \cdots$ satisfying the conditions $\nu(X_1)$ and $X_{i+1}\theta X_i$ ($i=1, 2, \cdots$.). Notice that any object P of this kind is a subobject, so also a constituent, of X_{n+1}. It is quite uncertain in our object theory whether there is a regular object formed by all the objects X_i ($i=1, 2, \cdots$.). However, if there is such an object, it, too, is basic by our definition. Accordingly, our definition of basic objects can not characterize the intended objects above described.

(3.1.2) $P\{T;\ \nu(T)\} \rightarrow \beta(P)$. (Any object formed by all the proto-numbers is basic.)

Proof. Assume $P\{T;\ \nu(T)\}$. Then, P is regular by (2.3.4). Take now any object X formed by satellites of a proto-member Y of P i.e. any object X formed by satellites of a proto-number. Then, by (1.8.3), X is an object formed by all the proto-numbers. So, $P\kappa X$ by (1.5.5) because $P \subseteq X$. Moreover, any proto-member X of P, i.e. any proto-number X, satisfies evidently $\nu(X) \vee X\theta \in P$. Hence P is basic.

(3.1.3) $\beta(P) \rightarrow \cdot\ \emptyset(P) \vee \exists X(\nu(X) \wedge X \in P)$. (Any non-empty basic object contains a proto-number as its proto-member.)

Proof. If a basic object P contains no proto-numers as its proto-members, then by defintion of basic objects any proto-member X of P is an object formed by satellites of a proto-member Y of P. Accordingly, for any proto-member X of P, there is a proto-member Y of X which is also a proto-member of P, since any object formed by satellites of Y contains Y as its proto-member by reflexivity of σ. Hence, by reflexivity of κ, P can not be regular, if P is not a null object; because $P\zeta P$ by (2.1.2) and reflexivity of κ.

(3.1.4) $\exists Q \cdot Q\{T;\ \nu(T) \vee T\theta \in P\}$ and $\beta(P) \wedge Q\{T;\ \nu(T) \vee T\theta \in P\} \cdot \rightarrow \beta(Q)$. (For any object P, there is an object Q formed by all the proto-numbers and all the objects T satisfying $T\theta \in P$. The object Q is basic if P is so.)

Proof. Take an object R formed by satellites of a given object P by (1.3.2), and take further an object Q satisfying $Q\{T;\ T\sigma R \wedge (\nu(T) \vee T\theta \in P)\}$. By definition of proto-numbers, $\nu(T)$ implies $T\sigma R$. Moreover, any object T formed by satellites of a proto-member Y of P is a satellite of R. For, Y is a satellite of P by (1.2.11), so T is a subobject of R by (2.3.5); consequently, T is also a satellite of R by (1.2.12). Hence, Q satisfies also $Q\{T;\ \nu(T) \vee T\theta \in P\}$.

Let us now discuss the case $\beta(P)$.

At first, we prove $\rho(Q)$: Namely, take an object H formed by satellites of R by (1.3.2). Any proto-member T of Q is a satellite of R, so $T \in H$. Accordingly, $Q \subseteq H$. On the other hand, the object R formed by satellites of the regular object P is regular by (2.3.2), so also the object H formed by satellites of the regular object R is regular, again by (2.3.2). Consequently, the sub-

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object Q of the regular object H is also regular by (2.2.5).

Secondly, we prove $\forall X(X\theta\in Q\rightarrow\cdot X\in Q\vee Q\kappa X)$: Take any object X formed by satellites of a proto-member Y of Q. Then, Y is either a proto-number or an object formed by satellites of a proto-member of P. At first, the case $\nu(Y)$: By (1.8.3), X is an object formed by all the proto-numbers. If P is a null object, then Q is an object formed by all the proto-numbers. So, $Q\kappa X$ by (1.5.5). If P is not a null object, then by (3.1.3) P contains a proto-number, say Z, as a proto-member. So, any object formed by satellites of Z i.e., by (1.8.3), any object formed by all the proto-numbers is a proto-member of Q, especially the object X is a proto-member of Q. Next, the case $Y\theta\in P$: In this case either $Y\in P$ or $P\kappa Y$ hold, because P is basic. In the case $Y\in P$, X is an object formed by satellites of the proto-member Y of P, so $X\in Q$. Accordingly, we have further only to discuss the case $P\kappa Y$.

Now we prove $Q\kappa X$ when Y is an object formed by satellites of a proto-member W of the object P which is a constituent of Y: Take any proto-member S of Q. Then, we can show $S\in X$. Namely, S is either a proto-number or an object formed by satellites of a proto-member of P. If S is a proto-number, S is surely a satellite of Y, so $S\in X$. Accordingly, we discuss the case where S is an object formed by satellites of a proto-member V of P. By (1.7.1), the constituent P of Y is either a subobject of Y or a constituent of a proto-member of Y. In the case $P\subseteq Y$, the proto-member V of P is a proto-member of Y by (1.2.2), so $V\sigma W$. Since any proto-member U of S is a satellite of V, U is also a satellite of W by transitivity of σ, so $U\in Y$. Consequently $S\subseteq Y$, so $S\sigma Y$ by (1.2.12). Accordingly $S\in X$. On the other hand, when P is a constituent of a proto-member M of Y, M is a satellite of W. Since any proto-member U of S is a satellite of the proto-member V of the object P which is a constituent of the satellite M of W, U is a satellite of W by (1.2.11), (1.5.3), and transitivity of σ. So $U\in Y$. Consequently $S\subseteq Y$, so $S\sigma Y$ by (1.2.12). Accordingly $S\in X$. Thus in any way we can prove that any proto-member S of Q is a proto-member of X. Accordingly $Q\subseteq X$, so $Q\kappa X$ by (1.5.5).

Hence, any object X formed by satellites of a proto-member of Q is a proto-member of Q unless Q is a constituent of X.

Thirdly, we prove $\forall X(X\in Q\rightarrow\cdot\nu(X)\vee X\theta\in Q)$: Namely, take any proto-member X of Q. Then, X is an object formed by satellites of a proto-member

Y of P, unless X is a proto-number. Since P is basic, Y is either a proto-number or an object formed by satellites of a proto-member of P. Anyway Y is a proto-member of Q, so X is an object formed by satellites of the proto-member Y of Q, unless X is a proto-number.

Hence Q is basic.

(**3.1.5**) $\beta(P) \to \exists Q(\beta(Q) \wedge P \subseteq Q \wedge \forall T(\nu(T) \to T \in Q))$. (For any basic object P, there is a basic object which includes P as a subobject and which contains every proto-number as a proto-member.)

Proof. Let P be any basic object. Then, we can take a basic object Q satisfying $Q\{T;\ \nu(T) \vee T\theta \in P\}$ by (3.1.4).

Firstly $P \subseteq Q$: To prove this, take any proto-member T of P. Then, T satisfies $\nu(T) \vee T\theta \in P$, because P is basic. So, $T \in Q$. Hence $P \subseteq Q$. Secondly, it is evident that every proto-number is a proto-member of Q. Consequently, Q is a basic object which includes P as a subobject and which contains every proto-number as a proto-member.

(**3.1.6**) $\beta(P) \wedge \beta(Q) \cdot \to \exists R(\beta(R) \wedge P\kappa R \wedge Q\kappa R)$. (For any pair of basic objects P and Q, there is such a basic object R that P as well as Q is a constituent of R.)

Proof. Let P and Q be any two basic objects. By (3.1.5) take a basic object U which includes Q as a subobject and which contains every proto-number as a proto-member.

We assert that either $P \subseteq U$ or $U\kappa P$ hold: Namely, if there is a proto-member Y of P which is not a proto-member of U but which is an object formed by satellites of a proto-member of U, then $U\kappa Y$ because U is basic. So, by (1.5.4) and transitivity of κ, U is a constituent of P. On the other hand, if there is no such proto-member of P, then take an object V satisfying $V\{T;\ T \in P \wedge \neg T \in U\}$ by (1.3.1). Clearly $V \varsubsetneq P$ by (2.1.2). Take now any proto-member X of V. Then, $X \in P$, and X can never be a proto-number, because every proto-number is a proto-member of U. Consequently, the proto-member X of the basic object P is an object formed by satellites of a proto-member Z of P. $Z \in X$ by reflexivity of σ. Moreover $\neg Z \in U$; for, the proto-member X of P is not a proto-member of U, so the object X formed by satellites of Z can never be an object formed by satellites of any proto-member of U by

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assumption. Thus we know $Z \in V$. Accordingly, for any proto-member X of V, there exists a common proto-member, Z for instance, of X and V. However, by reflexivity of κ, this can not be true for the basic object P, unless the sub-constituent V of P is a null object; since the basic object P is regular. So, $P \subseteqq U$.

When $P \subseteqq U$, P as well as Q is a constituent of the basic object U by (1.5.5), and when $U\kappa P$, P as well as Q is a constituent of the basic object P by (1.5.5) and reflexivity and transitivity of κ.

(3.1.7) $\beta(P) \wedge X \in P \wedge Y \in P \cdot \rightarrow \cdot X\sigma Y \vee Y\sigma X$. (Among any two proto-members of a basic object, one is a satellite of the other.)

Proof. Let P be any basic object. By (1.3.1), take an object Q satisfying $Q\{T;\ T \in P \wedge \neg \forall U(U \in P \rightarrow \cdot T\sigma U \vee U\sigma T)\}$. Then, $Q\zeta P$ by (2.1.2). Moreover, any proto-number S can never be a proto-member of Q. For, the proto-number S surely satisfies $\forall U(U \in P \rightarrow \cdot S\sigma U \vee U\sigma S)$, since any proto-number is a satellite of every object. Accordingly, any proto-member Z of Q is not a proto-number. Hence, the proto-member Z of the basic object P is an object formed by satellites of a proto-member W of P.

Now we assert $W \in Q$: To prove this, we have only to show $\neg \forall U(U \in P \rightarrow \cdot W\sigma U \vee U\sigma W)$. Namely, if $\forall U(U \in P \rightarrow \cdot W\sigma U \vee U\sigma W)$ holds, we assert that any proto-member U of P satisfies $Z\sigma U \vee U\sigma Z$ contradictory to the assumption $Z \in Q$. For, the proto-member U of the basic object P is either a proto-number or an object formed by satellites of a proto-member V of P. In the case $\nu(U)$, U is a satellite of any object, especially of Z. In the case $U\theta V \wedge V \in P$, either $W\sigma V$ or $V\sigma W$ hold by assumption. If $W\sigma V$, then hold $U\theta V$ and $Z\theta\sigma V$; consequently, $Z \subseteqq U$ by (2.3.5), so $Z\sigma U$ by (1.2.12). If $V\sigma W$, then hold $Z\theta W$ and $U\theta\sigma W$; consequently $U \subseteqq Z$ by (2.3.5), so, $U\sigma Z$ by (1.2.12).

The object W is a proto-member of Z by reflexivity of σ.

Hence, for any proto-member Z of Q, there is a proto-member W of Q which is also a proto-member of Z. Since P is a basic object, P is regular. So, the sub-constituent Q of P is a null object by reflexivity of κ. Namely, any proto-member X of P satisfies $X\sigma Y \vee Y\sigma X$ for any proto-member Y of P.

(3.2) Producible objects.

(3.2.1) Definition: $\pi(X) \overset{df}{\equiv} \exists P(\beta(P) \wedge X\sigma \in P)$. Any object X satisfying $\pi(X)$

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is called a *producible object.*

Illustration. In defining producible objects, we have intended to take up only such objects whose existence can be affirmed by our axioms. As such objects, we adopt those objects which are satellites of any terms X_k in an infinite sequence of objects X_1, X_2, $\cdots$ satisfying the conditions: $\nu(X_1)$ and $X_{i+1}\theta X_i$ $(i=1, 2, \cdots.)$. However, it is quite uncertain whether there is an object formed by all the terms of any sequence of this kind. So, we had instead to introduce the notion of basic objects, any one of which is supposed to be formed by first finite terms of any sequence of this kind. Although our definition of basic objects can not characterize the said kind of objects as is pointed out in (3.1.1), the definition works well for our purpose of introducing producible objects.

(3.2.2) $\nu(X)\to\pi(X)$. (Any proto-number is producible.)

Proof. By (1.8.5) and (3.1.2), there is a basic object formed by all the proto-numbers, so any proto-number is producible by reflexivity of σ.

(3.2.3) $X\sigma Y\wedge\pi(Y)\cdot\to\pi(X)$; especially $X\in Y\wedge\pi(Y)\cdot\to\pi(X)$, $X\subseteq Y\wedge\pi(Y)\cdot\to\pi(X)$, and $X\{Y\}\wedge\pi(Y)\cdot\to\pi(X)$. (Any satellite of a producible object is also producible. Especially, any proto-member, any subobject, and any unit object of any producible object are also producible.)

Proof. By (1.2.11), (1.2.12), (1.4.9), and transitivity of σ.

(3.2.4) $X\theta Y\wedge\pi(Y)\cdot\to\pi(X)$. (Any object formed by satellites of a producible object is also producible.)

Proof. Let X be any object formed by satellites of a producible object Y. Then, Y is a satellite of a proto-member U of a basic object P. By (3.1.4), take a basic object Q satisfying $Q\{T;\ \nu(T)\vee T\theta\in P\}$. Now, take an object V formed by satellites of U by (1.3.2); then, V is a proto-member of Q. Since $X\theta\sigma U$ and $V\theta U$ hold, $X\subseteq V$ by (2.3.5); so, $X\sigma V$ by (1.2.12). Hence, X satisfies $X\sigma\in Q$ for the basic object Q, so X is producible.

(3.2.5) $X\{T;\ T\subseteq Y\}\wedge\pi(Y)\cdot\to\pi(X)$ and $X\{T;T\in\in Y\}\wedge\pi(Y)\cdot\to\pi(X)$. (Any power object as well as any sum object of a producible object is also producible.)

Proof. By (1.2.11), (1.2.12), (1.3.2), (3.2.3), (3.2.4), and transitivity of σ,

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(**3.2.6**) $X\{T;\ T\simeq\in Y\}\wedge\pi(Y)\cdot\rightarrow\pi(X)$.

Proof. By (1.2.6), (1.2.11), (1.2.12), (1.3.2), (3.2.3), (3.2.4) and transitivity of σ.

(**3.2.7**) $\pi(X)\rightarrow\rho(X)$. (Any producible object is regular.)

