Proceedings of the Edinburgh Mathematical Society (2007) **50**, 229–249 ©
DOI:10.1017/S0013091504001233 Printed in the United Kingdom

ALMOST-PERIODIC SOLUTIONS FOR AN ECOLOGICAL MODEL WITH INFINITE DELAYS

YONGHUI XIA[1] AND JINDE CAO[2]

[1] *College of Mathematics and Computer Science, Fuzhou University, Fuzhou 350002, People's Republic of China* (yhxia@fzu.edu.cn)
[2] *Department of Mathematics, Southeast University, Nanjing 210096, People's Republic of China* (jdcao@seu.edu.cn)

(Received 6 November 2004)

Abstract By using Lebesgue's dominated convergence theorem and constructing a suitable Lyapunov functional, we study the following almost-periodic Lotka–Volterra model with M predators and N prey of the integro-differential equations

$$\begin{aligned}\dot{x}_i(t) = x_i(t)\biggl[b_i(t) - a_{ii}(t)x_i(t) - \sum_{k=1,k\neq i}^{N} a_{ik}(t)\int_{-\infty}^{t} H_{ik}(t-\sigma)x_k(\sigma)\,\mathrm{d}\sigma \\ - \sum_{l=1}^{M} c_{il}(t)\int_{-\infty}^{t} K_{il}(t-\sigma)y_l(\sigma)\,\mathrm{d}\sigma\biggr], \qquad i = 1,2,\dots,N,\\ \dot{y}_j(t) = y_j(t)\biggl[-r_j(t) - e_{jj}(t)y_j(t) + \sum_{k=1}^{N} d_{jk}(t)\int_{-\infty}^{t} P_{jk}(t-\sigma)x_k(\sigma)\,\mathrm{d}\sigma \\ - \sum_{l=1,l\neq j}^{M} e_{jl}(t)\int_{-\infty}^{t} Q_{jl}(t-\sigma)y_l(\sigma)\,\mathrm{d}\sigma\biggr], \quad j = 1,2,\dots,M.\end{aligned}$$

Some sufficient conditions are obtained for the existence of a unique almost-periodic solution of this model. Several examples show that the obtained criteria are new, general and easily verifiable.

Keywords: predator–prey system; Lyapunov functional; almost-periodic solution

2000 *Mathematics subject classification:* Primary 34D10; 34C25; 34D23; 34K14; 34K20

1. Introduction

The Lotka–Volterra system is a rudimentary model in mathematical ecology. The asymptotic behaviour of the Lotka–Volterra competition system with *almost-periodic* (*periodic*) coefficients has been studied extensively in [**1**, **2**, **5**, **6**, **8**–**11**, **14**, **15**, **18**–**20**, **24**, **28**]. Some sufficient conditions are obtained for the uniform persistence, existence and uniqueness of the asymptotic stable *almost-periodic* (*periodic*) solution for the Lotka–Volterra competition system.

The two-species predator–prey Lotka–Volterra system has been investigated extensively in [**3**, **16**, **17**], and the references cited therein. Some results were obtained for checking existence of the periodic solution and the asymptotic behaviour of these systems. However, few papers have considered the multi-species model. Yang and Xu [**25**] studied the following periodic system with M predators and N prey:

$$\left.\begin{aligned} \dot{x}_i(t) &= x_i(t)\biggl[b_i(t) - \sum_{k=1}^{N} a_{ik}(t)x_k(t) - \sum_{l=1}^{M} c_{il}(t)y_l(t)\biggr], \qquad i = 1,2,\dots,N. \\ \dot{y}_j(t) &= y_j(t)\biggl[-r_j(t) + \sum_{k=1}^{N} d_{jk}(t)x_k(t) - \sum_{l=1}^{M} e_{jl}(t)y_l(t)\biggr], \quad j = 1,2,\dots,M. \end{aligned}\right\} \tag{1.1}$$

where $x_i(t)$ denotes the density of prey species X_i at time t, and $y_j(t)$ denotes the density of predator species Y_j at time t. The coefficients b_i, r_j, a_{ik}, c_{il}, d_{jk} and e_{jl} $(i,k = 1,\dots,N;\ j,l = 1,\dots,M)$ are non-negative continuous periodic functions defined on $\mathbb{R}$. If (1.1) is autonomous, that is, if these coefficient functions are constants, then b_i is the intrinsic growth rate of prey species X_i, r_j is the death rate of the predator species Y_j, a_{ik} measures the amount of competition between the prey species X_i and X_k $(k \neq i$, $i,k = 1,\dots,N)$, e_{jl} measures the amount of competition between the predator species Y_j and Y_k $(k \neq j$, $j,k = 1,\dots,M)$, and the constant $\tilde{k}_{ij} \triangleq d_{ij}/c_{ij}$ measures how many of the prey species X_i convert into predator species Y_j $(i = 1,\dots,N;\ j = 1,\dots,M)$. Sufficient conditions for existence and global attractivity of a unique positive periodic solution of system (1.1) were obtained in [**25**].

Recently, Zhao and Chen [**27**] have investigated system (1.1) again, allowing the intrinsic growth rate of the prey species to be negative while the total intrinsic growth rate in a period is positive. By using differential inequalities and constructing a Lyapunov function, some sufficient conditions were obtained for existence and global attractivity of a unique positive periodic solution of (1.1). Recently, Xia *et al*. [**23**] studied (1.1) with almost-periodic coefficients. Sufficient conditions were obtained for existence and global attractivity of a unique positive almost-periodic solution of (1.1).

The system (1.1) with delay has not been studied so often. Wen [**21**] considered system (1.1) with several delays, that is

$$\left.\begin{aligned} \dot{x}_i(t) &= x_i(t)\biggl[b_i(t) - a_i(t)x_i(t) - \sum_{k=1}^{N} a_{ik}(t)x_k(t-\tau_{ik}) - \sum_{l=1}^{M} c_{il}(t)y_l(t-\sigma_{il})\biggr], \\ &\qquad\qquad i = 1,2,\dots,N, \\ \dot{y}_j(t) &= y_j(t)\biggl[-r_j(t) - e_j(t)y_j(t) + \sum_{k=1}^{N} d_{jk}(t)x_k(t-\xi_{jk}) - \sum_{l=1}^{M} e_{jl}(t)y_l(t-\eta_{jl})\biggr], \\ &\qquad\qquad j = 1,2,\dots,M. \end{aligned}\right\} \tag{1.2}$$

By means of the comparison theorem and the Lyapunov functional, some sufficient conditions were obtained for existence of the global attractivity of a unique positive periodic solution of system (1.2).

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

Recently, Xia and Cao [**22**] have considered the almost-periodic Lotka–Volterra model with M predators and N prey by 'pure-delay type', that is

$$\left.\begin{aligned}\dot{x}_i(t) &= x_i(t)\biggl[b_i(t) - \sum_{k=1}^{N} a_{ik}(t)x_k(t-\tau_{ik}(t)) - \sum_{l=1}^{M} c_{il}(t)y_l(t-\sigma_{il}(t))\biggr],\\ &\qquad\qquad i = 1,2,\dots,N,\\ \dot{y}_j(t) &= y_j(t)\biggl[-r_j(t) + \sum_{k=1}^{N} d_{jk}(t)x_k(t-\xi_{jk}(t)) - \sum_{l=1}^{M} e_{jl}(t)y_l(t-\eta_{jl}(t))\biggr],\\ &\qquad\qquad j = 1,2,\dots,M.\end{aligned}\right\} \tag{1.3}$$

By using the concept of *eventually uniform M-matrix* and constructing a suitable Lyapunov functional, a set of sufficient conditions for the existence and global attractivity of a unique positive almost-periodic (periodic) solution of system (1.3) were obtained.

To the best of the authors' knowledge, though Burton and Hutson [**4**] consider a similar system, there is no paper considering the almost-periodic solutions of the multiple-species predator–prey model with infinite delays. Therefore, in this paper, we consider the almost-periodic Lotka–Volterra model with M predators and N prey of the integro-differential equations

$$\left.\begin{aligned}\dot{x}_i(t) &= x_i(t)\biggl[b_i(t) - a_{ii}(t)x_i(t) - \sum_{k=1,k\neq i}^{N} a_{ik}(t)\int_{-\infty}^{t} H_{ik}(t-\sigma)x_k(\sigma)\,\mathrm{d}\sigma\\ &\qquad - \sum_{l=1}^{M} c_{il}(t)\int_{-\infty}^{t} K_{il}(t-\sigma)y_l(\sigma)\,\mathrm{d}\sigma\biggr], \quad i = 1,2,\dots,N,\\ \dot{y}_j(t) &= y_j(t)\biggl[-r_j(t) - e_{jj}(t)y_j(t) + \sum_{k=1}^{N} d_{jk}(t)\int_{-\infty}^{t} P_{jk}(t-\sigma)x_k(\sigma)\,\mathrm{d}\sigma\\ &\qquad - \sum_{l=1,\neq j}^{M} e_{jl}(t)\int_{-\infty}^{t} Q_{jl}(t-\sigma)y_l(\sigma)\,\mathrm{d}\sigma\biggr], \quad j = 1,2,\dots,M.\end{aligned}\right\} \tag{E}$$

By using Lebesgue's dominated convergence theorem and constructing a suitable Lyapunov functional, some sufficient conditions are obtained for the existence of a unique almost-periodic solution of system (E).