Proof. Let X be any producible object. Then, X is a satellite of a proto-member of a basic object P. By (1.2.11) and transitivity of σ, X is a satellite of the basic object P, which is naturally regular, so X is also regular by (2.2.7).

(**3.2.8**) $\pi(X)\wedge\pi(Y)\cdot\rightarrow\exists Z(\pi(Z)\wedge X\sigma Z\wedge Y\sigma Z)$. (For any two producible objects, there is a producible object which has these objects as its satellites.)

Proof. Let X and Y be any two producible objects. Then, there are such basic objects P and Q that X is a satellite of a proto-member W of P and Y is a satellite of a proto-member of Q. By (3.1.6), there is such a basic object R that P as well as Q is a constituent of R. Then, by (1.7.1), P is either a subobject of R, or a constituent of a proto-member T of R. In the case $P\subseteq R$, X is a satellite of the proto-member W of the basic object R by (1.2.2), and in the case $P\kappa T\wedge T\in R$, X is a satellite of the proto-member T of the basic object R by (1.2.11), (1.5.3), and transitivity of σ. Anyway, X is a satellite of a proto-member U of the basic object R. Similarly, we can prove that Y is also a satellite of a proto-member V of R.

According to (3.1.7), one of the two proto-members U and V of the basic object R is a satellite of the other, say Z, so by reflexivity and transitivity of σ, X and Y are both satellites of the same proto-member Z of the basic object R.

The object Z is producible by reflexivity of σ.

(**3.2.9**) **Definition:** $U\{X, Y\}\overset{df}{\equiv}U\{T;\ T\equiv X\vee T\equiv Y\}$. Any object U satisfying $U\{X, Y\}$ is called a ***pair object*** of X and Y.

(**3.2.10**) $\pi(X)\wedge\pi(Y)\cdot\rightarrow\exists U\cdot U\{X, Y\}$ and $\pi(X)\wedge\pi(Y)\wedge U\{X, Y\}\cdot\rightarrow\pi(U)$. (For any two producible objects, there is a pair object of them; and any pair object of any two producible objects is also producible.)

Proof. Let X and Y be any two producible objects. Then, by (3.2.8), there is a producible object W for which $X\sigma W$ and $Y\sigma W$ hold. Accordingly, if we take an object U satisfying $U\{T;\ T\sigma W\wedge(T\equiv X\vee T\equiv Y)\}$, then by (1.3.5), it satisfies also $U\{T;\ T\equiv X\vee T\equiv Y\}$ i.e. $U\{X, Y\}$.

Now, let U be any object satisfying $U\{X, Y\}$ for two producible objects X and Y, and by (1.3.2), take an object H formed by satellites of the producible object W. Then, by (1.3.5), U is a subobject of H. Since H is producible by (3.2.4), U itself is producible by (3.2.3).

(3.2.11) $\pi(X) \wedge \pi(Y) \cdot \rightarrow \exists Z \cdot Z\{T;\ T \epsilon X \vee T \epsilon Y\}$ and
$\pi(X) \wedge \pi(Y) \wedge Z\{T;\ T \epsilon X \vee T \epsilon Y\} \cdot \rightarrow \pi(Z)$.

(For any two producible objects, there is a union object of them; and any union object of any two producible objects is also producible. Here we call any object Z satisfying $Z\{T;\ T \epsilon X \vee T \epsilon Y\}$ a *union object* of X and Y.)

Proof. By (1.3.5), (1.4.5), (3.2.5), (3.2.10), and reflexivity of identity.

(4) Membership and Equality

In our theory of objects, we do not assume the extensionality axiom of the set theory for our objects with respect to proto-membership and identity, nor we can not expect the second equality axiom holds with respect to proto-membership and proto-equality for our objects. However, even if it is impossible to prove the axioms with respect to proto-membership and identity, or with respect to proto-membership and proto-equality, for our objects in general, it may be still possible to prove them with respect to other suitably defined notions, "membership" and "equality", in a suitably defined range of objects, the range of producible objects. In the following, we shall show that this is the case.

(4.1) Before going into details, we describe here shortly our plan. By modifying proto-membership, we introduce a new notion "membership" (notation: "$\in$"), and by modifying the notions, identity and proto-equality, we introduce a new unified notion "equality" (notaion: "$=$"), in such a way that the axiom of extensionality together with the equality axioms holds with respect to them. However, we try to minimize the modification as far as these axioms hold. Perhaps, "membership" and "equality" should be weaker than proto-membership and identity respectively, but we try to keep the mutual relation $X \in P \equiv X = \epsilon P$, a modification of the first formula of (1.2.8). As far as this relation should be kept, we have only to define "equality".

It would be very easy to define "equality", if an object formed by all the

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pair objects of X and Y satisfying $X=Y$ could be considered. However, it is impossible to consider such an object, if $X=X$ should hold for every X. For, such an object must contain all the unit objects as its proto-members, so all the objects must be its satellites.

An approach to define "equality" is to consider objects formed by only those pair objects of X and Y satisfying $X=Y$. If "$\varepsilon(P)$" is a condition for that P is an object of this kind, then every pair-object proto-member of P satisfying "$\varepsilon(P)$" must be a pair object of X and Y satisfying $X=Y$ and also satisfying $\forall S(S\in X \equiv S\in Y)$. We try to express the condition by introducing new relations "$X \underset{P}{=} Y$" and "$X \underset{P}{\cong} Y$", which are defined by $X \equiv Y \vee \exists U(U\{X, Y\} \wedge U \in P)$ and $\forall Z(Z \underset{P}{=} \in X \equiv Z \underset{P}{=} \in Y)$ respectively. (See (4.2.1), (4.3.1), and (4.5.1).) Then, "$\varepsilon(P)$" is defined for trial as $\pi(P) \wedge \forall XY(X \underset{P}{=} Y \rightarrow X \underset{P}{\cong} Y)$. (See (4.6.1).). In the following, we show that this trial definition works well for our purpose.

(4.2) *P*-equality.

(4.2.1) Definition: $X \underset{P}{=} Y \overset{df}{\equiv} \cdot X \equiv Y \vee \exists U(U\{X, Y\} \wedge U \in P)$. Any two objects X and Y satisfying $X \underset{P}{=} Y$ are called ***P*-equal** to each other. P-equality can be taken as a binary relation regarding P as a parameter.

Remark. We define P-equality to introduce a way of weakening the notion of identity with respect to an object P. Only for a special kind of objects P, P-equality can be considered as closely related to "equality".

(4.2.2) $X \equiv Y \rightarrow X \underset{P}{=} Y$. (Identity implies P-equality.)

(4.2.3) $P \subseteq Q \rightarrow (X \underset{P}{=} Y \rightarrow X \underset{Q}{=} Y)$. (If P is a subobject of Q, P-equality implies Q-equality.)

Proof. By (1.2.2) and (4.2.2).

(4.2.4) $\emptyset(P) \wedge X \underset{P}{=} Y \cdot \rightarrow X \equiv Y$. (For any null object P, P-equality implies identity.)

(4.2.5) $X \underset{P}{=} X$. (Reflexivity of P-equality.)

Proof. By reflexivity of identity.

(4.2.6) $X \underset{P}{=} Y \rightarrow Y \underset{P}{=} X$. (Symmetricity of P-equality.)

Proof. By symmetricity of identity.

(4.2.7) $P\{T;\ T \simeq \in Q\} \to \cdot X \underset{P}{=} Y \equiv X \underset{Q}{=} Y$.

Proof. Let P be an object satisfying $P\{T;\ T \simeq \in Q\}$. Then, $Q \subseteq P$ by reflexivity of proto-equality, so Q-equality implies P-equality by (4.2.3).

To show that also P-equality implies Q-equality, take any two objects X and Y which are P-equal to each other. Then, either $X \equiv Y$ or a proto-member U of P is a pair object of X and Y. In the case $X \equiv Y$, holds $X \underset{Q}{=} Y$ by (4.2.2). Also in the case $U \in P \wedge U\{X,\ Y\}$, the proto-member U of P is proto-equal to a proto-member V of Q, so V is also a pair object of X and Y. Hence $X \underset{Q}{=} Y$.

(4.2.8) $\exists Q(\forall XY(X \underset{P}{=} Y \equiv X \underset{Q}{=} Y) \wedge Q\{T;\ T \simeq \in Q\})$. (For any object P, there is such an object Q that Q-equality is equivalent to P-equality and $Q\{T;\ T \simeq \in Q\}$ holds.)

Proof. For any object P, take an object Q satisfying $Q\{T;\ T \simeq \in P\}$ by (1.3.3). Then, P-equality is equivalent to Q-equality by (4.2.7). Moreover, we can prove $Q\{T;\ T \simeq \in Q\}$: Namely, take any T which is proto-equal to a proto-member S of Q. Then, S is proto-equal to a proto-member R of P. By transitivity of proto-equality, T is proto-equal to the proto-member R of P, so $T \in Q$. Conversely, any proto-member U of Q is an object which is proto-equal to the proto-member U of Q by reflexivity of proto-equality. Hence $Q\{T;\ T \simeq \in Q\}$.

(4.3) *P*-membership.

(4.3.1) Definition: $X \underset{P}{\in} Y \overset{df}{\equiv} X \underset{P}{=} \in Y$. Any object X satisfying $X \underset{P}{\in} Y$ is called a *P-member* of Y.

(4.3.2) $X \in Y \to X \underset{P}{\in} Y$. (Proto-membership implies P-membership.)

Proof. By reflexivity of P-equality.

(4.3.3) $P \subseteq Q \to (X \underset{P}{\in} Y \to X \underset{Q}{\in} Y)$. (For any subobject P of an object Q, P-membership implies Q-membership.)

Proof. By (4.2.3).

(4.3.4) $P\{T;\ T \simeq \in Q\} \to \cdot X \underset{P}{\in} Y \equiv X \underset{Q}{\in} Y$.

Proof. By (4.2.7).

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(4.4) Transitive objects.

(4.4.1) Definition: $\tau(P) \overset{df}{\equiv} \cdot P\{T;\ T \simeq \in P\} \wedge \forall XY(X \underset{P}{=} \underset{P}{=} Y \rightarrow X \underset{P}{=} Y)$. Any object P satisfying $\tau(P)$ is called a ***transitive object***.

(4.4.2) $\tau(P) \wedge X \underset{P}{=} Y \cdot \rightarrow \cdot X \equiv Y \vee \forall U(U\{X, Y\} \rightarrow U \in P)$. (If two objects X and Y are P-equal for a transitive object P, then any pair object of X and Y is a proto-member of P unless $X \equiv Y$.)

Proof. Let P be a transitive object, and X and Y be P-equal but not identical. Then, there is a proto-member V of P satisfying $V\{X, Y\}$. Any pair object U of X and Y is evidently proto-equal to the proto-member V of P, so $U \in P$ because P is transitive.

(4.4.3) $\tau(F) \wedge \tau(G) \wedge P\{T;\ T \in F \wedge T \in G\} \cdot \rightarrow \tau(P)$. (Any object formed by all the common proto-members of two transitive objects is also transitive.)

Proof. Let P be any object satisfying $P\{T;\ T \in F \wedge T \in G\}$ for any two transitive objects F and G.

Firstly, we prove $P\{T;\ T \simeq \in P\}$: Namely, take any object T which is an object proto-equal to a proto-member S of P. Then, S is a proto-member of F as well as of G. Since F and G are transitive, T is a proto-member of F as well as of G, so $T \in P$. Conversely, any proto-member T of P is proto-equal to the proto-member T of P by reflexivity of proto-equality.

Secondly, we prove that $X \underset{P}{=} \underset{P}{=} Y$ implies $X \underset{P}{=} Y$: Let any object X be P-equal to an object Z and Z be P-equal to another object Y. Since P is a sub-object of F as well as of G, so by (4.2.3), X is F-equal as well as G-equal to Z, and Z is F-equal as well as G-equal to Y. Because F and G are both transitive, X is F-equal as well as G-equal to Y. Consequently, there is a pair object U of X and Y which is a proto-member of F, unless $X \equiv Y$. Since G is transitive, the pair object U of X and Y is a proto-member of G by (4.4.2), unless $X \equiv Y$. Accordingly U is a common proto-member of F and G i.e. $U \in P$, unless $X \equiv Y$. Hence $X \underset{P}{=} Y$.

(4.4.4) $\tau(P) \rightarrow (X \underset{P}{=} Y \rightarrow \cdot X \underset{P}{\in} Z \equiv Y \underset{P}{\in} Z)$. (For any transitive object P, any two P-equal objects are either both P-members or both no P-members of any object.)

Proof. Let X be P-equal to Y for a transitive object P, and X be a P-member of Z. Then, X is P-equal to a proto-member U of Z. By symmetricity of P-equality, Y is P-equal to the proto-member U of Z, since P is transitive. Hence $Y \underset{P}{\in} Z$.

Similarly, we can prove that $Y \underset{P}{\in} Z$ implies $X \underset{P}{\in} Z$.

(4.5) Equal *P*-extension.

(4.5.1) Definition: $X \underset{P}{\approxeq} Y \overset{df}{\equiv} \forall S(S \underset{P}{\in} X \equiv S \underset{P}{\in} Y)$. Any two objects X and Y satisfying $X \underset{P}{\approxeq} Y$ are called *objects of equal P-extent.*

(4.5.2) $X \equiv Y \rightarrow X \underset{P}{\approxeq} Y$. (Identity implies equal P-extension.)

Proof. By (1.3.5).

(4.5.3) $P \simeq Q \rightarrow \cdot X \underset{P}{\approxeq} Y \equiv X \underset{Q}{\approxeq} Y$. (If P and Q are proto-equal, equal P-extension is equivalent to equal Q-extension.)

Proof. Let P and Q be any two proto-equal objects, and X and Y be any two objects of equal P-extent. Take any Q-member S of X. Then, $S \underset{P}{\in} X$ by (1.2.6) and (4.3.3). Since $X \underset{P}{\approxeq} Y$, holds $S \underset{P}{\in} Y$. So, again by (1.2.6) and (4.3.3), $S \underset{Q}{\in} Y$. By the same reasoning, we can show that any Q-member of Y is a Q-member of X.

Similarly, we can prove that any two objects are of equal P-extent if they are objects of equal Q-extent.

(4.5.4) $P \subseteq Q \wedge \tau(Q) \cdot \rightarrow \cdot X \underset{P}{\approxeq} Y \rightarrow X \underset{Q}{\approxeq} Y$. (If X and Y are objects of equal P-extent for a subobject P of a transitive object Q, they are also objects of equal Q-extent.)