This paper is organized as follows. In § 2 we use Lebesgue's dominated convergence theorem to show that there exists a bounded solution of system (E) on $\mathbb{R}$. In § 3, by constructing a suitable Lyapunov functional, we shall prove that the solution of system (E) is relatively totally stable. Then some sufficient conditions are obtained for the existence of a unique almost-periodic solution of system (E). Finally, some suitable examples are given to illustrate the main results of this paper.

Throughout this paper, we shall use the following notation.

We always use $i, k = 1,\dots,N$; $j, l = 1,\dots,M$, unless otherwise stated.

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

If f is an almost-periodic function defined on $\mathbb{R}$, we set

$$f^{\iota} = \inf_{t\in\mathbb{R}} f(t), \qquad f^{\mu} = \sup_{t\in\mathbb{R}} f(t).$$

Denote

$$\begin{aligned}
p_i &= \frac{b_i^{\mu}}{a_{ii}^{\iota}},\\
q_j &= \frac{1}{e_{jj}^{\iota}}\left(\sum_{k=1}^{N} d_{jk}^{\mu} p_k - r_j^{\iota}\right),\\
\alpha_i &= \frac{1}{a_{ii}^{\mu}}\left(b_i^{\iota} - \sum_{k=1,k\neq i}^{N} a_{ik}^{\mu} p_k - \sum_{l=1}^{M} c_{il}^{\mu} q_l\right),\\
\beta_j &= \frac{1}{e_{jj}^{\mu}}\left(-r_j^{\mu} + \sum_{k=1}^{N} d_{jk}^{\iota}\alpha_k - \sum_{l=1,l\neq j}^{M} c_{jl}^{\mu} q_l\right),\\
x^* &= \max_{i,j}\{p_i, q_j\}, \qquad x_* = \min_{i,j}\{\alpha_i, \beta_j\}.
\end{aligned}$$

It is obvious that if $q_j > 0$, then $\alpha_i < p_i$, and if $q_j > 0$, $\alpha_i > 0$, then $\beta_j < q_j$.

We denote by $\mathbb{R}^{N+M}$ the $(N+M)$-dimensional real Euclidean space and by $|x|$ the norm of $x \in \mathbb{R}^{N+M}$.

Let

$$\boldsymbol{B} = \{\Phi = (\phi, \psi) \mid \Phi : (-\infty, 0] \mapsto \mathbb{R}^{N+M},\ \Phi(t) \text{ is a bounded and uniformly continuous function}\}$$

(see [**12**, **13**]). For $\Phi \in \boldsymbol{B}$, we set $\|\Phi\| = \sup_{s\leqslant 0} |\Phi(s)|$.

For any $\Phi, \Psi \in \boldsymbol{B}$, we set

$$\rho_m(\Phi, \Psi) = \sup_{-m\leqslant s\leqslant 0} |\Phi(s) - \Psi(s)|,$$

$$\rho(\Phi, \Psi) = \sum_{m=1}^{\infty} \frac{\rho_m(\Phi, \Psi)}{2^m(1 + \rho_m(\Phi, \Psi))}.$$

Obviously, $\rho(\Phi_n, \Phi) \mapsto 0$, as $n \to \infty$, if and only if $\Phi_n(s) \to \Phi(s)$ as $n \to \infty$ uniformly on each bounded subset of $(-\infty, 0]$.

For any function $x : R \mapsto \mathbb{R}^{N+M}$ and $t \in \mathbb{R}$, we define a function $x^t : (-\infty, 0] \mapsto \mathbb{R}^{N+M}$ by $x^t(s) = x(t+s)$ for $s \leqslant 0$.

Throughout this paper, we suppose that the following conditions are satisfied.

(H_1) b_i, r_j, a_{ik}, c_{il}, d_{jk} and e_{jl} are non-negative almost-periodic functions defined on $\mathbb{R}$ with $\inf_{t\in\mathbb{R}} a_{ii}(t) > 0$, $\inf_{t\in\mathbb{R}} e_{jj}(t) > 0$.

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

(H_2) H_{ik}, K_{il}, P_{jk} and $Q_{jl}(t)$ are non-negative functions, and

$$\int_0^\infty H_{ik}(s)\,\mathrm{d}s = 1, \qquad \int_0^\infty sH_{ik}(s)\,\mathrm{d}s < \infty;$$
$$\int_0^\infty K_{il}(s)\,\mathrm{d}s = 1, \qquad \int_0^\infty sK_{il}(s)\,\mathrm{d}s < \infty;$$
$$\int_0^\infty P_{jk}(s)\,\mathrm{d}s = 1, \qquad \int_0^\infty sP_{jk}(s)\,\mathrm{d}s < \infty;$$
$$\int_0^\infty Q_{jl}(s)\,\mathrm{d}s = 1, \qquad \int_0^\infty sQ_{jl}(s)\,\mathrm{d}s < \infty.$$

2. Existence of bounded solutions

In the following we will state some lemmas that will be used in the proof of Theorem 2.4. Since we are interested in the positive solutions of the system, we assume that system (E) is supplemented with initial conditions of the form

$$\left.\begin{aligned} x_i(s) = \phi_i(s) \geqslant 0, \quad s \leqslant 0; \quad \sup_{s\leqslant 0}\phi_i(s) < \infty; \quad \phi_i(0) > 0.\\ y_j(s) = \psi_j(s) \geqslant 0, \quad s \leqslant 0; \quad \sup_{s\leqslant 0}\psi_j(s) < \infty; \quad \psi_j(0) > 0.\end{aligned}\right\} \tag{2.1}$$

Lemma 2.1. *Both the positive and non-negative cones of $\mathbb{R}^{N+M}$ are invariant with respect to (E).*

Proof. Since

$$\begin{aligned} x_i(t) = x_i(t_0)\exp\biggl\{\int_{t_0}^t\biggl[b_i(s) - a_{ii}(s)x_i(s) - \sum_{k=1,k\neq i}^N a_{ik}(s)\int_{-\infty}^s H_{ik}(s-\sigma)x_k(\sigma)\,\mathrm{d}\sigma \\ - \sum_{l=1}^M c_{il}(s)\int_{-\infty}^s K_{il}(s-\sigma)y_l(\sigma)\,\mathrm{d}\sigma\biggr]\mathrm{d}s\biggr\}, \\ y_j(t) = y_j(t_0)\exp\biggl\{\int_{t_0}^t\biggl[-r_j(s) - e_{jj}(s)y_j(s) + \sum_{k=1}^N d_{jk}(s)\int_{-\infty}^s P_{jk}(s-\sigma)x_k(\sigma)\,\mathrm{d}\sigma \\ - \sum_{l=1,\neq j}^M e_{jl}(s)\int_{-\infty}^s Q_{jl}(s-\sigma)y_l(\sigma)\,\mathrm{d}\sigma\biggr]\mathrm{d}s\biggr\},\end{aligned}$$

the assertion of Lemma 2.1 follows immediately, for all $t \in [t_0, +\infty)$. □

Lemma 2.2. *If $a > 0$, $b > 0$ and $\mathrm{d}x(t)/\mathrm{d}t \leqslant x(t)[b - ax(t)]$, for $t \geqslant t_0$, then*

$$x(t) \leqslant \frac{b}{a}\biggl[1 + \biggl(\frac{b}{ax(t_0)} - 1\biggr)\mathrm{e}^{-bt}\biggr]^{-1}, \quad t \geqslant t_0.$$

Moreover, if $0 < x(t_0) \leqslant b/a$, then $0 < x(t) \leqslant b/a$.

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

If $a > 0$, $b > 0$ and $\mathrm{d}x(t)/\mathrm{d}t \geqslant x(t)[b - ax(t)]$, for $t \geqslant t_0$, then

$$x(t) \geqslant \frac{b}{a}\left[1 + \left(\frac{b}{ax(t_0)} - 1\right)\mathrm{e}^{-bt}\right]^{-1}, \quad t \geqslant t_0.$$

Moreover, if $x(t_0) \geqslant b/a > 0$, then $x(t) \geqslant b/a > 0$.