Proof. Let X and Y be objects of equal P-extent for a subobject P of a transitive object Q, and let S be any Q-member of X. Then, S is Q-equal to a proto-member T of X. By (4.3.2), $T \underset{P}{\in} X$, so $T \underset{P}{\in} Y$ by assumption. Accordingly, T is P-equal to a proto-member U of Y. By (4.2.3) $T \underset{Q}{=} U$, so $S \underset{Q}{=} U$ because Q is transitive. Hence $S \underset{Q}{\in} Y$.

Similarly, we can prove that any Q-member of Y is a Q-member of X.

(4.5.5) $X \underset{P}{\approxeq} X$. (Reflexivity of equal P-extension.)

(4.5.6) $X \underset{P}{\approxeq} Y \rightarrow Y \underset{P}{\approxeq} X$. (Symmetricity of equal P-extension.)

(4.5.7) $X \underset{P}{\approxeq} \underset{P}{\approxeq} Y \rightarrow X \underset{P}{\approxeq} Y$. (Transitivity of equal P-extension.)

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(4.6) ε-objects.

(4.6.1) Definition: $\varepsilon(P) \overset{df}{\equiv} \cdot \pi(P) \wedge \forall XY(X \underset{P}{=} Y \rightarrow X \underset{P}{\approx} Y)$. Any object P satisfying $\varepsilon(P)$ is called an *ε-object.*

(4.6.2) $\emptyset(P) \rightarrow \varepsilon(P)$. (Any null object is an ε-object.)

Proof. By (1.8.2), (3.2.2), (4.2.4), and (4.5.2).

(4.6.3) $\varepsilon(P) \wedge Q \simeq P \cdot \rightarrow \varepsilon(Q)$. (Any object which is proto-equal to an ε-obect is also an ε-object.)

Proof. Let Q be an object proto-equal to an ε-object P.

Firstly, Q is a producible object by (1.2.6) and (3.2.3), because the ε-object P is naturally producible.

Secondly, we assert that Q-equality implies equal Q-extension: To show this, take any two mutually Q-equal objects X and Y. Then, by (1.2.6) and (4.2.3), they are mutually P-equal; so, they are also objects of equal P-extent, because P is an ε-object. Consequently, by (4.5.3) $X \underset{Q}{\approx} Y$. Hence Q is an ε-object.

(4.6.4) $\pi(P) \wedge \forall U(U \in P \rightarrow \exists X \cdot U\{X\}) \cdot \rightarrow \varepsilon(P)$. (Any producible object formed by exclusively unit objects is an ε-object.)

Proof. Let P be a producible object which is formed by unit objects only. To show that P is an ε-object, take any two mutually P-equal objects X and Y. Then, $X \equiv Y$ because any pair object of X and Y can be a proto-member of P only when $X \equiv Y$ by reflexivity, symmetricity, and transitivity of identity. Since identity implies equal P-extension by (4.5.2), holds $X \underset{P}{\approx} Y$. Hence $\varepsilon(P)$.

(4.7) $\varepsilon(P) \wedge \varepsilon(Q) \cdot \rightarrow \exists M(P \subseteq M \wedge Q \subseteq M \wedge \varepsilon(M) \wedge \tau(M))$. (For any two ε-objects, there is a transitive ε-object which includes them as subobjects.)

Proof. The proof consists of two parts. In the first part, we prove that for any two ε-objects P and Q there exists a producible object M which satisfies $M\{U;\ \forall S(\tau(S) \wedge P \subseteq S \wedge Q \subseteq S \cdot \rightarrow U \in S)\}$, and that M is a transitive object including P and Q as subobjects. In the second part, we show $\varepsilon(M)$ by proving that M is proto-equal to any object N satisfying $N\{V;\ V \in M \wedge \forall ST(V\{S, T\} \rightarrow S \underset{M}{\approx} T)\}$. There is surely an obect N of this kind by (1.3.1).

Now, let P and Q be any two ε-objects. By (1.4.5), we can take sum

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objects F and G of P and Q respectively. Since ε-objects P and Q are producible, F and G are also producible objects by (3.2.5). By (3.2.11), we can take a union object H of the producible objects F and G, which is also producible. Namely, H is a producible object satisfying $H\{T;\ T \in\in P \vee T \in\in Q\}$. By (1.4.6) and (3.2.5), we can take a producible object K satisfying $K\{U;\ U \leqq H\}$.

We assert that the producible object K is transitive and that P and Q are subobjects of K. At first, we prove $K\{U;\ U \simeq \in K\}$: Namely, take any object U which is proto-equal to a proto-member V of K. Then, $V \leqq H$, so, by (1.2.6) and transitivity of proto-inclusion, also $U \leqq H$. Accordingly, $U \in K$. Conversely, any proto-member U of K satisfies $U \simeq \in K$ by reflexivity of proto-equality. Hence $K\{U;\ U \simeq \in K\}$. Secondly, we prove that $X \underset{K}{=} \underset{K}{=} Y$ implies $X \underset{K}{=} Y$ for any X and Y: Namely, let X be K-equal to Z and Z be K-equal to Y. Then, either $X \equiv Z$ or there is a proto-member U of K which is a pair object of X and Z, and also either $Z \equiv Y$ or there is a proto-member V of K which is a pair object of Z and Y. When one of the two identities $X \equiv Z$ and $Z \equiv Y$ holds, we can conclude $X \underset{K}{=} Y$ by (1.3.5), so we have only to discuss the case $\neg X \equiv Z \cdot \wedge \cdot \neg Z \equiv Y$. In this case, there are two proto-members U and V of K satisfying $U\{X, Z\}$ and $V\{Z, Y\}$. The proto-members U and V of K are subobjects of H, so X and Y are proto-members of H by (1.2.2) and reflexivity of identity. Accordingly, if we take an object W satisfying $W\{T;\ T \in H \wedge (T \equiv X \vee T \equiv Y)\}$ by (1.3.1), W is a pair object of X and Y by (1.3.5). Moreover $W \in K$, because $W \leqq H$ again by (1.3.5). Consequently $X \underset{K}{=} Y$. Hence K is transitive. To prove further that P as well as Q is a subobject of K, take any proto-member S of P or Q. Then, any proto-member T of S is surely a proto-member of H, so $S \leqq H$ i.e. $S \in K$. Hence $P \leqq K$ and $Q \leqq K$.

Now we assert that there exists a producible object M satisfying $M\{U;\ \forall S(\tau(S) \wedge P \leqq S \wedge Q \leqq S \cdot \rightarrow U \in S)\}$. To show this, take an object M satisfying $M\{U;\ U \in K \wedge \forall S(\tau(S) \wedge P \leqq S \wedge Q \leqq S \cdot \rightarrow U \in S)\}$ by (1.3.1). Clearly, M is a subobject of the producible object K, so M is also a producible object by (3.2.3). Now, any U satisfying $\forall S(\tau(S) \wedge P \leqq S \wedge Q \leqq S \cdot \rightarrow U \in S)$ satisfies especially $\tau(K) \wedge P \leqq K \wedge Q \leqq K \cdot \rightarrow U \in K$, so $U \in K$, because K is a transitive object and satisfies $P \leqq K$ and $Q \leqq K$. Consequently, the producible object M satisfies also $M\{U;\ \forall S(\tau(S) \wedge P \leqq S \wedge Q \leqq S \cdot \rightarrow U \in S)\}$.

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Next we prove that M is a transitive object and P as well as Q is a sub-object of M. To show that M is transitive, we prove at first $M\{U;\ U \simeq \epsilon M\}$: Namely, let any object U be proto-equal to a proto-member V of M. To show that U is also a proto-member of M, take any transitive object S including P and Q as its subobjects. Then, V is surely a proto-member of S, so also U is a proto-member of S, because S is transitive. Conversely, any proto-member U of M satisfies $U \simeq \epsilon M$ by reflexivity of proto-equality. Hence $M\{U;\ U \simeq \epsilon M\}$. Next we prove that $X \underset{M}{\overline{\overline{=}}} Z$ and $Z \underset{M}{\overline{\overline{=}}} Y$ imply $X \underset{M}{\overline{\overline{=}}} Y$ for any three objects X, Y, and Z. Namely, take again any transitive object S for which $P \subseteq S$ and $Q \subseteq S$ hold. Then, M is a subobject of S, because any proto-member U of M satisfies $\forall S(\tau(S) \wedge P \subseteq S \wedge Q \subseteq S \cdot \rightarrow U \epsilon S)$. Accordingly, by (4.2.3), $X \underset{S}{\overline{\overline{=}}} Z$ as well as $Z \underset{S}{\overline{\overline{=}}} Y$ holds; so, $X \underset{S}{\overline{\overline{=}}} Y$ holds too, because S is transitive. If $X \equiv Y$, then $X \underset{M}{\overline{\overline{=}}} Y$ holds by (4.2.2). If $\neg X \equiv Y$, then we can find a proto-member V of S which is a pair object of X and Y. Any pair object U of X and Y is evidently proto-equal to V, so $U \epsilon S$, since S is transitive. Accordingly, U is a proto-member of any transitive object S which includes P and Q as subobjects, i.e. $U \epsilon M$. Namely, any pair object U of X and Y is a proto-member of M, especially $V \epsilon M$. Consequently, X and Y are M-equal even in the case $\neg X \equiv Y$. Hence M is transitive. Lastly, we prove that P and Q are subobjects of M: Namely, take any proto-member U of P or Q. Then, by (1.2.2) U is a proto-member of any transitive object including P and Q as subobjects, so $U \epsilon M$. Hence $P \subseteq M$ and $Q \subseteq M$.

Next step is to show $\varepsilon(M)$. To prove this, take an object N satisfying $N\{V;\ V \epsilon M \wedge \forall ST(V\{S, T\} \rightarrow S \underset{M}{\overset{\approx}{=}} T)\}$ by (1.3.1). Then, surely $N \subseteq M$.

We assert $\tau(N)$. To show this, we prove at first $N\{V;\ V \simeq \epsilon N\}$: Namely, take any object V which is proto-equal to a proto-member W of N. $V \epsilon M$ because W is surely a proto-member of M and V is proto-equal to the proto-member W of the transitive object M. Moreover, if V is a pair object of two objects S and T, W is also a pair object of these two objects, because V is proto-equal to W. Since W is a proto-member of N, $S \underset{M}{\overset{\approx}{=}} T$ holds by assumption. Consequently $V \epsilon N$. Conversely, any proto-member V of N satisfies $V \simeq \epsilon N$ by reflexivity of proto-equality. Hence $N\{V;\ V \simeq \epsilon N\}$. Next we assert that $X \underset{N}{\overline{\overline{=}}} \underset{N}{\overline{\overline{=}}} Y$ implies $X \underset{N}{\overline{\overline{=}}} Y$ for any X and Y: To show this, assume $X \underset{N}{\overline{\overline{=}}} Z$ and

$Z \underset{N}{\equiv} Y$. Then, we can take a proto-member A of N being a pair object of X and Z and a proto-member B of N being a pair object of Z and Y, unless one of the two identities $X \equiv Z$ and $Z \equiv Y$ holds. When one of these two identities holds, X is N-equal to Y by (1.3.5), so we have only to discuss the case where we have such pair objects A and B which are proto-members of N. Since N is a subobject of the transitive object M, hold $X \underset{M}{\equiv} Z$ and $Z \underset{M}{\equiv} Y$ by (4.2.3), and accordingly $X \underset{M}{\equiv} Y$ holds, i.e. there is a proto-member C of M which is a pair object of X and Y, unless $X \equiv Y$. When $X \equiv Y$, holds $X \underset{N}{\equiv} Y$ by (4.2.2). On the other hand, when there is a pair object C of X and Y which is a proto-member of M, C is also a proto-member of N. To show this, we remark that $X \underset{M}{\cong} Y$ holds by (4.5.7), because $X \underset{M}{\cong} Z$ and $Z \underset{M}{\cong} Y$ hold by assumption. This implies that for any S and T satisfying $C\{S, T\}$ holds $S \underset{M}{\cong} T$ by (1.3.5) and (4.5.6), because we can show that, either $S \equiv X$ and $T \equiv Y$, or $S \equiv Y$ and $T \equiv X$ hold for such S and T by (1.2.4) and (1.3.5). Accordingly $X \underset{N}{\equiv} Y$ anyway. Hence, N is a transitive object.

Next we prove that P as well as Q is a subobject of N: Namely, take any proto-member U of P. The proto-member U of the subobject P of M is a proto-member of M by (1.2.2). If U is a pair object of S and T, clearly $S \underset{P}{\equiv} T$, so $S \underset{P}{\cong} T$, because P is an ε-object. Moreover $S \underset{P}{\cong} T$ implies $S \underset{M}{\cong} T$ by (4.5.4), since P is a subobject of the transitive object M. Consequently $U \in N$. Hence $P \subseteq N$. Similarly, we can show $Q \subseteq N$.

Since every proto-member U of M is a proto-member of the transitive object N including P and Q as its subobjects, holds $M \subseteq N$.

Lastly, we prove that the producible object M is an ε-object: Namely, take any two mutually M-equal objects X and Y. Then, there is a proto-member U of M which is a pair object of X and Y, unless $X \equiv Y$. When $X \equiv Y$, holds $X \underset{M}{\cong} Y$ by (4.5.2). Also in the case $\neg X \equiv Y$, the pair object U of X and Y, being a proto-member of M, is a proto-member of N by (1.2.2), because $M \subseteq N$. Accordingly $X \underset{M}{\cong} Y$. Hence, M is an ε-object.

(4.8) **Equality.**

(4.8.1) **Definition**: $A = B \overset{df}{\equiv} \exists P(\varepsilon(P) \wedge A \underset{P}{\equiv} B)$. Two objects are called mutually *equal* if and only if they are P-equal for a suitable ε-object P.

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(4.8.2) $A \equiv B \to A = B$. (Identity implies equality.)

(4.8.3) $A = A$. (Refelxivity of equality.)

Proof of (4.8.2) *and* (4.8.3). There is an ε-object P by (1.4.2) and (4.6.2). If $A \equiv B$, then $A \underset{P}{=} B$ for the ε-object P by (4.2.2), so $A = B$. Also, by reflexivity of P-equality, holds $A \underset{P}{=} A$ for the ε-object P, so $A = A$.

(4.8.4) $A = B \to B = A$. (Symmetricity of equality.)

Proof. Assume $A = B$. Then, $A \underset{P}{=} B$ for a suitable ε-object P. By symmetricity of P-equality holds $B \underset{P}{=} A$ for the ε-object P. Hence $B = A$.

(4.8.5) $A == B \to A = B$. (Transitivity of equality.)

Proof. Let any object A be equal to an object T which is equal to another object B. Then, $A \underset{P}{=} T$ for a suitable ε-object P and $T \underset{Q}{=} B$ for another suitable ε-object Q. By (4.7), there exists a transitive ε-object M which includes these two ε-objects P and Q as subobjects. Accordingly, by (4.2.3) hold $A \underset{M}{=} T$ and $T \underset{M}{=} B$. Since M is transitive, holds $A \underset{M}{=} B$. Since M is an ε-object, holds $A = B$.

(4.9) Membership.

To define membership "$\in$", two ways are possible. Of course, definitions by these two ways are logically equivalent to each other.

(4.9.1) **Definition**: $A \in B \overset{df}{\equiv} A = \epsilon B$ or $A \in B \overset{df}{\equiv} \exists P(\varepsilon(P) \wedge A \underset{P}{\in} B)$. Any object A is called a *member* of an object B, if and only if A is equal to a proto-member of B; or, if and only if A is a P-member of B for a suitable ε-object P.

Proof of equivalence of the two definitions. At first, assume that A is equal to a proto-member T of B. Then, $A \underset{P}{=} T$ for a suitable ε-object P. $A \underset{P}{=} T$ and $T \epsilon B$ implies $A \underset{P}{\in} B$. So, $\exists P(\varepsilon(P) \wedge A \underset{P}{\in} B)$. Next, assume conversely $A \underset{P}{\in} B$ for an ε-object P. Then, A is P-equal to a proto-member T of B. Since $A \underset{P}{=} T$ and $\varepsilon(P)$ implies $A = T$, A is equal to the proto-member T of B.

(4.9.2) $A \epsilon B \to A \in B$. (Proto-membership implies membership.)

Proof. By reflexivity of equality.

(4.10) Equality principle.

Theorems corresponding to the equality axioms of the set theory hold in our system with respect to equality and membership.

(**4.10.1**) $A=B\to\cdot A\in P\equiv B\in P$. (Two equal objects are either simultaneously members of an object or simultaneously non-members of the object.)

Proof. Assume $A=B$. If A is a member of an object P, A is equal to a suitable proto-member C of P. By symmetricity and transitivity of equality, B is also equal to the proto-member C of P. Hence $B\in P$.

Similarly, we can prove that A is a member of P whenever B is so.

(**4.10.2**) $A=B\to\cdot X\in A\equiv X\in B$. (Equal objects have all their members in common.)

Proof. Let X be a member of an object A which is equal to another object B. Then, $X\underset{P}{\Subset}A$ for a suitable ε-object P, and $A\underset{Q}{\equiv}B$ for another suitable ε-object Q. By (4.7), we can take a transitive ε-object M which includes P and Q as subobjects. By (4.3.3) $X\underset{M}{\Subset}A$ and by (4.2.3) $A\underset{M}{\equiv}B$. Since M is an ε-object, $A\underset{M}{\equiv}B$ implies $A\underset{M}{\approx}B$. So, $X\underset{M}{\Subset}A$ implies $X\underset{M}{\Subset}B$. Hence $X\in B$, because $X\underset{M}{\Subset}B$ for the ε-object M.

Similarly, we can prove that any member of B is a member of A.

(**4.10.3**) $X=Y\to\cdot\mathfrak{A}(X)\equiv\mathfrak{A}(Y)$, if $\mathfrak{A}(T)$ can be expressed by the notions of membership and equality only. (A theorem corresponding to the second axiom of equality with respect to membership and equality.)

Proof. By (4.10.1), (4.10.2), and symmetricity and transitivity of equality.

(4.11) Equal extension.

(**4.11.1**) **Definition**: $X\simeq Y\overset{df}{\equiv}\forall Z(Z\in X\equiv Z\in Y)$. Any two objects are called *objects of equal extent* if and only if they have all their members in common.

(**4.11.2**) $A=B\to A\simeq B$. (Equality implies equal extension.)

Proof. By (4.10.2).

(**4.11.3**) $X\simeq X$. (Reflexivity of equal extension.)

(**4.11.4**) $X\simeq Y\to Y\simeq X$. (Symmetricity of equal extension.)

(**4.11.5**) $X\simeq\simeq Y\to X\simeq Y$. (Transitivity of equal extension.)

(4.12) Extensionality principle.

(**4.12.1**) $\pi(A)\wedge\pi(B)\wedge A\simeq B\cdot\to A=B$. (A theorem corresponding to the axiom

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of extensionality of the set theory. Producible objects of equal extent are equal to each other.)

Proof. Let A and B be any two producible objects of equal extent. Then, by (3.2.11) we can take a producible object H which is a union object of A and B.

We assert first that there is a producible object P which satisfies $P\{U;\ \exists ST(S\kappa H \wedge T\kappa H \wedge U\{S, T\} \wedge S \simeq T)\}$: To show this, take a producible object K formed by satellites of H by (1.3.2) and (3.2.4), and then take an object P satisfying $P\{U;\ U\sigma K \wedge \exists ST(S\kappa H \wedge T\kappa H \wedge U\{S, T\} \wedge S \simeq T)\}$. Any constituents S and T of H are proto-members of K by (1.5.3), so any pair object U of the proto-members S and T of K is a subobject of K by (1.3.5); therefore $U\sigma K$ by (1.2.12). Consequently, P satisfies $P\{U;\ \exists ST(S\kappa H \wedge T\kappa H \wedge U\{S, T\} \wedge S \simeq T)\}$. Moreover, P is a producible object by (3.2.3), because we can prove by (1.3.2) and (3.2.4) that P is a subobject of a producible object formed by satellites of the producible object K.

We prove next that P-equality implies equal extension: Namely, take any two mutually P-equal objects X and Y. Then, either X and Y are identical or a pair object U of X and Y is a proto-member of P. In the case $X \equiv Y$, holds surely $X \simeq Y$ by (1.3.5). In the case $\neg X \equiv Y$, there is a pair object U of X and Y which is a proto-member of P. The proto-member U of P is a pair object of two suitable objects S and T of equal extent, so one of the two objects X and Y is identical with S and the other is identical with T by reflexivity, symmetricity, and transitivity of identity. Hence, by (1.3.5), $X \simeq Y$ holds also in this case.

Now we assert $\varepsilon(P)$: In order to show this, we have only to prove that P-equality implies equal P-extension, because we know already that P is producible. Take now any two mutually P-equal objects X and Y. Then, X and Y are objects of equal extent, as we have shown above. If $X \equiv Y$, X and Y are objects of equal P-extent by (4.5.2), so we have only to discuss the case $\neg X \equiv Y$. In this case there is a proto-member M of P which is a pair object of X and Y. Since M is also a pair object of two suitable constituents S and T of H, X as well as Y is also a constituent of H by (1.3.5) and reflexivity of identity. Take now any P-member Z of X. Then, Z is P-equal to a proto-member W of X. Since P-equality implies equal extension for the object P, holds $Z \simeq W$. Moreover, $W \in X$ since proto-membership implies membership by (4.9.2). Because $X \simeq Y$, the member W of X is also a member of Y; so, W is equal to a proto-member V of Y. Since equality implies equal extension by (4.11.2), holds $W \simeq V$. So, by transitivity of equal extension, holds $Z \simeq V$. If $Z \equiv W$, Z is a proto-member of the constituent X of H by (1.3.5), so Z is also a constituent of H by (1.5.4) and transitivity of κ. If $\neg Z \equiv W$, Z is a proto-member of a pair-object proto-member N of P by reflexivity of identity, and every proto-member of N is a constituent of H by (1.3.5), so Z itself is a constituent of H. Consequently, $Z \kappa H$ anyway. Moreover, the proto-member V of the constituent Y of H is also a constituent of H by (1.5.4) and transitivity of κ. Accordingly, by (1.3.5) and (1.5.3), there exists a pair object of Z and V, which is a proto-member of P. So, Z is P-equal to the proto-member V of Y. Hence $Z \underset{P}{\in} Y$. Similarly, we can prove that any P-member of Y is also a P-member of X. Hence $X \underset{P}{\simeq} Y$.

We prove now $A \underset{P}{=} B$: Because A and B are subobjects of their union object H, they are constituents of H by (1.5.5). By assumption, $A \simeq B$. Moreover, there is a pair object R of the producible objects A and B by (3.2.10). Consequently, the pair object R is a proto-member of P, so $A \underset{P}{=} B$.

Because $A \underset{P}{=} B$ holds for the ε-object P, holds $A = B$.

(4.12.2) $\pi(A) \wedge \pi(B) \wedge \forall Z(\pi(Z) \rightarrow \cdot Z \in A \equiv Z \in B) \cdot \rightarrow A \simeq B$. (If any two producible objects have all their producible members in common, they are objects of equal extent.)

Proof. Let A and B be two producible objects which satisfy $\forall Z(\pi(Z) \rightarrow \cdot Z \in A \equiv Z \in B)$. To prove $A \simeq B$, take any member T of A. Then, T is equal to a proto-member Z of A. The proto-member Z of the producible object A is surely producible by (3.2.3), and $Z \in A$ by (4.9.2). So, $Z \in B$ by assumption. Consequently, by (4.10.1), $T \in B$ too. Similarly, we can prove that any member of B is a member of A.

(5) Properties modulo equality

Producible objects can be regarded as sets if we draw no distinction between mutually equal objects. To treat objects formally without drawing any distinction between mutually equal objects is to consider only "properties modulo equality" whose notion shall be introduced in this Chapter. If we restrict ourselves only to properties modulo equality exclusively in the field of producible objects, our object theory can be regarded as a set theory. Indeed, with any proposition, we can associate a set-theoretical proposition which is called the set-theoretical image of it. Fundamental properties of set-theoretical images are also studied in this Chapter.

(5.1) Properties modulo equality.

(5.1.1) **Definition**: Any condition $\mathfrak{A}(X)$ is called to define a *property of X modulo equality* if and only if $\forall XY(X=Y\rightarrow(\mathfrak{A}(X)\equiv\mathfrak{A}(Y)))$ holds, where X and Y do not occur in $\mathfrak{A}(Z)$. The condition $\mathfrak{A}(X)$ is also called shortly a *condition on X modulo equality.*

(5.1.2) If X does not occur in $\mathfrak{A}$, $\mathfrak{A}$ is a condition on X modulo equality.

(5.1.3) $X\in Y$ defines a property of X as well as of Y modulo equality.

(5.1.4) $X=Y$ defines a property of X as well as of Y modulo equality.

Proof of (5.1.3) *and* (5.1.4). By (4.10.3).

(5.1.5) $\pi(X)$ defines a property of X modulo equality.

Proof. Let X and Y be mutually equal objects, and moreover, one of them, say X, be producible. Then, either these two objects are identical or there is a pair object U of them which is a proto-member of an ε-object P. In the former case, $\pi(X)$ implies $\pi(Y)$ by (1.3.5). In the latter case, the ε-object P is naturally producible, so the proto-member U of P is also a producible object by (3.2.3). Since Y is a proto member of the producible object U by reflexivity of identity, Y is also a producible object again by (3 2.3).

(5.1.6) If $\mathfrak{A}(X, Y)$ defines a property of X and Y modulo equality, $\mathfrak{A}(X, X)$ defines a property of X modulo equality, assuming that X and Y do not occur in $\mathfrak{A}(U, V)$.

(5.1.7) If $\mathfrak{A}(X)$ defines a property of X modulo equality, $\neg\mathfrak{A}(X)$ defines a property of X modulo equality.

(5.1.8) If $\mathfrak{A}(X)$ and $\mathfrak{B}(X)$ are conditions on X modulo equality, $\mathfrak{A}(X)\wedge\mathfrak{B}(X)$, $\mathfrak{A}(X)\vee\mathfrak{B}(X)$, and $\mathfrak{A}(X)\rightarrow\mathfrak{B}(X)$ are also conditions on X modulo equality.

(5.1.9) If $\mathfrak{A}(X)$ defines a property of X modulo equality, $\forall Z \cdot \mathfrak{A}(X)$ as well as $\exists Z \cdot \mathfrak{A}(X)$ defines a property of X modulo equality.

(5.1.10) Any condition, expressible in terms of membership "$\in$", equality "$=$", and producibility "$\pi(\cdot)$", defines a property of any variable modulo equality.

Proof. By (5.1.2)—(5.1.9).

(5.1.11) A necessary and sufficient condition for that $\mathfrak{A}(X)$ defines a property of X modulo equality is that $\mathfrak{A}(X)$ is equivalent to $\exists S(X = S \wedge \mathfrak{A}(S))$, assuming that X and S do not occur in $\mathfrak{A}(T)$.

Proof. Assume first that $\mathfrak{A}(X)$ defines a property of X modulo equality. Then, $\mathfrak{A}(X)$ holds whenever X is equal to any object S satisfying $\mathfrak{A}(S)$. On the other hand, $\mathfrak{A}(X)$ implies $\exists S(X = S \wedge \mathfrak{A}(S))$ by reflexivity of equality.

Next, assume conversely that $\mathfrak{A}(X)$ is equivalent to $\exists S(X = S \wedge \mathfrak{A}(S))$ for every X. By symmetricity of equality, $X = Y$ and $\mathfrak{A}(X)$ imply $\exists S(Y = S \wedge \mathfrak{A}(S))$, so also $\mathfrak{A}(Y)$ by assumption. Similarly, $X = Y$ and $\mathfrak{A}(Y)$ imply $\mathfrak{A}(X)$. Hence, $\mathfrak{A}(X)$ defines a property of X modulo equality.

(5.1.12) $\exists S(X = S \wedge \mathfrak{A}(S))$ is one of the strongest conditions under those conditions on X modulo equality, which are weaker than the condition $\mathfrak{A}(X)$, assuming that X and S do not occur in $\mathfrak{A}(T)$.

Proof. Let $\mathfrak{A}(X)$ be any condition on X, where X and S do not occur in $\mathfrak{A}(T)$.

$\exists S(X = S \wedge \mathfrak{A}(S))$ defines a property of X modulo equality. For, $\exists S(X = S \wedge \mathfrak{A}(S))$ is equivalent to $\exists T(X = T \wedge \exists S(T = S \wedge \mathfrak{A}(S)))$ by reflexivity and transitivity of equality, so $\exists S(X = S \wedge \mathfrak{A}(S))$ defines a porperty of X modulo equality by (5.1.11).