Proof. We give a proof for the first case. From $\mathrm{d}x(t)/\mathrm{d}t \leqslant x(t)[b - ax(t)]$, we have

$$\frac{\mathrm{d}x^{-1}(t)}{\mathrm{d}t} \geqslant a - bx^{-1}(t), \qquad \frac{\mathrm{d}(\mathrm{e}^{bt}x^{-1}(t))}{\mathrm{d}t} \geqslant a\mathrm{e}^{bt}.$$

Hence, we have

$$\mathrm{e}^{bt}x^{-1}(t) - x^{-1}(t_0) \geqslant \frac{a}{b}(\mathrm{e}^{bt} - 1).$$

Moreover, we have

$$x^{-1}(t) \geqslant \mathrm{e}^{-bt}x^{-1}(t_0) + \frac{b}{a}(1 - \mathrm{e}^{-bt}) = \frac{a}{b}\left[1 + \left(\frac{b}{ax(t_0)} - 1\right)\mathrm{e}^{-bt}\right].$$

Therefore, we obtain

$$x(t) \leqslant \frac{b}{a}\left[1 + \left(\frac{b}{ax(t_0)} - 1\right)\mathrm{e}^{-bt}\right]^{-1}.$$

Then it is not difficult to derive that if $0 < x(t_0) \leqslant b/a$, then $0 < x(t) \leqslant b/a$. This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let $\Phi = (\phi, \psi) \in \boldsymbol{B}$ satisfy $\alpha_i \leqslant \phi_i(s) \leqslant p_i$, $\beta_j \leqslant \psi_j(s) \leqslant q_j$ for all $s \leqslant 0$, and $F = (x, y)$ be the solution of (E) through (t_0, Φ). If (E) satisfies $\alpha_i > 0$, $\beta_j > 0$ and $q_j > 0$, then*

$$\alpha_i \leqslant x_i(t) \leqslant p_i, \quad \beta_j \leqslant y_j(t) \leqslant q_j, \quad \text{for all } t \geqslant t_0.$$

Proof. From the first equation of system (E), we have $\dot{x}_i \leqslant x_i(b_i^{\mu} - a_{ii}^{\iota}x_i)$, so if

$$0 < x_i(t_0) \leqslant p_i,$$

then from Lemma 2.2 we have

$$x_i(t) \leqslant p_i. \tag{2.2}$$

From (2.2) and the second equation of system (E), we obtain

$$\dot{y}_j \leqslant y_j\left[-r_j^{\iota} + \sum_{k=1}^{N} d_{jk}^{\mu} p_k \int_{-\infty}^{t} P_{jk}(t - \sigma)\,\mathrm{d}\sigma - e_{jj}^{\iota} y_j\right].$$

Since

$$\int_{-\infty}^{t} P_{jk}(t - \sigma)\,\mathrm{d}\sigma = \int_{0}^{\infty} P_{jk}(s)\,\mathrm{d}s = 1,$$

we obtain

$$\dot{y}_j \leqslant y_j\left[\left(-r_j^{\iota} + \sum_{k=1}^{N} d_{jk}^{\mu} p_k\right) - e_{jj}^{\iota} y_j\right].$$

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

Hence, if

$$0 < y_j(t_0) \leqslant q_j,$$

then from Lemma 2.2 we have

$$y_j(t) \leqslant q_j. \tag{2.3}$$

Now the first equation of (E), (2.2), (2.3) and (H_2) lead to

$$\begin{aligned}\dot{x}_i &\geqslant x_i\biggl[b_i^\iota - \sum_{k=1,k\neq i}^{N} a_{ik}^\mu p_k \int_{-\infty}^{t} H_{ik}(t-\sigma)\,\mathrm{d}\sigma - \sum_{l=1}^{M} c_{il}^\mu q_l \int_{-\infty}^{t} K_{il}(t-\sigma)\,\mathrm{d}\sigma - a_{ii}^\mu x_i\biggr] \\ &= x_i\biggl[b_i^\iota - \sum_{k=1,k\neq i}^{N} a_{ik}^\mu p_k \int_{0}^{\infty} H_{ik}(s)\,\mathrm{d}s - \sum_{l=1}^{M} c_{il}^\mu q_l \int_{0}^{\infty} K_{il}(s)\,\mathrm{d}s - a_{ii}^\mu x_i\biggr] \\ &= x_i\biggl[\biggl(b_i^\iota - \sum_{k=1,k\neq i}^{N} a_{ik}^\mu p_k - \sum_{l=1}^{M} c_{il}^\mu q_l\biggr) - a_{ii}^\mu x_i\biggr],\end{aligned}$$

which implies that if $x_i(t_0) \geqslant \alpha_i$ holds, from Lemma 2.2 we have

$$x_i(t) \geqslant \alpha_i. \tag{2.4}$$

If $\alpha_i > 0$, then from (E), (2.2), (2.3) and (H_2), by a similar discussion to that above, we obtain

$$\dot{y}_j \geqslant y_j\biggl[\biggl(-r_j^\mu + \sum_{k=1}^{N} d_{jk}^\iota \alpha_k - \sum_{l=1,l\neq j}^{M} e_{jl}^\mu q_l\biggr) - e_{jj}^\mu y_j\biggr],$$

which implies that if $y_j(t_0) \geqslant \beta_j$ holds, from Lemma 2.2 we obtain

$$y_j(t) \geqslant \beta_j. \tag{2.5}$$

If $\alpha_i > 0$, $\beta_j > 0$ and $q_j > 0$, it is obvious that $0 < \alpha_i < p_i$, $0 < \beta_j < q_j$. Therefore, we have $\alpha_i \leqslant x_i(t) \leqslant p_i$, $\beta_j \leqslant y_j(t) \leqslant q_j$, for all $t \geqslant t_0$. This completes the proof of Lemma 2.3. □

We denote by $S(E)$ the set of all solutions $F = (x, y)$ of (E) on $\mathbb{R}$ satisfying $\alpha_i \leqslant x_i(t) \leqslant p_i$, $\beta_j \leqslant y_j(t) \leqslant q_j$ for all $t \in \mathbb{R}$.

Theorem 2.4. *If (E) satisfies*

(H_3) $\alpha_i > 0$, $\beta_j > 0$, $q_j > 0$,

then $S(E) \neq \emptyset$.

Proof. By (H_1), there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that

$$\left.\begin{aligned} b_i(t+t_n) &\to b_i(t), \\ r_j(t+t_n) &\to r_j(t), \\ a_{ik}(t+t_n) &\to a_{ik}(t), \\ c_{il}(t+t_n) &\to c_{il}(t), \\ d_{jk}(t+t_n) &\to d_{jk}(t), \\ e_{jl}(t+t_n) &\to e_{jl}(t), \end{aligned}\right\} \quad \text{as } n \to \infty \text{ uniformly on } \mathbb{R}.$$

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

Let $F=(x,y)$ be a solution of (E) through $(t_0,\Phi)\in\mathbb{R}\times\boldsymbol{B}$ satisfying $\alpha_i\leqslant x_i(t)\leqslant p_i$, $\beta_j\leqslant y_j(t)\leqslant q_j$ for all $t\geqslant t_0$, whose existence was ensured by Lemma 2.3. Obviously, the sequence $\{F(t+t_n)\}$ is uniformly bounded and equicontinuous on each bounded subset of $\mathbb{R}$. Therefore, by Ascoli's theorem and a diagonalization procedure, we may assume that the sequence $\{F(t+t_n)\}$ converges to a continuous function $U(t)=(u(t),v(t))=(u_1(t),\dots,u_N(t),v_1(t),\dots,v_M(t))$ as $n\to\infty$ uniformly on each bounded subset of $\mathbb{R}$. Let a $\tau\in\mathbb{R}$ be given. We may assume that $t_n+\tau\geqslant t_0$ for all n. For $t\geqslant t_0$, we have

$$
\begin{aligned}
&x_i(t+t_n+\tau)-x_i(t_n+\tau)\\
&\quad=\int_{t_n+\tau}^{t+t_n+\tau}x_i(s)\biggl\{b_i(s)-a_{ii}(s)x_i(s)-\sum_{k=1,k\neq i}^{N}a_{ik}(s)\int_{-\infty}^{s}H_{ik}(s-\sigma)x_k(\sigma)\,\mathrm{d}\sigma\\
&\qquad\qquad-\sum_{l=1}^{M}c_{il}(s)\int_{-\infty}^{s}K_{il}(s-\sigma)y_l(\sigma)\,\mathrm{d}\sigma\biggr\}\,\mathrm{d}s\\
&\quad=\int_{\tau}^{t+\tau}x_i(\bar{s}+t_n)\biggl\{b_i(\bar{s}+t_n)-a_{ii}(\bar{s}+t_n)x_i(\bar{s}+t_n)\\
&\qquad\qquad-\sum_{l=1,l\neq i}^{N}a_{il}(\bar{s}+t_n)\int_{-\infty}^{\bar{s}+t_n}H_{ik}(\bar{s}+t_n-\sigma)x_k(\sigma)\,\mathrm{d}\sigma\\
&\qquad\qquad-\sum_{k=1}^{M}c_{ik}(s)\int_{-\infty}^{\bar{s}+t_n}K_{ik}(\bar{s}+t_n-\sigma)y_k(\sigma)\,\mathrm{d}\sigma\biggr\}\,\mathrm{d}\bar{s},
\end{aligned}\tag{2.6}
$$

$$
\begin{aligned}
&y_j(t+t_n+\tau)-y_j(t_n+\tau)\\
&\quad=\int_{t_n+\tau}^{t+t_n+\tau}y_j(s)\biggl\{-r_j(s)-e_{jj}(s)y_j(s)+\sum_{k=1}^{N}d_{jk}(s)\int_{-\infty}^{s}P_{jk}(s-\sigma)x_k(\sigma)\,\mathrm{d}\sigma\\
&\qquad\qquad-\sum_{l=1,l\neq i}^{M}e_{jl}(s)\int_{-\infty}^{s}Q_{jl}(s-\sigma)y_l(\sigma)\,\mathrm{d}\sigma\biggr\}\,\mathrm{d}s\\
&\quad=\int_{\tau}^{t+\tau}x_i(\bar{s}+t_n)\biggl\{-r_j(\bar{s}+t_n)-e_{jj}(\bar{s}+t_n)y_j(\bar{s}+t_n)\\
&\qquad\qquad+\sum_{k=1}^{N}d_{jk}(\bar{s}+t_n)\int_{-\infty}^{\bar{s}+t_n}P_{jk}(\bar{s}+t_n-\sigma)x_k(\sigma)\,\mathrm{d}\sigma\\
&\qquad\qquad-\sum_{l=1,l\neq j}^{M}e_{jl}(s)\int_{-\infty}^{\bar{s}+t_n}Q_{jl}(\bar{s}+t_n-\sigma)y_l(\sigma)\,\mathrm{d}\sigma\biggr\}\,\mathrm{d}\bar{s}.
\end{aligned}\tag{2.7}
$$

We note that

$$
\int_{-\infty}^{\bar{s}+t_n}H_{ik}(\bar{s}+t_n-\sigma)x_k(\sigma)\,\mathrm{d}\sigma=\int_{0}^{\infty}H_{ik}(\bar{\sigma})x_k(\bar{s}+t_n-\bar{\sigma})\,\mathrm{d}\bar{\sigma},
$$

$$
H_{ik}(\bar{\sigma})x_k(\bar{s}+t_n-\bar{\sigma})\to H_{ik}(\bar{\sigma})u_k(\bar{s}-\bar{\sigma})\quad\text{as } n\to\infty,
$$

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

and that

$$|H_{ik}(\bar{\sigma})x_k(\bar{s}+t_n-\bar{\sigma})| \leqslant H_{ik}(\bar{\sigma})p_k \quad \text{for } \bar{\sigma} \geqslant t_0,\ \bar{s} \in [\tau, t+\tau].$$

By (H_2) and Lebesgue's dominated convergence theorem, it follows that

$$\lim_{n\to\infty} \int_0^{\infty} H_{ik}(\bar{\sigma})x_k(\bar{s}+t_n-\bar{\sigma})\,\mathrm{d}\bar{\sigma} = \int_0^{\infty} H_{ik}(\bar{\sigma})u_k(\bar{s}-\bar{\sigma})\,\mathrm{d}\bar{\sigma} = \int_{-\infty}^{\bar{s}} H_{ik}(\bar{s}-\sigma)u_k(\sigma)\,\mathrm{d}\sigma. \tag{2.8}$$

Similarly, by (H_2) and Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{n\to\infty} \int_{-\infty}^{\bar{s}+t_n} K_{il}(\bar{s}+t_n-\sigma)y_l(\sigma)\,\mathrm{d}\sigma &= \int_0^{\infty} K_{il}(\bar{\sigma})v_l(\bar{s}-\bar{\sigma})\,\mathrm{d}\bar{\sigma} \\ &= \int_{-\infty}^{\bar{s}} K_{il}(\bar{s}-\sigma)v_l(\sigma)\,\mathrm{d}\sigma, \end{aligned}\tag{2.9}$$

$$\begin{aligned}\lim_{n\to\infty} \int_{-\infty}^{\bar{s}+t_n} P_{jk}(\bar{s}+t_n-\sigma)x_k(\sigma)\,\mathrm{d}\sigma &= \int_0^{\infty} P_{jk}(\bar{\sigma})u_k(\bar{s}-\bar{\sigma})\,\mathrm{d}\bar{\sigma} \\ &= \int_{-\infty}^{\bar{s}} P_{jk}(\bar{s}-\sigma)u_k(\sigma)\,\mathrm{d}\sigma, \end{aligned}\tag{2.10}$$

$$\begin{aligned}\lim_{n\to\infty} \int_{-\infty}^{\bar{s}+t_n} Q_{jl}(\bar{s}+t_n-\sigma)y_l(\sigma)\,\mathrm{d}\sigma &= \int_0^{\infty} Q_{jl}(\bar{\sigma})v_l(\bar{s}-\bar{\sigma})\,\mathrm{d}\bar{\sigma} \\ &= \int_{-\infty}^{\bar{s}} Q_{jl}(\bar{s}-\sigma)v_l(\sigma)\,\mathrm{d}\sigma. \end{aligned}\tag{2.11}$$

From (2.8)–(2.11), and letting $n \to \infty$ in (2.6) and (2.7), respectively, we have

$$\begin{aligned} u_i(t+\tau) - u_i(\tau) &= \int_{\tau}^{t+\tau} u_i(\bar{s})\biggl\{b_i(\bar{s}) - a_{ii}(\bar{s})u_i(\bar{s}) - \sum_{k=1,k\neq i}^{N} a_{ik}(\bar{s}) \int_{-\infty}^{\bar{s}} H_{ik}(\bar{s}-\sigma)u_k(\sigma)\,\mathrm{d}\sigma \\ &\qquad - \sum_{l=1}^{M} c_{il}(\bar{s}) \int_{-\infty}^{\bar{s}} K_{il}(\bar{s}-\sigma)v_l(\sigma)\,\mathrm{d}\sigma\biggr\}\,\mathrm{d}\bar{s}, \end{aligned}\tag{2.12}$$

$$\begin{aligned} v_j(t+\tau) - v_j(\tau) &= \int_{\tau}^{t+\tau} v_j(\bar{s})\biggl\{-r_j(\bar{s}) - e_{jj}(\bar{s})v_j(\bar{s}) + \sum_{k=1}^{N} d_{jk}(\bar{s}) \int_{-\infty}^{\bar{s}} P_{jk}(\bar{s}-\sigma)u_k(\sigma)\,\mathrm{d}\sigma \\ &\qquad - \sum_{l=1,l\neq j}^{M} e_{jl}(\bar{s}) \int_{-\infty}^{\bar{s}} Q_{jl}(\bar{s}-\sigma)v_l(\sigma)\,\mathrm{d}\sigma\biggr\}\,\mathrm{d}\bar{s}, \end{aligned}\tag{2.13}$$

for all $t \geqslant t_0$. Since $\tau \in \mathbb{R}$ is arbitrarily given,

$$U(t) = (u(t), v(t)) = (u_1(t), \dots, u_N(t), v_1(t), \dots, v_M(t))$$

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

is a solution of system (E) on $\mathbb{R}$. It is clear that $\alpha_i \leqslant u_i(t) \leqslant p_i$, $\beta_j \leqslant v_j(t) \leqslant q_j$ for all $t \in \mathbb{R}$. Thus, $U(t) \in S(E)$. This completes the proof of the theorem. □

By repeating almost the same argument as in the proof of Theorem 2.4, we can also prove the following theorem.