$\exists S(X = S \wedge \mathfrak{A}(S))$ is weaker than $\mathfrak{A}(X)$, i.e. the latter implies the former. For, if $\mathfrak{A}(X)$ holds, $\exists S(X = S \wedge \mathfrak{A}(S))$ holds by reflexivity of equality.

$\exists S(X = S \wedge \mathfrak{A}(S))$ is stronger than any condition on X modulo equality which is weaker than $\mathfrak{A}(X)$; namely, if the condition $\mathfrak{A}(X)$ on X implies any condition $\mathfrak{B}(X)$ on X modulo equality, also $\exists S(X = S \wedge \mathfrak{A}(S))$ implies $\mathfrak{B}(X)$ for any X. To show this, take any condition $\mathfrak{B}(X)$ on X modulo equality which is weaker than $\mathfrak{A}(X)$. If X is equal to an object S which satisfies $\mathfrak{A}(S)$, then X is equal to the object S which satisfies $\mathfrak{B}(S)$. Consequently, $\mathfrak{B}(X)$ holds too

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by (5.1.11), because $\mathfrak{B}(X)$ is assumed to define a property of X modulo equality.

(5.1.13) $A=B\wedge\pi(B)\cdot\rightarrow\pi(A)$. (Any object which is equal to a producible object is also producible.)

Proof. By (5.1.5).

(5.1.14) $A\in B\wedge\pi(B)\cdot\rightarrow\pi(A)$. (Any member of a producible object is also producible.)

Proof. Let A be a member of a producible object B. Then, A is equal to a proto-member C of B. The proto-member C of the producible object B is a producible object by (3.2.3), so A is also a producible object by (5.1.13).

(5.2) Set variables.

Together with capital Latin letters as variables for denoting objects in general, we employ small Latin letters as variables for denoting producible objects. Accordingly, whenever any small Latin letter x is employed as a free variable in a proposition $\mathfrak{A}(x)$, the proposition means that $\mathfrak{A}(X)$ holds for a corresponding free variable X which does not occur already in the course of the reasoning, having in mind that X denotes a producible object. $\forall x\cdot\mathfrak{A}(x)$ means naturally $\forall X(\pi(X)\rightarrow\mathfrak{A}(X))$, and $\exists x\cdot\mathfrak{A}(x)$ means $\exists X(\pi(X)\wedge\mathfrak{A}(X))$, where x and X are variables which do not occur in $\mathfrak{A}(S)$.

(5.2.1) *Remark.* Producible objects can be regarded as sets with respect to membership and equality, so we call variables denoted by small Latin letters ***set variables.*** Also, producible objects denoted by set variables are called sometimes ***sets.*** Any expression expressed by membership and equality with no bound capital-letter variables is called a ***set-theoretical expression.***

(5.2.2) *Remark.* Although the relation between the set notion and the object notion of our system is quite different from the relation between the set notion and the class notion of the Bernays-Gödel set-theory [5], [6], we can use our set variables just as the set variables of the Bernays-Gödel set-theory.

(5.2.3) Any proposition in our system can be expressed exclusively by the primitive notion "proto-membership" and without employing bound set variables. Any expression expressed exclusively by proto-membership and without employing bound set variables is called a ***proper expression.*** Any proposition has a proper expression of it.

(5.3) Set-theoretical images.

(5.3.1) **Definition.** Any set-theoretical expression $\mathfrak{B}$ is called the *set-theoretical image* of a proper expression $\mathfrak{A}$, if and only if we obtain $\mathfrak{B}$ by replacing proto-membership in $\mathfrak{A}$ by membership and all the bound variables X, Y, Z, $\cdots$ in $\mathfrak{A}$ by their corresponding set variables x, y, z, $\cdots$.

(5.3.2) Any proper expression has a definite set-theoretical image. For any proper expression $\mathfrak{A}$, we denote its set-theoretical image by $|\mathfrak{A}|$.

(5.3.3) Any set-theoretical expressions, especially the set-theoretical images of any proper expressions, define properties modulo equality with respect to any variables.

Proof. By (5.1.10).

(5.3.4) If $\mathfrak{A}$ and $\mathfrak{B}$ are proper expressions, so $\mathfrak{A}\wedge\mathfrak{B}$, $\mathfrak{A}\vee\mathfrak{B}$, and $\mathfrak{A}\to\mathfrak{B}$ are also proper expressions, and $|\mathfrak{A}\wedge\mathfrak{B}|$, $|\mathfrak{A}\vee\mathfrak{B}|$, and $|\mathfrak{A}\to\mathfrak{B}|$ are $|\mathfrak{A}|\wedge|\mathfrak{B}|$, $|\mathfrak{A}|\vee|\mathfrak{B}|$, and $|\mathfrak{A}|\to|\mathfrak{B}|$ respectively. If $\mathfrak{A}$ is a proper expression, so $\neg\mathfrak{A}$ is also a proper expression, and $|\neg\mathfrak{A}|$ is $\neg|\mathfrak{A}|$. If $\mathfrak{A}(T)$ is a proper expression, so $\forall X\cdot\mathfrak{A}(X)$ and $\exists X\cdot\mathfrak{A}(X)$ are also proper, and $|\forall X\cdot\mathfrak{A}(X)|$ and $|\exists X\cdot\mathfrak{A}(X)|$ are $\forall x|\mathfrak{A}(x)|$ and $\exists x|\mathfrak{A}(x)|$ respectively, where X and x do not occur in $\mathfrak{A}(T)$.

(5.3.5) *Remark.* Although set-theoretical images are defined only for definite expressions, they are practically defined whenever the expressions are definite except for nomination of bound variables. Accordingly, we can make use of notations such as $|A\subseteq B|$, $|A=B|$, $|\sigma(A)|$, $|A\sigma B|$, etc.

(5.4) Inclusion and set-theoretical inclusion.

(5.4.1) **Definition:** $X\subseteq Y\overset{df}{\equiv}\forall Z(Z\in X\to Z\in Y)$. Any object X is called a *minor object* of Y or to be *included* in Y if and only if every member of X is a member of Y.

(5.4.2) $X\subseteq Y\to X\subseteq Y$. (Proto-inclusion implies inclusion.)

Proof. Let X be any subobject of an object Y. Take any member S of X. Then, S is equal to a suitable proto-member T of X, which is surely a proto-member of Y by (1.2.2). Accordingly, S is equal to the proto-member T of Y, i.e. $S\in Y$. Hence $X\subseteq Y$.

(5.4.3) **Definition:** The binary relation $|X\subseteq Y|$ i.e. $\forall s(s\in X\to s\in Y)$ is called

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set-theoretical inclusion. Any object X satisfying $|X \subseteq Y|$ is called a *set-theoretical subobject* of Y.

(5.4.4) $X \subseteq Y \rightarrow |X \subseteq Y|$. (Inclusion implies set-theoretical inclusion.)

(5.4.5) $X \subseteq X$ and $|X \subseteq X|$. (Reflexivity of inclusion and of set-theoretical inclusion.)

(5.4.6) $X \subseteq \subseteq Y \rightarrow X \subseteq Y$ and $|X \subseteq Y| \wedge |Y \subseteq Z| \cdot \rightarrow |X \subseteq Z|$. (Transitivity of inclusion and of set-theoretical inclusion.)

(5.4.7) $X \simeq Y \equiv \cdot X \subseteq Y \wedge Y \subseteq X$. (Two objects are of equal extent if and only if each one of them is included in the other.)

(5.4.8) $X = Y \rightarrow X \subseteq Y$. (Equality implies inclusion.)

Proof. By (4.11.2) and (5.4.7).

(5.4.9) $X \subseteq Y \equiv X \simeq \subseteq Y$. (Any object X is a minor object of another object Y if and only if X is an object of equal extent with some subobject of Y.)

Proof. Let X be a minor object of Y. Take an object Z satisfying $Z\{T;\ T \in Y \wedge T \in X\}$ by (1.3.1). Evidently $Z \subseteq Y$. Moreover, we assert $X \simeq Z$: Namely, to show $X \subseteq Z$, take any member U of X. Then, U is a member of the object Y which includes X. Accordingly, U is equal to a proto-member T of Y. Moreover, $T \in X$ by (4.10.1). Accordingly $T \in Z$, so $U \in Z$. To show conversely $Z \subseteq X$, take any member U of Z. Then, U is equal to a proto-member T of Z. Since the proto-member T of Z is a member of X, U is equal to the member T of X i.e. $U \in X$ again by (4.10.1). Accordingly, $X \simeq Z$ holds by (5.4.7). Hence $X \simeq \subseteq Y$.

Conversely, let X be an object of equal extent with a subobject W of Y. Then, $W \subseteq Y$ by (5.4.2), so X itself is included in Y by (5.4.7) and transitivlty of inclusion.

(5.4.10) $x \subseteq y \equiv x = \subseteq y$ and $x \subseteq y \equiv \exists z(x = z \wedge z \subseteq y)$. (Any set x is included in another set y if and only if x is equal to a subobject Z of y, or, if and only if x is equal to a producible subobject z of y.)

Proof. If we assume in the proof of the preceding theorem further that X and Y are producible objects and denoted by x and y respectively, then the object Z is also a producible object by (3.2.3), since Z is a subobject of the producible object y. Consequently, $x = Z$ by (4.12.1). Hence, $x \subseteq y$ implies $x = \subseteqq y$ and $\exists z(x = z \wedge z \subseteqq y)$.

Conversely, if x is equal to a subobject Z of y, then $x \simeq Z$ by (4.11.2), so $x \subseteq y$ by (5.4.9).

(5.4.11) $x \subseteq y \equiv x \subseteqq = y$ and $x \subseteq y \equiv \exists z(x \subseteqq z \wedge z = y)$. (Any set x is a minor object of another set y if and only if x is a subobject of an object Z which is equal to y, or, if and only if x is a subobject of a set z which is equal to y.)

Proof. At first, let any set x be a minor object of another set y. By (3.2.11), take a producible union-object z of x and y. Clearly $y \subseteqq z$, so $y \subseteq z$ because proto-inclusion implies inclusion by (5.4.2). To show $z \subseteq y$, take any member S of z. Then, S is equal to a proto-member T of z. The proto-member T of the union object z of x and y is a proto-member of x unless it is a proto-member of y. So, $T \in x$ unless $T \in y$, because proto-membership implies membership by (4.9.2). Since $x \subseteq y$, T is a member of y even when $T \in x$. So, $T \in y$ anyway. Accordingly $S \in y$ by (4.10.1). Hence $z \simeq y$ by (5.4.7), so, $z = y$ by (4.12.1). Since clearly $x \subseteqq z$, x is a subobject of the set z which is equal to y.

Conversely, let x be any subobject of an object Z which is equal to y. Then, $x \subseteq Z$ and $Z \subseteq y$ hold, since proto-inclusion as well as equality implies incusion by (5.4.2) and (5.4.8). Hence $x \subseteq y$ by transitivity of inclusion.

(5.4.12) $x \subseteq Y \equiv |x \subseteqq Y|$. (Inclusion is equivalent to set-theoretical inclusion for sets. More precisely, any set is a minor object of an object if and only if the set is a set-theoretical subobject of the object.)

Proof. Since inclusion implies set-theoretical inclusion by (5.4.4), we have only to show that $|x \subseteqq Y|$ implies $x \subseteq Y$.

To show this, assume $|x \subseteqq Y|$. Any member Z of x is equal to a proto-member W of x. By (3.2.3), the proto-member W of the set x is producible, and it is a member of x by (4.9.2). Accordingly $W \in Y$, so also $Z \in Y$ by (4.10.1). Hence $x \subseteq Y$.

(5.4.13) $x = y$, $x \simeq y$, and $|x \approx y|$ are equivalent to each other.

Proof. By (4.10.2), (4.12.1), and (4.12.2).

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(5.5) Set-theoretical satellites.

(5.5.1) $|X \in \in y| \equiv X \in \in y$.

Proof. $|X \in \in y|$ implies $X \in \in y$ evidently. Conversely, let X be any member of a member U of y. Then, U is equal to a proto-member V of y. The proto-member V of the producible object y is a producible object by (3.2.3), so U is also a producible object by (5.1.13). Hence $|X \in \in y|$ holds.

(5.5.2) $X \in \in Y \equiv X \in \epsilon Y$. (Any object is a member of a member of another object if and only if the former is a member of a proto-member of the latter.)

Proof. Since proto-membership implies membership by (4.9.2), $X \in \epsilon Y$ implies $X \in \in Y$. Conversely, let X be any member of a member U of Y. Then, U is equal to a proto-member V of Y. Since the member X of U is also a member of V by (4.10.2), X is a member of the proto-member V of Y.

(5.5.3) $|x \subseteq \epsilon y| \equiv x \subseteq \in y$.

Proof. By (3.2.3), (4.9.2), (5.3.3), and (5.4.12). Similar to the proof of (5.5.1).

(5.5.4) $X \subseteq \in Y \equiv X \subseteq \epsilon Y$. (Any object is a minor object of a member of another object if and only if the former is a minor object of a proto-member of the other.)

Proof. By (4.9.2), (5.4.8), and transitivity of inclusion. Similar to the proof of (5.5.2).

(5.5.5) $\kappa(p) \rightarrow |\kappa(p)|$. (Any producible κ-object is also a set-theoretical κ-object. Here we call any object P satisfying $|\kappa(P)|$ a ***set-theoretical*** κ***-object.***)

Proof. Let a set p be any κ***-set*** i.e. any producible κ-object. To show that p is also a set-theoretical κ-object, take any set s satisfying $|s \epsilon \epsilon p|$. Then, by (5.5.1) and (5.5.2), s is a member of a proto-member of p, so s is equal to a proto-member T of a proto-member of p. Since p is a κ-set, T is a subobject of a proto-member U of p. The proto-member U of the set p is a producible object by (3.2.3). Since s is equal to subobject T of the producible object U, $|s \subseteq U|$ by (5.4.10) and (5.4.12). By (4.9.2), the proto-member U of p is also a member of p; so, s is a set-theoretical subobject of the producible member U of p. Hence, p is a set-theoretical κ-object by (5.3.4).

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(5.5.6) $\alpha(p) \rightarrow |\alpha(p)|$. (Any producible α-object is also a set-theoretical α-object. Here we call any object P satisfying $|\alpha(P)|$ a *set-theoretical* α-*object*.)

Proof. Let p be any α-set i.e. any producible α-object. To show that p is a set-theoretical α-object, take any set s satisfying $|s \subseteq \in p|$. Then, by (5.5.3) and (5.5.4), s is a minor object of a proto-member V of p. The proto-member V of the set p is a producible object by (3.2.3), so the minor object s of V is equal to a subobject T of V by (5.4.10). Since p is an α-set, the subobject T of the proto-member V of P is a proto-member of a proto-member W of p. So, $s \Subset W$ and $W \Subset p$ by (4.9.2). Accordingly $|s \in \in p|$ by (5.5.1). Hence p is a set-theoretical α-object by (5.3.4).

(5.5.7) $\sigma(p) \rightarrow |\sigma(p)|$. (Any producible σ-object is also a set-theoretical σ-object. Here we call any object P satisfying $|\sigma(P)|$ a *set-theoretical* σ-*object*.)

Proof. Let p be any σ-set i.e. any producible σ-object. Then, p is a κ-object as well as an α-object by (1.5.2) and (1.5.9), so the producible object p is a set-theoretical κ-object as well as a set-theoretical α-object by (5.5.5) and (5.5.6). Since any object which is a set-theoretical κ-object as well as a set-theoretical α-object is a set-theoretical σ-object by (5.3.4), the set p is a set-theoretical σ-object.

(5.5.8) $|x\kappa h| \rightarrow x = \kappa\, h$, $|y\alpha h| \rightarrow y = \alpha\, h$, and $|z\sigma h| \rightarrow z = \sigma\, h$. (Any producible set-theoretical constituent of a set is equal to a constituent of the set, any producible set-theoretical ancestor of a set is equal to an ancestor of the set, and any producible set-theoretical satellite of a set is equal to a satellite of the set. Here we call any object X satisfying $|X\kappa Y|$, $|X\alpha Y|$, or $|X\sigma Y|$ a *set-theoretical constituent*, a *set-theoretical ancestor*, or a *set-theoretical satellite* of Y respectively.)

Proof. Let three sets x, y, and z be any set-theoretical constituent, any set-theoretical ancestor, and any set-theoretical satellite of a set h respectively. Then, by (1.3.2) and (3.2.4), we can take a set p formed by satellites of the set h, and then we can take a set q formed by constituents of h by (1.5.3) and (3.2.3), and also a set r formed by ancestors of h by (1.5.10) and (3.2.3).

At first we assert $\kappa(q)$. To show this, take any proto-member S of a proto-member T of q. Then, $T\kappa h$, so, by (1.5.4) and transitivity of κ, $S\kappa h$ i.e. $S \in q$. Consequently, by reflexivity of proto-inclusion, S is a subobject of the proto-member S of p.

Next we assert $\alpha(r)$. To show this, take any subobject S of a proto-member T of r. Then, $T\alpha h$, so, by (1.5.11) and transitivity of α, $S\alpha h$. Take now by (1.4.4) any unit object K of S. Then, by (1.5.12) and transitivity of α, also $K\alpha h$ i.e. $K \in r$. Since $S \in K$ holds by reflexivity of identity, S is a proto-member of the proto-member K of r.

We assert thirdly $\sigma(p)$. We can prove this similarly to the above two proofs by employing (1.2.11) in place of (1.5.4), (1.2.12) in place of (1.5.11), (1.4.9) in place of (1.5.12), and transitivity of σ in places of transitivity of κ and of α.

h is a set-theortical subobject of a producible members of p, q, and r respectively. ($|h \subseteq \in p|$, $|h \subseteq \in q|$, and $|h \subseteq \in r|$.) For, h is a set-theoretical subobject of h by reflexivity of set-theoretical inclusion, and h is a member of p, q, and r by (4.9.2) and reflexivity of σ, κ, and α.

Since the κ-set q is a set-theoretical κ-object by (5.5.5), the α-set r is a set-theoretical α-object by (5.5.6), and the σ-set p is a set-theoretical σ-object by (5.5.7), x, y, and z are set-theoretical subobjects of some producible members of q, r, and p respectively, by (5.3.4). ($|x \subseteq \in q|$, $|y \subseteq \in r|$, and $|z \subseteq \in p|$.) By (3.2.3), (5.5.3), and (5.5.4), x, y, and z are minor objects of some producible proto-members of q, r, and p respectively. So, by (5.4.10), x, y, and z are equal to some subobjects of a constituent, of an ancestor, and of a satellite of h respectively. Accordingly, by (1.5.5), (1.5.11), (1.2.12), and transitivity of κ, of α, and of σ, the sets x, y, and z are equal to a constituent, to an ancestor, and to a satellite of h respectively.

(5.6) Some other set-theoretical images.

(5.6.1) $|\emptyset(p)| \equiv \emptyset(P)$. (Any set is a set-theoretical null object if and only if it is a null object. Here we call any object P satisfying $|\emptyset(P)|$ a *set-theoretical null object*.)

Proof. Evidently, $\emptyset(p)$ implies $|\emptyset(p)|$ for any set p. Conversely, if any set p is not a null object, then p must have some proto-members, which are producible members of p by (3.2.3) and (4.9.2), so p can never be a set-theoretical null object. Hence, $|\emptyset(p)|$ implies $\emptyset(p)$ for any set p.

(5.6.2) $U\{X, Y\} \rightarrow \forall s(s \in U \equiv \cdot\, s = X \vee s = Y)$, especially $U\{X\} \rightarrow \forall s(s \in U \equiv$

$s=X$). (Any pair object of two objects X and Y is a set-theoretical pair object of X and Y, especially any unit object of an object X is a set-theoretical unit object of X. Here we call any object U a *set-theoretical pair object* of X and Y or a *set-theoretical unit object* of X if and only if U satisfies $\forall s(s\in U \equiv \cdot\, s=X \vee s=Y)$ or $\forall s(s\in U \equiv s=X)$ respectively.)

Proof. Let U be any pair object of X and Y. To show that U is a set-theoretical pair object of X and Y, take at first any producible member s of U. Then, s is equal to a proto-member T of U. Since the proto-member T of the pair object U of X and Y must be identical with X or with Y, s must be equal to X or to Y by (1.3.5). Conversely, if we take any set s which is equal to one of the objects X and Y, then by reflexivity of identity, it is equal to a proto-member of U. Consequently $s\in U$. Hence $s\in U \equiv \cdot\, s=X \vee s=Y$ holds for any set s.

Especially, by taking Y as X, we know that any unit object U of X is also a set-theoretical unit object of X.

(6) A theory of sets

Any proposition, expressible in terms of membership and equality only with respect to exclusively set variables, can be regarded as a set-theoretical proposition. Any provable set-theoretical proposition can be regarded as a set-theoretical theorem. The set-theoretical image of any proposition containing no free variables other than set variables is evidently a set-theoretical proposition. Furthermore, we shall prove in this Chapter that the set-theoretical image of any theorem is a set-theoretical theorem, assuming that the original theorem contains no free variables at all. In this Chapter, we shall prove also that all the axioms of the Zermelo set-theory except the axiom of choice are set-theoretical theorems. We shall prove further that the axiom of fundierung is also a set-theoretical theorem.

(6.1) Set-theoretical images of the axioms.

(6.1.1) The set-theoretical image of any axiom is a set-theoretical theorem.

Proof. According to (5.3.4), the set-theoretical image of an axiom is a formula of the form $\exists p \forall x(x\in p \equiv \cdot\, |x\sigma m| \wedge |\mathfrak{A}(x)|)$.

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To show this formula for any set m, take an object P satisfying $P\{T;\ T\sigma m \wedge |T\sigma m| \wedge |\mathfrak{A}(T)|\}$. Because P is clearly a subobject of any object of satellites of the set m, it is a producible object by (1.3.2), (3.2.3), and (3.2.4), so it can be also denoted by a small letter p.

We assert $\forall x(x \in p \equiv \cdot\, |x\sigma m| \wedge |\mathfrak{A}(x)|)$: To show this, take any producible member x of p. Then, x is equal to a proto-member Z of p. Z satisfies $|Z\sigma m|$ and $|\mathfrak{A}(Z)|$. Since set-theoretical images define properties modulo equality by (5.3.3), so also x satisfies $|x\sigma m|$ and $|\mathfrak{A}(x)|$. Conversely, take any producible set-theoretical satellite x of the set m satisfying $|\mathfrak{A}(x)|$. Then, by (5.5.8), x is equal to a satellite T of m. Since set-theoretical images define properties modulo equality by (5.3.3), T satisfies $|T\sigma m|$ and $|\mathfrak{A}(T)|$. Accordingly $T \in p$. Since x is equal to the proto-member T of p, holds $x \in p$.

(6.1.2) The set-theoretical image of any provable theorem containing no free variables is a set-theoretical theorem.

Proof. Let $\mathfrak{T}$ be any provable theorem. Take any proof figure of $\mathfrak{T}$, and replace all the propositions in the proof figure by their set theoretical images and all the free variables by their corresponding set variables. Then, the proof figure is transformed into a right proof figure of the set-theoretical image $|\mathfrak{T}|$ of $\mathfrak{T}$. For, by virtue of (5.3.4), every right inference is transformed into a right inference, and moreover, by (6.1.1), every axiom is transformed into a provable proposition. Hence, the set-theoretical image $|\mathfrak{T}|$ of any provable theorem $\mathfrak{T}$ is also provable.

(6.1.3) The set-theoretical images of mutually equivalent propositions are also mutually equivalent with respect to free set variables.

Proof. If $\mathfrak{A}$ and $\mathfrak{B}$ are mutually equivalent, then $\mathfrak{A} \equiv \mathfrak{B}$ is provable. By (6.1.2), $|\mathfrak{A} \equiv \mathfrak{B}|^*$ is also provable, which implies $|\mathfrak{A}|^* \equiv |\mathfrak{B}|^*$ by (5.3.4) ; where $|\mathfrak{A} \equiv \mathfrak{B}|^*$, $|\mathfrak{A}|^*$, and $|\mathfrak{B}|^*$ are the formulas obtained by replacing all the free variables of $|\mathfrak{A} \equiv \mathfrak{B}|$, $|\mathfrak{A}|$, and $|\mathfrak{B}|$ by their corresponding free set variables respectively.

(6.1.4) *Remark.* By (6.1.3), we can now talk of the set-theoretical image of any proposition with respect to free set variables without assigning its definite expression.

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(6.2) Equality.

(6.2.1) $x=y \equiv \forall p(x \in p \equiv y \in p)$ i.e. $x=y \equiv |x \rightleftharpoons y|$. (Two sets are equal to each other if and only if they are, for every set, simultaneously members or no members of it.)

Proof. $x=y$ implies $\forall p(x \in p \equiv y \in p)$ by (4.10.1). Conversely, by the set-theoretical image of (1.3.4), $\forall p(x \in p \equiv y \in p)$ implies $|x \approx y|$ which is equivalent to $x=y$ by (5.4.13).

(6.2.2) $x=y \equiv \forall s(s \in x \equiv s \in y)$. (Two sets are equal to each other if and only if every set is always simultaneously a member or no member of both of them. This theorem implies the axiom of extensionality with respect to membership and equality.)

Proof. By (5.4.13).

(6.2.3) $x=y \rightarrow \cdot\, \mathfrak{A}(x) \equiv \mathfrak{A}(y)$, where $\mathfrak{A}(t)$ is a set-theoretical proposition. (The second axiom of equality with respect to membership and equality.)

Proof. By (6.2.1) and the set-theoretical image of (1.3.5).

(6.3) Term-symbols.

In the theory of objects, we can not introduce term-symbols of the form $\{T;\ \mathfrak{A}(T)\}$ as we have pointed out in (1.2.9). However, by virtue of (6.2.1), (6.2.2), and (6.2.3), we can adopt term-symbols of the form $\{t;\ \mathfrak{A}(t)\}$ in our theory of sets, assuming that there exists surely a set p satisfying $p\{t;\ \mathfrak{A}(t)\}$. Here we denote by $p\{t;\ \mathfrak{A}(t)\}$ the set-theoretical proposition $\forall t(t \in p \equiv \mathfrak{A}(t))$.

(6.3.1) $p\{t;\ \mathfrak{A}(t)\} \wedge q\{t;\ \mathfrak{A}(t)\} \cdot \rightarrow \cdot\, \mathfrak{B}(p) \equiv \mathfrak{B}(q)$, where $\mathfrak{B}(s)$ is a set-theoretical proposition.

Proof. By (6.2.2) and (6.2.3).

(6.3.2) When it is certain that there is at least one set p satisfying $p\{t;\ \mathfrak{A}(t)\}$, we know by (6.3.1) that, *in our theory of sets*, any set q satisfying $q\{t;\ \mathfrak{A}(t)\}$ does not show any distinction from the set p, so we can consider as if there is one and only one set p satisfying $p\{t;\ \mathfrak{A}(t)\}$.

Definition: When it is certain that there exists a set p satisfying $p\{t;\ \mathfrak{A}(t)\}$, we denote the set p by $\{t;\ \mathfrak{A}(t)\}$. In this case, we call $\{t;\ \mathfrak{A}(t)\}$

an *admissible term-symbol.*

(6.3.3) Any symbol of the form $\{t;\ |t\sigma m| \wedge \mathfrak{A}(t)\}$ is an admissible term-symbol, assuming that $\mathfrak{A}(t)$ defines a property of t modulo equality.

Proof. Similar to the proof of (6.1.1), by making use of (1.3.2), (3.2.3), (3.2.4), (5.3.4), and (5.5.8).

(6.3.4) *Remark.* Here we would like to fix the meaning of set-theoretical images of propositions, in which term-symbols occur. Namely, the set-theoretical image $|\mathfrak{A}(\mathfrak{t})|$ containing a term-symbol $\mathfrak{t}$ denotes the proposition obtained by replacing all the free variables t of $|\mathfrak{A}(t)|$ by $\mathfrak{t}$, where the free variable t does not occur in $|\mathfrak{A}(s)|$ if s is different from t.

(6.3.5) *Remark.* It is possible that free variables occur in a term as parameters. To denote explicitely parameters $x, \cdots, z$ of a term, we denote them like $\mathfrak{t}(x, \cdots, z)$.

(6.3.6) $x = y \rightarrow \mathfrak{t}(x) = \mathfrak{t}(y)$.

Proof. By (6.2.3) and reflexivity of equality.

(6.4) The null set.