Theorem 2.5. *Let $U(t) \in S(E)$, and let a sequence $\{t_n\}$, $t_n \geqslant 0$, be given. Suppose that, for some functions $\bar{b}_i$, $\bar{r}_j$, $\bar{a}_{ik}$, $\bar{c}_{il}$, $\bar{d}_{jk}$ and $\bar{e}_{jl}$,*

(i) *$b_i(t+t_n) \to \bar{b}_i(t)$, $r_j(t+t_n) \to \bar{r}_j(t)$, $a_{ik}(t+t_n) \to \bar{a}_{ik}(t)$, $c_{il}(t+t_n) \to \bar{c}_{il}(t)$, $d_{jk}(t+t_n) \to \bar{d}_{jk}(t)$, $e_{jl}(t+t_n) \to \bar{e}_{jl}(t)$, as $n \to \infty$, uniformly on $t \in \mathbb{R}$;*

(ii) *$U(t+t_n) \to \bar{U}(t)$ as $n \to \infty$ uniformly on a subset of $\mathbb{R}$.*

Then $\bar{U}(t) \in S(\bar{E})$, where $S(\bar{E})$ is the set of solutions $W = (w, z)$ of the system

$$\left.\begin{aligned}\dot{w}_i(t) = w_i(t)\bigg[\bar{b}_i(t) - \bar{a}_{ii}(t)w_i(t) - \sum_{k=1,k\neq i}^{N} \bar{a}_{ik}(t)\int_{-\infty}^{t} H_{ik}(t-\sigma)w_k(\sigma)\,\mathrm{d}\sigma \\ - \sum_{l=1}^{M} \bar{c}_{il}(t)\int_{-\infty}^{t} K_{il}(t-\sigma)z_l(\sigma)\,\mathrm{d}\sigma\bigg], \\ \dot{z}_j(t) = z_j(t)\bigg[-\bar{r}_j(t) - \bar{e}_{jj}(t)z_j(t) + \sum_{k=1}^{N} \bar{d}_{jk}(t)\int_{-\infty}^{t} P_{jk}(t-\sigma)w_k(\sigma)\,\mathrm{d}\sigma \\ - \sum_{l=1,l\neq j}^{M} \bar{e}_{jl}(t)\int_{-\infty}^{t} Q_{jl}(t-\sigma)z_l(\sigma)\,\mathrm{d}\sigma\bigg]\end{aligned}\right\} \qquad (\bar{E})$$

on $\mathbb{R}$ satisfying $\alpha_i \leqslant w_i(t) \leqslant p_i$, $\beta_j \leqslant z_j(t) \leqslant q_j$ for all $t \in \mathbb{R}$.

When (i) and (ii) hold, we write $(\bar{U}, \bar{E}) \in \Omega(U, E)$. Based on Theorems 2.4 and 2.5, we will discuss the existence of the almost-periodic solution of system (E) in § 3.

3. Existence of a unique almost-periodic solution

We now state some definitions and lemmas which will be used in the proof of our main theorem (Theorem 3.8). The method is as follows: by using a series of definitions (especially the concept of *asymptotic almost-periodicity*), we prove first that system (E) has an asymptotic almost-periodic solution and then that system (E) has a unique almost-periodic solution.

Definition 3.1. A function $U \in S(E)$ is said to be *relatively uniformly stable* (RUS) in $\Omega(E)$ if, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ with the property that, for any $t_0 \geqslant 0$, any $(\bar{U}, \bar{E}) \in \Omega(U, E)$ and any $\bar{W} \in S(\bar{E})$ satisfying $\rho(\bar{U}^{t_0}, \bar{W}^{t_0}) < \delta(\varepsilon)$, we have $\rho(\bar{U}^t, \bar{W}^t) < \varepsilon$ for all $t \geqslant t_0$.

Definition 3.2. A function $U \in S(E)$ is said to be *relatively weakly uniformly asymptotically stable* (RWUAS) in $\Omega(E)$ if $U(t)$ is RUS in $\Omega(E)$, and if $\rho(\bar{U}^t, \bar{W}^t) \to 0$ as $t \to \infty$ for all $(\bar{U}, \bar{E}) \in \Omega(U, E)$ and all $\bar{W} \in S(\bar{E})$.

Definition 3.3. A function $U \in S(E)$ is said to be *relatively totally stable* (RTS) for (E) if, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ with the property that if $t_0 \geqslant 0$, $\rho(F^{t_0}, U^{t_0}) < \delta(\varepsilon)$ and $G = (f, g) : R \mapsto \mathbb{R}^{N+M}$ is any continuous function satisfying $\sup_{t\in\mathbb{R}} |G(t)| < \delta(\varepsilon)$, then we have $\rho(F^t, U^t) < \delta(\varepsilon)$ for all $t \geqslant t_0$, where $F = (x, y)$ is any solution of the system

$$\left.\begin{aligned} \dot{x}_i(t) &= x_i(t)\biggl[b_i(t) - a_{ii}(t)x_i(t) - \sum_{k=1,k\neq i}^{N} a_{ik}(t)\int_{-\infty}^{t} H_{ik}(t-\sigma)x_k(\sigma)\,\mathrm{d}\sigma \\ &\qquad - \sum_{l=1}^{M} c_{il}(t)\int_{-\infty}^{t} K_{il}(t-\sigma)y_l(\sigma)\,\mathrm{d}\sigma\biggr] + f_i(t), \\ \dot{y}_j(t) &= y_j(t)\biggl[-r_j(t) - e_{jj}(t)y_j(t) + \sum_{k=1}^{N} d_{jk}(t)\int_{-\infty}^{t} P_{jk}(t-\sigma)x_k(\sigma)\,\mathrm{d}\sigma \\ &\qquad - \sum_{l=1,\neq j}^{M} e_{jl}(t)\int_{-\infty}^{t} Q_{jl}(t-\sigma)y_l(\sigma)\,\mathrm{d}\sigma\biggr] + g_i(t) \end{aligned}\right\} \tag{E_G}$$

on $\mathbb{R}$ satisfying $\alpha_i \leqslant x_i(t) \leqslant p_i$, $\beta_j \leqslant y_j(t) \leqslant q_j$ for all $t \in \mathbb{R}$.

Definition 3.4 (Fink [7]). A continuous function is *asymptotic almost periodic* if and only if there is an almost-periodic function p and a continuous function q defined on $\mathbb{R}^+$ with $\lim_{t\to\infty} q(t) = 0$ such that

$$f(t) = p(t) + q(t).$$

Definition 3.5 (Fink [7]). For every sequence $\{h_k\}$, where $h_k > 0$ and $h_k \to \infty$ as $k \to \infty$, if there exists a subsequence $\{h_{kj}\}$ such that $f(t + h_{kj})$ is uniformly convergent on $[0, +\infty)$, then it is said that f has property $\boldsymbol{L}$.

Lemma 3.6. *If $U \in S(E)$ is RWUAS in $\Omega(E)$, then it is RTS for (E).*

Proof. We give the proof for completeness, although it is similar to the one for [**18**, Lemma 4]. Suppose the contrary. There then exists an $\varepsilon > 0$, sequences $\{\varepsilon_n\}$, $0 < \varepsilon_n < \varepsilon$, and $\varepsilon_n \to 0$ as $n \to \infty$, $\{s_n\}$, $\{t_n\}$, $t_n \geqslant s_n \geqslant 0$, $\{G_n\}$ and $\{F^n\}$ such that $G_n : R \to R^{N+M}$ is a continuous function satisfying $\sup_{t\in\mathbb{R}} |G_n(t)| < \varepsilon$ and that

$$\rho(U^{s_n}, (F^n)^{s_n}) < \varepsilon_n, \quad \rho(U^{t_n}, (F^n)^{t_n}) = \varepsilon, \quad \rho(U^t, (F^n)^t) < \varepsilon, \quad t \in [s_n, t_n), \tag{3.1}$$

where F^n is a solution of (E_{G_n}) on $\mathbb{R}$ satisfying $\alpha_i \leqslant (x^n)_i(t) \leqslant p_i$, $\beta_j \leqslant (y^n)_j(t) \leqslant q_j$ for all $t \in \mathbb{R}$.