(6.4.1) $\emptyset(p) \wedge \emptyset(q) \cdot \rightarrow p = q$. (All the null objects are equal to each other.)

Proof. If p as well as q has no proto-members at all, they are sets of equal extent, so they are mutually equal by (5.4.13).

We do not need to assume the two null objects are producible, because every null object must be a producible object by (1.8.2) and (3.2.2).

(6.4.2) $\emptyset(p) \equiv p\{t;\ |t\sigma m| \wedge \neg |t\sigma m|\}$.

Proof. By (3.2.3) and (4.9.2).

(6.4.3) Definition: The term-symbol $\{t;\ |t\sigma m| \wedge \neg |t\sigma m|\}$, which is proved to be admissible by (6.3.3) and which denotes a definite set independent of m by (6.4.1) and (6.4.2), is called the *null set* and is denoted by $\emptyset$.

(6.4.4) $\emptyset(\emptyset)$ and $s \notin \emptyset$.

Compare with (5.6.1). *Proof.* By (6.4.2).

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(6.5) Unit sets and pair sets.

(6.5.1) $\{t;\ t=x\}$ is an admissible term-symbol.

Proof. By (1.4.4) and (3.2.3), there exists a set u satisfying $u\{x\}$. By (5.6.2), $u\{x\}$ implies $u\{t;\ t=x\}$, so $\{t;\ t=x\}$ is an admissible term-symbol.

(6.5.2) Definition: The set $\{t;\ t=x\}$ is denoted by $\{x\}$ and is called the ***unit set*** of x.

(6.5.3) $t\in\{x\} \equiv t=x$.

(6.5.4) $\{t;\ t=x\vee t=y\}$ is an admissible term-symbol. (The pair-set axiom.)

Proof. For any two sets x and y, there is a set u satisfying $u\{x, y\}$ by (3.2.10). By (5.6.2), $u\{x, y\}$ implies $u\{t;\ t=x\vee t=y\}$, so $\{t;\ t=x\vee t=y\}$ is an admissible term-symbol.

(6.5.5) Definition: The set $\{t;\ t=x\vee t=y\}$ is denoted by $\{x, y\}$ and is called the *pair set* of x and y.

(6.5.6) *Remark.* Similarly, we can also define the set $\{x_1, \cdots, x_n\}$ of n-members $x_1, \cdots, x_n$.

(6.5.7) $t\in\{x, y\} \equiv \cdot\, t=x\vee t=y$.

(6.5.8) $\{x, x\}=\{x\}$.

Proof. By (6.2.2), (6.5.3), and (6.5.7).

(6.5.9) $\{x\}=\{y\} \equiv x=y$.

Proof. By (6.2.2), (6.5.3), and reflexivity, symmetricity, and transitivity of equality.

(6.5.10) $\{x, y\}=\{u, v\} \equiv (x=u\wedge y=v\cdot\vee\cdot x=v\wedge y=u)$.

Proof. By (6.2.2), (6.5.7), and reflexivity, symmetricity, and transitivity of equality.

(6.6) Union.

(6.6.1) $\{t;\ t\in x\vee t\in y\}$ is an admissible term-symbol.

Proof. For any two sets x and y, there is a set z satisfying $z\{T;\ T\in x \vee T\in y\}$ by (3.2.11). To show $z\{t;\ t\in x\vee t\in y\}$, take any producible member

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t of z. Then, t is equal to a proto-member T of z, which must be a proto-member of one of the two sets x and y. Accordingly, t is equal to the proto-member T of x or of y, i.e. t is a member of x or a member of y. Conversely, if any set t is a member of x or of y, t must be equal to an object T which is a proto-member of x or of y, so T is a proto-member of z. Accordingly, t is equal to the proto-member T of z, i.e. $t \in z$.

Hence $\{t;\ t \in x \vee t \in y\}$ is an admissible term-symbol.

(**6.6.2**) **Definition**: The set $\{t;\ t \in x \vee t \in y\}$ is denoted by $x \cup y$ and is called the *union* of x and y.

(**6.6.3**) $t \in (x \cup y) \equiv \cdot\, t \in x \vee t \in y$.

(**6.7**) **Aussonderung.**

(**6.7.1**) $x \in y \rightarrow |x \sigma y|$. (Any member of a set is a set-theoretical satellite of the set.)

(**6.7.2**) $|x \subseteq y| \rightarrow |x \sigma y|$. (Any subset of a set is a set-theoretical satellite of the set. Here we call any producible set-theoretical subobject of a set simply a *subset* of the set.)

(**6.7.3**) $|x \sigma y| \wedge |y \sigma z| \cdot \rightarrow |x \sigma z|$. (Any set-theoretical satellite of a set-theoretical satellite of a set is a set-theoretical satellite of the set.)

Proof of (6.7.1), (6.7.2), *and* (6.7.3). These are the set-theoretical images of (1.2.11), (1.2.12), and a modification of (1.2.14), respectively.

(**6.7.4**) $\{t;\ t \in m \wedge \mathfrak{A}(t)\}$ is an admissible term-symbol, assuming that $\mathfrak{A}(t)$ is a set-theoretical proposition. (The axiom of aussonderung with respect to membership and equality.) Especially, $\{t;\ t \in x \wedge t \in y\}$ is an admissible term-symbol.

Proof. By (6.3.3) and (6.7.1).

(**6.7.5**) **Definition**: The set $\{t;\ t \in x \wedge t \in y\}$ is denoted by $x \cap y$ and is called the *intersection* of x and y.

(**6.7.6**) $t \in (x \cap y) \equiv \cdot\, t \in x \wedge t \in y$.

(**6.7.7**) $\{t;\ |t \subseteq m| \wedge \mathfrak{A}(t)\}$ is an admissible term-symbol, assuming that $\mathfrak{A}(t)$ is a set-theoretical proposition. Especially, $\{t;\ |t \subseteq x|\}$ is an admissible term-

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symbol. (The power-set axiom.)

Proof. By (6.3.3) and (6.7.2).

(6.7.8) **Definition**: The set $\{t; |t \subseteq x|\}$ is called the *power set* of x and is denoted by $\mathfrak{P}(x)$.

(6.7.9) $t \in \mathfrak{P}(x) \equiv |t \subseteq x|$, and $t \in \mathfrak{P}(x) \equiv t \subseteq x$.

Proof of the second formula. By (5.4.12).

(6.7.10) $\{t; \exists y(t \in y \wedge y \in x)\}$ is an admissible term-symbol. (The sum-set axiom.)

Proof. By (6.3.3), (6.7.1), and (6.7.3).

(6.7.11) **Definition**: The set $\{t; \exists y(t \in y \wedge y \in x)\}$ is called the *sum set* of x and is denoted by $\mathfrak{S}(x)$.

(6.7.12) $t \in \mathfrak{S}(x) \equiv \exists y(t \in y \wedge y \in x)$, and $t \in \mathfrak{S}(x) \equiv t \in\in x$

Proof of the second formula. By (5.5.1).

(6.8) **Relations and functions.**

Any set-theoretical proposition containing some free variables can be considered as a relation, and also any term containing some free variables can be considered as a set-theoretical function. However, these relations and functions can not be taken as sets in general. In order to consider sets which represent relations and functions, we introduce the notions, ordered pairs, ordered n-tuples, and direct products of sets, as usual.

(6.8.1) *Remark.* We can define the ordered pair $\langle x, y\rangle$ as the term denoting the set $\{\{x\}, \{x, y\}\}$ as usual. Moreover, we can further define the ordered n-tuple $\langle x_1, \cdots, x_n\rangle$ in a natural way. We can also prove the formula $\langle x, y\rangle = \langle u, v\rangle \equiv \cdot\, x = u \wedge y = v$ by (6.5.8), (6.5.9), (6.5.10), and symmetricity and transitivity of equality. According to this definition, we can prove that x as well as y is a set-theoretical satellite of $\langle x, y\rangle$. Namely, by (6.5.7) and reflexivity of equality, x as well as y is a member of the member $\{x, y\}$ of $\langle x, y\rangle$. So, x as well as y is a set-theoretical satellite of $\langle x, y\rangle$ by (6.7.1) and (6.7.3). Furthermore, we can prove also that $\langle x, y\rangle$ is a set-theoretical subobject of the power set of any set which contains x and y as members. Namely, take any set w which contains x and y. Then, by (4.10.1), (6.5.3), and (6.5.7), $\{x\}$ as

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well as $\{x, y\}$ is a set-theoretical subobject of w, so $\langle x, y\rangle$ is a set-theoretical subobject of $\mathfrak{P}(w)$ by (4.10 1), (6.5.7), and (6.7.9). Accordingly, $\langle x, y\rangle$ is also a set-theoretical satellite of $\mathfrak{P}(w)$ by (6.7.2).

(6.8.2) The term-symbol $\{\langle u, v\rangle ; u\in x \wedge v\in y\}$ is admissible. Here the term-symbol naturally stands for $\{t ; \exists uv(t=\langle u, v\rangle \wedge u\in x \wedge v\in y)\}$.

Proof. Take any ordered pair $\langle u, v\rangle$ of a member u of x and a member v of x. Then, $\langle u, v\rangle$ is a set-theoretical satellite of $\mathfrak{P}(x\cup y)$ by (6.6.3) and (6.8.1). Accordingly, by (6.2.3) and (6.3.3), $\{\langle u, v\rangle ; u\in x\wedge v\in y\}$ is an admissible term-symbol.

(6.8.3) Definition: The set $\{\langle u, v\rangle ; u\in x \wedge v\in y\}$ is called the *direct product* of x and y and is denoted by $x\times y$.

(6.8.4) $\langle u, v\rangle\in(x\times y)\equiv\cdot u\in x\wedge v\in y$, and
$t\in(x\times y)\equiv\exists uv(t=\langle u, v\rangle \wedge u\in x\wedge v\in y)$.

Proof. The second formula holds evidently. The first formula can be proved by the second formula, (4.10.1), (6.8.1), and reflexivity of equality.

(6.8.5) *Remark.* Similarly, we can define the direct product of n-factors.

(6.9) Natural numbers.

In our theory of sets, natural numbers are introduced quite naturally by taking $\emptyset$ as the number *zero*, and the unit set $\{x\}$ of a number x as the number next to the number x. Since we can consider a set of all the natural numbers as shown in (6.9.6), the axiom of infinity holds in our set theory.

(6.9.1) Definition: Any set x satisfying $|\nu(x)|$ is called a *natural number.*

(6.9.2) $|\nu(\emptyset)|$. (Zero is a natural number.)

Proof. By (5.6.1), (6.4.4), and the set-theoretical image of (1.8.2).

(6.9.3) $|\nu(x)|\rightarrow|\nu(\{x\})|$. (The unit set of any natural number or, in other words, the number next to any natural number is also a natural number.)

Proof. By (1.4.4) and (3.2.3), we can take a set u satisfying $u\{x\}$. By (5.6.2) and (6.2.1), u satisfies $\forall s(s\in u\equiv|s=x|)$, so, also $|u\{x\}|$ by (5.3.4). Consequently, the set-theoretical image of (1.8.4) shows that u is a natural number if x is so. On the other hand, by (5.6.2) and (6.5.3), we have $|u\simeq\{x\}|$

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which implies $u=\{x\}$ by (5.4.13). Hence, by (6.2.3), $\{x\}$ is a natural number if x is so.

(6.9.4) $\emptyset \neq \{x\}$. (Zero is a number next to none.)

Proof. x is a member of $\{x\}$ by (6.5.3) and reflexivity of equality, whereas x is not a member of $\emptyset$ by (6.4.4), so $\emptyset \neq \{x\}$ by (6.2.2).

(6.9.5) $|\nu(t)| \equiv |t\sigma\emptyset|$. (Any set is a natural number if and only if it is a set-theoretical satellite of the null set.)

Proof. By (6.9.2) and the set-theoretical image of (1.8.3).

(6.9.6) $\{t\,;\,|\nu(t)|\}$ is an admissible term-symbol. (This theorem implies the axiom of infinity with respect to membership and equality.)

Proof. By the set-theoretical image of (1.8.5).

(6.9.7) Definition: The set $\{t\,;\,|\nu(t)|\}$ is called the *set of natual numbers* and is denoted by $\mathbf{N}$.

(6.9.8) $x\in\mathbf{N} \equiv |\nu(x)|$.

(6.9.9) $\emptyset\in p \wedge \forall x(x\in p \rightarrow \{x\}\in p)\cdot\rightarrow|\mathbf{N}\subseteq p|$. (For any set p, if $\emptyset$ is a member of p, and $\{x\}$ is a member of p for any member x of p, then $\mathbf{N}$ is a subset of p.)

Proof. Take any set p satisfying $\emptyset\in p$ and $\forall x(x\in p\rightarrow\{x\}\in p)$. Then, by (4.10.1), (5.6.1), (6.4.1) and (6.4.4), we can show $\forall x(|\emptyset(x)|\rightarrow x\in p)$, and by (4.10.1), (5.3.4), (5.4.13), and (6.5.3), we can also show $\forall yz(|y\langle z\rangle| \wedge z\in p\cdot\rightarrow y\in p)$. Consequently, by (5.3.4) and the set-theoretical image of (1.8.10), $|\nu(u)|\rightarrow u\in p$ holds for every u, which implies $|\mathbf{N}\subseteq p|$ by (6.9.8).

(6.9.10) *Complete induction.* $\mathfrak{A}(\emptyset)$ and $\forall x(\mathfrak{A}(x)\rightarrow\mathfrak{A}(\{x\}))$ imply $\mathfrak{A}(u)$ for any natural number u, assuming that $\mathfrak{A}(u)$ is a set-theoretical proposition.

Proof. Let $\mathfrak{A}(u)$ be any set-theoretical proposition satisfying $\mathfrak{A}(\emptyset)$ and $\forall x(\mathfrak{A}(x)\rightarrow\mathfrak{A}(\{x\}))$. Then, by (6.3.3), $\{t\,;\,|t\sigma\emptyset|\wedge\mathfrak{A}(t)\}$ is an admissible term-symbol, which we will denote by $\mathfrak{m}$. By the set-theoretical image of (1.2.13), $\emptyset$ is a member of $\mathfrak{m}$, and by (5.3.4), (6.5.3), (6.7.3), and the set-theoretical image of (1.4.9), $\{x\}$ is a member of $\mathfrak{m}$ for any member x of $\mathfrak{m}$. Consequently, by (6.9.9), $\mathbf{N}$ is a set-theoretical subobject of $\mathfrak{m}$, so every natural number u

satisfies $\mathfrak{A}(u)$ by (6.9.8).

(**6.9.11**) *Remark.* (6.5.9), (6.9.2), (6.9.3), (6.9.4), and (6.9.10) show that we can construct a complete theory of natural numbers in our theory of sets.

(**6.9.12**) $x \in \mathbf{N} \equiv \cdot\, x = \emptyset \vee \exists y(y \in \mathbf{N} \wedge x = \{y\})$. (Any set is a natural number if and only if is equal to the null set or to the unit set of a natural number.)

Proof. By (4.10.2), (5.3.4), (5.4.13), (5.6.1), (6.2.3), (6.4.1), (6.4.4), (6.5.3), (6.9.8), and the set-theoretical image of (1.8.9).

(**6.9.13**) $x \in \mathbf{N} \equiv x \in \in \mathbf{N}$, i.e. $\mathbf{N} = \mathfrak{S}(\mathbf{N})$. (Any set is a natural number if and only if it is a member of a natural number. Namely, the sum set of **N** is **N**.)

Proof. By (5.3.4), (5.4.13), (5.5.1), (6.5.3), (6.7.1), (6.7.3), (6.7.12), (6.9.3), (6.9.8), and reflexivity of equality.

(**6.9.14**) *Remark.* The ordering of the natural numbers can be defined very easily as $x \leq y$ stands for $|x\kappa y|$. We do not go into details in this matter.

(**6.9.15**) *Remark.* We do not discuss here in detail on recursive functions and recursively defined relations. However, it should be remarked here that we can define functions as sets recursively, if it is certain that their value domains can be considered as definite sets. Moreover, we can take the parameters of these functions as variables of these functions, if they are restricted to definite sets. The same holds also for recursively defined relations.

There is no difficulty to use the usual function notations such as $f(x)$, $f(x, y)$, etc. as terms. However, it should be noticed here that in our theory of sets, we can not define functions or relations as sets recursively in general. For example, it seems impossible to prove that there is a function $f(x)$ of a natural-number parameter x satisfying $f(\emptyset) = \emptyset$ and $f(\{x\}) = \{u;\ |u\sigma f(x)|\}$.

(6.10) Fundierung.

In our theory of sets, also the axiom of fundierung holds in a somewhat generalized form.

(**6.10.1**) $\forall x \exists y(x \in p \to \cdot\, |y\kappa \in x| \wedge y \in p) \to p = \emptyset$, especially
$\forall x \exists y(x \in p \to \cdot\, y \in x \wedge y \in p) \to p = \emptyset$. (The second formula is the axiom of fundierung. As for the meaning of the first formula, see (2.2.2).)

Proof. The first formula implies the second one by the set-theoretical

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image of (1.5.6). We shall prove the first formula in our *theory of objects.*

Take namely any set p satisfying $\forall x \exists y(x \in p \rightarrow \cdot |y \kappa \epsilon x| \wedge y \in p)$, and then take an object Q satisfying $Q\{T;\ T\sigma p \wedge T \in p\}$. Q is surely a subobject of any object formed by satellites of p, so Q is producible by (1.3.2), (3.2.''), and (3.2.4). Consequently, Q is regular by (3.2.7).

Now we prove $\forall X \exists U(X \epsilon Q \rightarrow \cdot U \kappa \epsilon X \wedge U \epsilon Q)$: Namely, take any proto-member X of Q. Then, the proto-member X of the set Q is a producible object by (3.2.3), and $X \in p$ by definition. Accordingly, by our assumption, there is a member y of p, which is also a set-theoretical constituent of a member s of X. The member s of X is equal to a proto-member T of X. Since $|y\kappa s|$ defines a property of s modulo equality by (5.3.3), holds $|y\kappa T|$. Moreover, the proto-member T of the set X is a producible object by (3.2.3); so, by (5.5.8), the set-theoretical constituent y of T is equal to a constituent U of T. By (4.10.1), $U \in p$. On the other hand, the proto-member X of Q is a satellite of p by definition, so the constituent U of the proto-member T of X is a satellite of p by (1.2.11), (1.5.3), and transitivity of σ. Consequently, $U \epsilon Q$ by definition. Hence, for any proto-member X of Q, there is a proto-member of Q (U for example) which is a constituent of the proto-member T of X.

Because $Q \zeta Q$ by (2.1.2) and reflexivity of κ, the regular object Q must be a null object.

Now we prove $\emptyset(p)$: Namely, if there were a proto-member W of p, W would be a satellite as well as a member of p by (1.2.11) and (4.9.2), so W would be a proto-member of Q contradictory to the fact that Q is a null object.

$\emptyset(p)$ implies $p = \emptyset$ by (6.4.1) and (6.4.4).

(6.10.2) $|\rho(p)|$. (Every set is set-theoretically regular, i.e. every set p satisfies $|\rho(p)|$.)

Proof. This follows trivially from (4.10.2), (5.3.4), (5.4.13), (6.4.4), and (6.10.1).

(7) Supplementary remark

To show relative consistency of our object theory with respect to the Fraenkel set-theory, we prove that all the interpreted propositions of our axioms are provable in the Fraenkel system Φ without the axiom of choice. Objects and proto-membership are interpreted as sets and membership respectively. We

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use small letters for sets of Φ. *Inclusion* "$\subseteq$" is defined naturally by $x\subseteq y \overset{df}{\equiv} \forall s(s\in x\rightarrow s\in y)$, and the notion of *satellites* in Φ is defined by $x\hat{\sigma}y \overset{df}{\equiv} \forall p(\hat{\sigma}(p)\wedge y\subseteq\in p\cdot\rightarrow x\subseteq\in p)$, where $\hat{\sigma}(p)$ stands for $\forall s(s\in\in p\equiv s\subseteq\in p)$.

We show that the following theorem holds in Φ.

Theorem. $\exists p\forall x(x\in p\equiv\cdot x\hat{\sigma}m\wedge\mathfrak{A}(x))$.

Proof. For any set x of Φ, there are well defined sum set $\mathfrak{S}(x)$ and power set $\mathfrak{P}(x)$ of x. Also, for any pair of sets x and y of Φ, there is a well defined pair set $\{x, y\}$ of them. Accordingly, we can define the ordered pair $\langle x, y\rangle$ and the unit set $\{x\}$. Furthermore, in Φ there is also the set of all the natural numbers $\mathbf{N}=\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \cdots\}$. We can also define the transitive relation "$<$" on $\mathbf{N}$ which satisfies $x<\{y\}\equiv\cdot x<y\vee x=y$.

On the domain $\mathbf{N}$, we define functions $f(x, t)$ and $g(x, t)$ satisfying the conditions: $f(x,\emptyset)=x$, $f(x,\{t\})=\{f(x,t)\}$; and $g(x, \emptyset)=\{x\}$, $g(x,\{t\})=\mathfrak{S}(g(x,t))$. Namely, $f(x,t)$ and $g(x, t)$ are y and z defined by the following conditions respectively:

$$\forall h(\langle\emptyset, x\rangle\in h\wedge\forall uv(u<t\wedge\langle u, v\rangle\in h\cdot\rightarrow\langle\{u\},\{v\}\rangle\in h)\cdot\rightarrow\langle t, y\rangle\in h),$$

$$\forall h(\langle\emptyset,\{x\}\rangle\in h\wedge\forall uv(u<t\wedge\langle u, v\rangle\in h\cdot\rightarrow\langle\{u\},\mathfrak{S}(v)\rangle\in h)\cdot\rightarrow\langle t, z\rangle\in h).$$

Existence and uniqueness of y and z for every natural number t can be proved by complete induction.

By the axiom of replacement, we define $\mathbf{A}(x)\overset{df}{=}\{f(x,t);\ t\in\mathbf{N}\}$, and $\mathbf{B}(x)\overset{df}{=}\{g(x, t);\ t\in\mathbf{N}\}$. Accordingly, $w\in\mathbf{A}(z)$ holds if and only if $w=z$ or w is a unit set of a member of $\mathbf{A}(z)$. Evidently, $w\in\mathbf{A}(z)$ implies $\{w\}\in\mathbf{A}(z)$. Also holds $m\in\mathfrak{S}(\mathbf{B}(m))$ and that $x\in\in\mathfrak{S}(\mathbf{B}(m))$ implies $x\in\mathfrak{S}(\mathbf{B}(m))$.

Again by the axiom of replacement, we define $\mathbf{C}(m)\overset{df}{=}\{\mathfrak{P}(x);\ x\in\mathfrak{S}(\mathbf{B}(m))\}$. Evidently, $s\subseteq\in\mathfrak{S}(\mathbf{C}(m))$ as well as $s\in\in\mathfrak{S}(\mathbf{C}(m))$ implies $s\in\mathfrak{S}(\mathbf{C}(m))$. Once again by the axiom of replacemement, we define $\mathbf{D}(m)\overset{df}{=}\{\mathbf{A}(z);\ z\in\mathfrak{S}(\mathbf{C}(m))\}$.

Now we prove that $s\subseteq\in\mathfrak{S}(\mathbf{D}(m))$ implies $s\in\mathfrak{S}(\mathbf{D}(m))$ as well as $s\in\in\mathfrak{S}(\mathbf{D}(m))$: Namely, take any subset s of a member w of $\mathfrak{S}(\mathbf{D}(m))$. Then, w is a member of a member $\mathbf{A}(z)$ of $\mathbf{D}(m)$, z being a member of $\mathfrak{S}(\mathbf{C}(m))$. The member w of $\mathbf{A}(z)$ is a unit set unless $w=z$. If w is a unit set, the subset s of w is $\emptyset$ or w itself. In the case $s=\emptyset$, hold $s\subseteq z$ and

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$z \in \mathfrak{S}(\mathbf{C}(m))$, so $s \in \mathfrak{S}(\mathbf{C}(m))$. Accordingly $\mathbf{A}(s) \in \mathbf{D}(m)$, so $s \in \in \mathbf{D}(m)$ i.e. $s \in \mathfrak{S}(\mathbf{D}(m))$. In the case $s = w$, holds $s \in \mathfrak{S}(\mathbf{D}(m))$ because $w \in \mathfrak{S}(\mathbf{D}(m))$. On the other hand, if $w = z$, then $s \subseteq z$ and $z \in \mathfrak{S}(\mathbf{C}(m))$, so $s \in \mathfrak{S}(\mathbf{C}(m))$ since $s \subseteq \in \mathfrak{S}(\mathbf{C}(m))$ implies $s \in \mathfrak{S}(\mathbf{C}(m))$. Because $s \in \mathbf{A}(s)$, holds $s \in \in \mathbf{D}(m)$ i.e. $s \in \mathfrak{S}(\mathbf{D}(m))$. Hence $s \in \mathfrak{S}(\mathbf{D}(m))$ anyway. Now, the member s of $\mathfrak{S}(\mathbf{D}(m))$ is a member of a member $\mathbf{A}(z)$ of $\mathbf{D}(m)$, z being a member of $\mathfrak{S}(\mathbf{C}(m))$. Because $s \in \mathbf{A}(z)$ implies $\{s\} \in \mathbf{A}(z)$, so $s \in \in \mathfrak{S}(\mathbf{D}(m))$.

Next we prove that $s \in \in \mathfrak{S}(\mathbf{D}(\mathrm{m}))$ implies $s \subseteq \in \mathfrak{S}(\mathbf{D}(m))$: Namely, let s be a member of a member w of $\mathfrak{S}(\mathbf{D}(m))$. Then, w is a member of a member $\mathbf{A}(z)$ of $\mathbf{D}(m)$, z being a member of $\mathfrak{S}(\mathbf{C}(m))$. The member w of $\mathbf{A}(z)$ is a unit set of a member of $\mathbf{A}(z)$ unless $w = z$. If w is a unit set of a member of $\mathbf{A}(z)$, also its member s belongs to $\mathbf{A}(z)$, so $s \in \in \mathbf{D}(m)$ i.e. $s \in \mathfrak{S}(\mathbf{D}(m))$. On the other hand, if $w = z$, then $s \in \in \mathfrak{S}(\mathbf{C}(m))$, so $s \in \mathfrak{S}(\mathbf{C}(m))$. Because $s \in \mathbf{A}(s)$, $s \in \in \mathbf{D}(m)$ i.e. $s \in \mathfrak{S}(\mathbf{D}(m))$. Hence $s \in \mathfrak{S}(\mathbf{D}(m))$ holds anyway. Since $s \subseteq s$, holds $s \subseteq \in \mathfrak{S}(\mathbf{D}(m))$.

Thus we obtain $\hat{\sigma}(\mathfrak{S}(\mathbf{D}(m)))$. Moreover, holds $m \subseteq \in \mathfrak{S}(\mathbf{D}(m))$. For: $m \in \in \mathbf{B}(m)$ i.e. $m \in \mathfrak{S}(\mathbf{B}(m))$, because $\{m\} \in \mathbf{B}(m)$ by definition. Since $m \subseteq m$, holds $m \in \in \mathbf{C}(m)$ i.e. $m \in \mathfrak{S}(\mathbf{C}(m))$. Because $m \in \mathbf{A}(m)$, holds $m \in \in \mathbf{D}(m)$ i.e. $m \in \mathfrak{S}(\mathbf{D}(m))$. Hence $m \subseteq \in \mathfrak{S}(\mathbf{D}(m))$, since $m \subseteq m$.

Assume now $x \hat{\sigma} m$. Then, $x \subseteq \in \mathfrak{S}(\mathbf{D}(m))$ holds, because $\hat{\sigma}(\mathfrak{S}(\mathbf{D}(m)))$ and $m \subseteq \in \mathfrak{S}(\mathbf{D}(m))$. However, $x \subseteq \in \mathfrak{S}(\mathbf{D}(m))$ implies $x \in \mathfrak{S}(D(m))$, so $x \hat{\sigma} m$ implies $x \in \mathfrak{S}(\mathbf{D}(m))$.

Because the aussonderung axiom holds in Φ, we can take a set p satisfying $\forall x(x \in p \equiv \cdot\, x \in \mathfrak{S}(\mathbf{D}(m)) \wedge x \hat{\sigma} m \wedge \mathfrak{A}(x))$. The set p satisfies evidently $\forall x(x \in p \equiv \cdot\, x \hat{\sigma} m \wedge \mathfrak{A}(x))$, because $x \hat{\sigma} m$ implies $x \in \mathfrak{S}(\mathbf{D}(m))$.

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Mathematical Institute
Nagoya University