Furthermore, by (3.1) we can choose a sequence $\{\tau_n\}$, $s_n < \tau_n < t_n$, so that

$$\rho(U^{\tau_n}, (F^n)^{\tau_n}) = \tfrac{1}{2}\delta(\tfrac{1}{2}\varepsilon) \tag{3.2}$$

and

$$\tfrac{1}{2}\delta(\tfrac{1}{2}\varepsilon) \leqslant \rho(U^t, (F^n)^t) \leqslant \varepsilon, \quad t \in [\tau_n, t_n], \tag{3.3}$$

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

where $\delta(\cdot)$ is the number given in Definition 3.1. We may assume that $U(t+\tau_n) \to \bar{U}(t)$ as $n \to \infty$ on each bounded subset of $\mathbb{R}$ for a continuous function $\bar{U}(t)$ and that $(\bar{U}, \bar{E}) \in \Omega(U, E)$. Moreover, we may assume that $F^n(\tau_n + t) \to \bar{W}(t)$ as $n \to \infty$ uniformly on any bounded subset of $\mathbb{R}$ for a continuous function $\bar{W}$, since the sequence $\{F^n(\tau_n + t)\}$ is uniformly bounded and equicontinuous on $\mathbb{R}$. Then, the same argument as in the proof of Theorem 2.4 shows that $\bar{W} \in S(\bar{E})$. Now, suppose that $t_n - \tau_n \to \infty$ as $n \to \infty$. Letting $n \to \infty$ in (3.3), we have $\frac{1}{2}\delta(\frac{1}{2}\varepsilon) \leqslant \rho(\bar{U}^t, \bar{W}^t) \leqslant \varepsilon$ for all $t \geqslant 0$. On the other hand, $\rho(\bar{U}^t, \bar{W}^t) \to 0$ as $t \to \infty$, since U is RWUAS in $\Omega(E)$. This is a contradiction. Thus, taking a subsequence if necessary, we may assume that $t_n - \tau_n \to r < \infty$ as $n \to \infty$. Let $n \to \infty$ in (3.2). We obtain $\rho(\bar{U}^0, \bar{W}^0) = \frac{1}{2}\delta(\frac{1}{2}\varepsilon) < \delta(\frac{1}{2}\varepsilon)$, and hence $\rho(\bar{U}^t, \bar{W}^t) < \frac{1}{2}\varepsilon$ for all $t \geqslant 0$, because U is RUS in $\Omega(E)$. On the other hand, from (3.1) we have $\rho(\bar{U}^r, \bar{W}^r) = \varepsilon$, which is a contradiction. This completes the proof of Lemma 3.6. □

Lemma 3.7 (see Chapter 1 in Fink [7] or Theorem 9.3 (1) in Burton and Hutson [4]). *The asymptotic almost-periodicity of $f(t)$ is equivalent to $f(t)$ having property **L**.*

We now state our main result on the existence of a unique almost-periodic solution of system (E).

Theorem 3.8. *If system (E) satisfies* (H_1)–(H_3) *and*

(H_4) *there exist strictly positive constants s_i, θ_j such that*

$$s_i a_{ii}^{\iota} > \sum_{k=1,k\neq i}^{N} s_k a_{ki}^{\mu} + \sum_{l=1}^{M} \theta_l c_{li}^{\mu},$$

$$\theta_j e_{jj}^{\iota} > \sum_{k=1}^{N} s_k d_{kj}^{\mu} + \sum_{l=1,l\neq j}^{M} \theta_l e_{lj}^{\mu},$$

then system (E) has a unique almost-periodic solution $Q(t)$ in $\Omega(E)$.

Proof. Let $U \in S(E)$. First of all, we shall prove that U is RTS for system (E). By Lemma 3.6, it suffices to show that U is RWUAS in $\Omega(E)$. For arbitrary $(\bar{U}, \bar{E}) \in \Omega(U, E)$ and $\bar{W} \in S(\bar{E})$, let

$$\begin{aligned} V(t) &= V(t, \bar{U}(\cdot), \bar{W}(\cdot)) \\ &= \sum_{i=1}^{N} s_i \biggl[|\ln \bar{u}_i(t) - \ln \bar{w}_i(t)| \\ &\qquad + \sum_{k=1,k\neq i}^{N} \int_0^{\infty} H_{ik}(s) \biggl\{ \int_{t-s}^{t} \bar{a}_{ik}(s+\sigma) |\bar{u}_k(\sigma) - \bar{w}_k(\sigma)| \,\mathrm{d}\sigma \biggr\} \mathrm{d}s \\ &\qquad\qquad + \sum_{l=1}^{M} \int_0^{\infty} K_{il}(s) \biggl\{ \int_{t-s}^{t} \bar{c}_{il}(s+\sigma) |\bar{v}_l(\sigma) - \bar{z}_l(\sigma)| \,\mathrm{d}\sigma \biggr\} \mathrm{d}s \biggr] \end{aligned}$$

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

$$+\sum_{j=1}^{M}\theta_j\bigg[|\ln\bar v_j(t)-\ln\bar z_j(t)| + \sum_{k=1}^{N}\int_0^\infty P_{jk}(s)\bigg\{\int_{t-s}^{t}\bar d_{jk}(s+\sigma)|\bar u_k(\sigma)-\bar w_k(\sigma)|\,\mathrm{d}\sigma\bigg\}\,\mathrm{d}s + \sum_{l=1,l\neq j}^{M}\int_0^\infty Q_{jl}(s)\bigg\{\int_{t-s}^{t}\bar e_{jl}(s+\sigma)|\bar v_l(\sigma)-\bar z_l(\sigma)|\,\mathrm{d}\sigma\bigg\}\,\mathrm{d}s\bigg]. \tag{3.4}$$

We denote

$$\lambda=\min_{i,j}\{s_i,\theta_j\},\qquad \Lambda=\max_{i,j}\{s_i,\theta_j\}.$$

Calculating the right derivative $D^+V(t)$ of V along system (E)–(2.1), we obtain

$$\begin{aligned}D^+V(t)=\sum_{i=1}^{N}s_i\bigg\{&\operatorname{sgn}\{\bar u_i(t)-\bar w_i(t)\}\\ &\times\bigg[-\bar a_{ii}(t)(\bar u_i(t)-\bar w_i(t))\\ &\quad-\sum_{k=1,k\neq i}^{N}\bar a_{ik}(t)\int_{-\infty}^{t}H_{ik}(t-\sigma)(\bar u_k(\sigma)-\bar w_k(\sigma))\,\mathrm{d}\sigma\\ &\quad-\sum_{l=1}^{M}\bar c_{il}(t)\int_{-\infty}^{t}K_{il}(t-\sigma)(\bar v_l(\sigma)-\bar z_l(\sigma))\,\mathrm{d}\sigma\\ &\quad+\sum_{k=1,k\neq i}^{N}\int_0^\infty H_{ik}(s)\bar a_{ik}(s+t)\,\mathrm{d}s|\bar u_k(t)-\bar w_k(t)|\\ &\quad+\sum_{k=1,k\neq i}^{N}\bar a_{ik}(t)\int_{-\infty}^{t}H_{ik}(t-\sigma)|\bar u_k(\sigma)-\bar w_k(\sigma)|\,\mathrm{d}\sigma\\ &\quad+\sum_{l=1}^{M}\int_0^\infty K_{il}(s)\bar c_{il}(s+t)\,\mathrm{d}s|\bar v_l(t)-\bar z_l(t)|\\ &\quad+\sum_{l=1}^{M}\bar c_{il}(t)\int_{-\infty}^{t}K_{il}(t-\sigma)|\bar v_l(\sigma)-\bar z_l(\sigma)|\,\mathrm{d}\sigma\bigg]\bigg\}\\ +\sum_{j=1}^{M}\theta_j\bigg\{&\operatorname{sgn}\{\bar v_j(t)-\bar z_j(t)\}\\ &\times\bigg[-\bar e_{jj}(t)(\bar v_j(t)-\bar z_j(t))\\ &\quad+\sum_{k=1}^{N}\bar d_{jk}(t)\int_{-\infty}^{t}P_{jk}(t-\sigma)(\bar u_k(\sigma)-\bar w_k(\sigma))\,\mathrm{d}\sigma\end{aligned}$$

$$
\begin{aligned}
&- \sum_{l=1,l\neq j}^{M} \bar{e}_{jl}(t) \int_{-\infty}^{t} Q_{jl}(t-\sigma)(\bar{v}_l(\sigma)-\bar{z}_l(\sigma))\,\mathrm{d}\sigma \\
&+ \sum_{k=1}^{N} \int_{0}^{\infty} P_{jk}(s)\bar{d}_{jk}(s+t)\,\mathrm{d}s|\bar{u}_k(t)-\bar{w}_k(t)| \\
&- \sum_{k=1}^{N} \bar{d}_{jk}(t) \int_{-\infty}^{t} P_{jk}(t-\sigma)|\bar{u}_k(\sigma)-\bar{w}_k(\sigma)|\,\mathrm{d}\sigma \\
&+ \sum_{l=1,l\neq j}^{M} \int_{0}^{\infty} Q_{jl}(s)\bar{e}_{jl}(s+t)|\bar{v}_l(t)-\bar{z}_l(t)| \\
&\qquad + \sum_{l=1,l\neq j}^{M} \bar{e}_{jl}(t) \int_{-\infty}^{t} Q_{jl}(t-\sigma)|\bar{v}_l(\sigma)-\bar{z}_l(\sigma)|\,\mathrm{d}\sigma \Bigg] \Bigg\}
\end{aligned}
$$

By simplifying, one obtains

$$
\begin{aligned}
D^+V(t) &\leqslant \sum_{i=1}^{N} \Bigg[\sum_{k=1,k\neq i}^{N} s_k a_{ki}^{\mu} + \sum_{l=1}^{M} \theta_l c_{li}^{\mu} - s_i a_{ii}^{\iota} \Bigg] |\bar{u}_i(t)-\bar{w}_i(t)| \\
&\quad + \sum_{j=1}^{M} \Bigg[\sum_{k=1}^{N} s_k d_{kj}^{\mu} + \sum_{l=1,l\neq j}^{M} \theta_l e_{lj}^{\mu} - \theta_j e_{jj}^{\iota} \Bigg] |\bar{v}_j(t)-\bar{z}_j(t)| \\
&\leqslant -\gamma \Bigg[\sum_{i=1}^{N} |\bar{u}_i(t)-\bar{w}_i(t)| + \sum_{j=1}^{M} |\bar{v}_j(t)-\bar{z}_j(t)| \Bigg], \qquad (3.5)
\end{aligned}
$$

where

$$
\gamma = \min_{i,j} \Bigg\{ s_i a_{ii}^{\iota} - \sum_{k=1,k\neq i}^{N} s_k a_{ki}^{\mu} - \sum_{l=1}^{M} \theta_l c_{li}^{\mu}, \theta_j e_{jj}^{\iota} - \sum_{k=1}^{N} s_k d_{kj}^{\mu} - \sum_{l=1,l\neq j}^{M} \theta_l e_{lj}^{\mu} \Bigg\}.
$$

From (H_4), we know that $\gamma > 0$. Integrating (3.5) over $[0, t]$, it follows that

$$
V(t) + \gamma \int_0^t \Bigg[\sum_{i=1}^{N} |\bar{u}_i(s)-\bar{w}_i(s)| + \sum_{j=1}^{M} |\bar{v}_j(s)-\bar{z}_j(s)| \Bigg] \mathrm{d}s < V(0) < +\infty \quad \text{for } t \geqslant 0. \qquad (3.6)
$$

Consequently,

$$
\int_0^t \Bigg[\sum_{i=1}^{N} |\bar{u}_i(s)-\bar{w}_i(s)| + \sum_{j=1}^{M} |\bar{v}_j(s)-\bar{z}_j(s)| \Bigg] \mathrm{d}s < \infty.
$$

Since the function

$$
\sum_{i=1}^{N} |\bar{u}_i(t)-\bar{w}_i(t)| + \sum_{j=1}^{M} |\bar{v}_j(t)-\bar{z}_j(t)|
$$

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

is uniformly continuous on $[0,+\infty)$, we have

$$\sum_{i=1}^{N}|\bar{u}_i(t)-\bar{w}_i(t)|+\sum_{j=1}^{M}|\bar{v}_j(t)-\bar{z}_j(t)|\to 0,$$

and thus $\rho(\bar{U}^t,\bar{W}^t)\to 0$ as $t\to\infty$. Moreover, from (3.4) and (3.6), it follows that

$$\begin{aligned}
&\sum_{i=1}^{N}|\ln\bar{u}_i(t)-\ln\bar{w}_i(t)|+\sum_{j=1}^{M}|\ln\bar{v}_j(t)-\ln\bar{z}_j(t)|\\
&\quad\leqslant\frac{V(t)}{\lambda}\leqslant\frac{V(t_0)}{\lambda}\\
&\quad\leqslant\frac{2x^*\Lambda}{\lambda}\sum_{i=1}^{n}\biggl[\sum_{k=1,k\neq i}^{N}a_{ik}^{\mu}\int_L^\infty sH_{ik}(s)\,\mathrm{d}s+\sum_{l=1}^{M}c_{il}^{\mu}\int_L^\infty sK_{il}(s)\,\mathrm{d}s\biggr]\\
&\qquad+\frac{2x^*\Lambda}{\lambda}\sum_{j=1}^{M}\biggl[\sum_{k=1}^{N}d_{jk}^{\mu}\int_L^\infty sP_{jk}(s)\,\mathrm{d}s+\sum_{l=1,l\neq j}^{M}e_{jl}^{\mu}\int_L^\infty sQ_{jl}(s)\,\mathrm{d}s\biggr]\\
&\qquad+\frac{1}{\lambda}\sum_{i=1}^{N}s_i\biggl\{|\ln\bar{u}_i(t_0)-\ln\bar{w}_i(t_0)|\\
&\qquad\qquad+\sum_{k=1,k\neq i}^{N}a_{ik}^{\mu}\int_0^\infty sH_{ik}(s)\,\mathrm{d}s\sup_{t_0-L\leqslant\sigma\leqslant t_0}|\bar{u}_k(\sigma)-\bar{w}_k(\sigma)|\\
&\qquad\qquad\quad+\sum_{l=1}^{M}c_{il}^{\mu}\int_0^\infty sK_{il}(s)\,\mathrm{d}s\sup_{t_0-L\leqslant\sigma\leqslant t_0}|\bar{v}_l(\sigma)-\bar{z}_l(\sigma)|\biggr\}\\
&\qquad+\frac{1}{\lambda}\sum_{j=1}^{M}\theta_j\biggl\{|\ln\bar{v}_j(t_0)-\ln\bar{z}_j(t_0)|\\
&\qquad\qquad+\sum_{k=1}^{N}d_{jk}^{\mu}\int_0^\infty sP_{jk}(s)\,\mathrm{d}s\sup_{t_0-L\leqslant\sigma\leqslant t_0}|\bar{u}_k(\sigma)-\bar{w}_k(\sigma)|\\
&\qquad\qquad\quad+\sum_{l=1,l\neq j}^{M}e_{jl}^{\mu}\int_0^\infty sQ_{jl}(s)\,\mathrm{d}s\sup_{t_0-L\leqslant\sigma\leqslant t_0}|\bar{v}_l(\sigma)-\bar{z}_l(\sigma)|\biggr\}.
\end{aligned}\tag{3.7}$$

for all $t\geqslant t_0\geqslant 0$, and all $L\geqslant 0$. For each $\varepsilon>0$ we set

$$\begin{aligned}
\bar{\delta}(\varepsilon)=\inf\biggl\{&\sum_{i=1}^{N}|\ln u_i(t)-\ln w_i(t)|+\sum_{j=1}^{M}|\ln v_j(t)-\ln z_j(t)|:|U-W|\geqslant\varepsilon\\
&\text{and }\alpha_i\leqslant u_i(t),\ w_i(t)\leqslant p_i,\ \beta_j\leqslant v_j(t),\ z_j(t)\leqslant q_j\biggr\}.
\end{aligned}\tag{3.8}$$

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

Obviously, $\bar{\delta}(\varepsilon) > 0$. We select a number $L > 0$ so large that

$$\begin{aligned}\bigg\{2x^*\Lambda\sum_{i=1}^{N}\bigg[\sum_{k=1,k\neq i}^{N}a_{ik}^{\mu}\int_L^{\infty}sH_{ik}(s)\,\mathrm{d}s+\sum_{l=1}^{M}c_{il}^{\mu}\int_L^{\infty}sK_{il}(s)\,\mathrm{d}s\bigg]\\+2x^*\Lambda\sum_{j=1}^{M}\bigg[\sum_{k=1}^{N}d_{jk}^{\mu}\int_L^{\infty}sP_{jk}(s)\,\mathrm{d}s+\sum_{l=1,l\neq j}^{M}e_{jl}^{\mu}\int_L^{\infty}sQ_{jl}(s)\,\mathrm{d}s\bigg]\bigg\}<\tfrac{1}{2}\lambda\bar{\delta}(\varepsilon),\end{aligned}$$

which is possible by (H_2). Moreover, we select a $\delta(\varepsilon) \in (0, \varepsilon)$ so that

$$\begin{aligned}\sum_{i=1}^{N}s_i\bigg[|\ln\phi_i(0)-\ln\xi_i(0)|+\sum_{k=1,k\neq i}^{N}a_{ik}^{\mu}\int_0^{\infty}sH_{ik}(s)\,\mathrm{d}s\sup_{t_0-L\leqslant\sigma\leqslant t_0}|\phi_k(\sigma)-\xi_k(\sigma)|\\+\sum_{l=1}^{M}c_{il}^{\mu}\int_0^{\infty}sK_{il}(s)\,\mathrm{d}s\sup_{t_0-L\leqslant\sigma\leqslant t_0}|\psi_l(\sigma)-\eta_l(\sigma)|\bigg]\\+\sum_{j=1}^{M}\theta_j\bigg[|\ln\psi_j(0)-\ln\eta_j(0)|+\sum_{k=1}^{N}d_{jk}^{\mu}\int_0^{\infty}sP_{jk}(s)\,\mathrm{d}s\sup_{t_0-L\leqslant\sigma\leqslant t_0}|\phi_k(\sigma)-\xi_k(\sigma)|\\+\sum_{l=1,l\neq j}^{M}e_{jl}^{\mu}\int_0^{\infty}sQ_{jl}(s)\,\mathrm{d}s\sup_{t_0-L\leqslant\sigma\leqslant t_0}|\psi_l(\sigma)-\eta_l(\sigma)|\bigg]<\tfrac{1}{2}\lambda\bar{\delta}(\varepsilon),\end{aligned}$$

whenever $\rho(\Phi,\Psi) < \delta(\varepsilon)$. Hence, if $\rho(\bar{U}^{t_0}, \bar{W}^{t_0}) < \delta(\varepsilon)$, we have

$$\sum_{i=1}^{N}|\ln\bar{u}_i(t)-\ln\bar{w}_i(t)|+\sum_{j=1}^{M}|\ln\bar{v}_j(t)-\ln\bar{z}_j(t)|<\bar{\delta}(\varepsilon).$$

By (3.7) and (3.8), consequently, $|\bar{U}(t) - \bar{W}(t)| < \varepsilon$ for all $t \geqslant t_0$. Thus, if $\rho(\bar{U}^{t_0}, \bar{W}^{t_0}) < \delta(\varepsilon)$, then

$$\begin{aligned}\rho(\bar{U}^t,\bar{W}^t)&\leqslant\sum_{n=1}^{\infty}\frac{\rho_n(\bar{U}^{t_0},\bar{W}^{t_0})+\varepsilon}{2^n(1+\rho_n(\bar{U}^{t_0},\bar{W}^{t_0})+\varepsilon}\\&\leqslant\sum_{n=1}^{\infty}2^{-n}\frac{\rho_n(\bar{U}^{t_0},\bar{W}^{t_0})}{(1+\rho_n(\bar{U}^{t_0},\bar{W}^{t_0})+(\varepsilon/(1+\varepsilon))}\\&\leqslant\delta(\varepsilon)+\varepsilon\\&<2\varepsilon,\end{aligned}$$

for all $t \geqslant t_0$. Note that $\delta(\cdot)$ is independent of the particular choice of $\bar{U}, \bar{W} \in S(\bar{E})$. Therefore, each $U \in S(E)$ is RWUAS in $\Omega(E)$.

Next, we shall prove that each $U \in S(E)$ is *asymptotically almost periodic*. Let $\{t_n\}$ be any sequence satisfying $t_n \to \infty$ as $n \to \infty$. We may assume that the sequence $\{U(t + t_n)\}_{n=1}^{\infty}$ is uniformly convergent on each bounded subset of $\mathbb{R}$ and that the

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

sequences

$$\{b_i(t+t_n)\}_{n=1}^{\infty},\qquad \{r_j(t+t_n)\}_{n=1}^{\infty},\qquad \{a_{ik}(t+t_n)\}_{n=1}^{\infty},$$
$$\{c_{il}(t+t_n)\}_{n=1}^{\infty},\qquad \{d_{jk}(t+t_n)\}_{n=1}^{\infty},\qquad \{e_{jl}(t+t_n)\}_{n=1}^{\infty}$$

are uniformly convergent on $\mathbb{R}$. Set $U^m(t) = U(t+t_m)$, $t \in \mathbb{R}$, for each positive integer m. Clearly, U^m is a solution of the system

$$\left.\begin{aligned} \dot{x}_i(t) &= x_i(t)\biggl[b_i(t+t_m) - a_{ii}(t+t_m)x_i(t) - \sum_{k=1,k\neq i}^{N} a_{ik}(t+t_m) \\ &\qquad \times \int_{-\infty}^{t} H_{ik}(t-\sigma)x_k(\sigma)\,\mathrm{d}\sigma - \sum_{l=1}^{M} c_{il}(t+t_m)\int_{-\infty}^{t} K_{il}(t-\sigma)y_l(\sigma)\,\mathrm{d}\sigma\biggr], \\ \dot{y}_j(t) &= y_j(t)\biggl[-r_j(t+t_m) - e_{jj}(t+t_m)y_j(t) + \sum_{k=1}^{N} d_{jk}(t+t_m) \\ &\qquad \times \int_{-\infty}^{t} P_{jk}(t-\sigma)x_k(\sigma)\,\mathrm{d}\sigma - \sum_{l=1,l\neq j}^{M} e_{jl}(t+t_m)\int_{-\infty}^{t} Q_{jl}(t-\sigma)y_l(\sigma)\,\mathrm{d}\sigma\biggr] \end{aligned}\right\} \tag{E^m}$$

on $\mathbb{R}$ and it is RTS for system (E^m) with the common number $\delta(\cdot)$, since U is RTS for system (E) with the common number $\delta(\cdot)$. For any positive integers m and n, we define a continuous function $G_{mn} : R \to R^{N+M}$ by

$$G_{mn}(t) = (f_{mn1}(t), \dots, f_{mnN}(t), g_{mn1}(t), \dots, g_{mnM}(t)),$$

where

$$\begin{aligned} f_{mni}(t) = u_i(t+t_n)\biggl[&b_i(t+t_n) - b_i(t+t_m) - (a_{ii}(t+t_n) - a_{ii}(t+t_m))u_i(t+t_n) \\ &- \sum_{k=1,k\neq i}^{N} (a_{ik}(t+t_n) - a_{ik}(t+t_m))\int_{-\infty}^{t} H_{ik}(t-\sigma)u_k(\sigma+t_n)\,\mathrm{d}\sigma \\ &- \sum_{l=1}^{M} (c_{il}(t+t_n) - c_{il}(t+t_m))\int_{-\infty}^{t} K_{il}(t-\sigma)v_l(\sigma+t_n)\,\mathrm{d}\sigma\biggr], \\ g_{mnj}(t) = v_j(t+t_n)\biggl[&-(r_j(t+t_n) - r_j(t+t_m)) - (e_{jj}(t+t_n) - e_{jj}(t+t_m))v_j(t+t_n) \\ &+ \sum_{k=1}^{N} (d_{jk}(t+t_n) - d_{jk}(t+t_m))\int_{-\infty}^{t} P_{jk}(t-\sigma)u_k(\sigma+t_n)\,\mathrm{d}\sigma \\ &- \sum_{l=1,l\neq j}^{M} (e_{jl}(t+t_n) - e_{jl}(t+t_m))\int_{-\infty}^{t} Q_{jl}(t-\sigma)v_l(\sigma+t_n)\,\mathrm{d}\sigma\biggr]. \end{aligned}$$

Now, for any $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that $\sup_{t\in R}|G_{mn}(t)| < \delta(\varepsilon)$ and $\rho((U^m)^0, (U^n)^0) < \delta(\varepsilon)$ if $m, n \geqslant n_0(\varepsilon)$. Then, the fact that U^m is RTS for

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

system (E_m) implies that $\rho((U^m)^t,(U^n)^t)<\varepsilon$ for all $t\geqslant 0$ if $m,n\geqslant n_0(\varepsilon)$, since U^m is a solution of system $(E^m_{G_{mn}})$ on $\mathbb{R}$ and $\alpha_i\leqslant (u_i^n)(t)\leqslant p_i$, $\beta_j\leqslant (v_j)^n(t)\leqslant q_j$ for all $t\in\mathbb{R}$. Thus, the sequence $\{U(t+t_n)\}_{n=1}^{\infty}$ is uniformly convergent on $[0,\infty)$, which shows that $U(t)$ is asymptotically almost-periodic, by Lemma 3.7, that is, $U(t)$ is the sum of an almost-periodic function $Q(t)$ and a continuous function $\gamma(t)$ defined on $\mathbb{R}$ such that $U(t)=Q(t)+\gamma(t)$, $t\in\mathbb{R}$, and $\gamma(t)\to 0$ as $t\to\infty$.

Finally, we show that Q is a unique almost-periodic solution in $S(E)$. We choose a sequence $\{s_n\}$, $s_n\to\infty$, as $n\to\infty$ such that

$$b_i(t+s_n)\to b_i(t),\quad r_j(t+s_n)\to r_j(t),\quad a_{ik}(t+s_n)\to a_{ik}(t),\quad c_{il}(t+s_n)\to c_{il}(t),$$
$$d_{jk}(t+s_n)\to d_{jk}(t),\quad e_{jl}(t+s_n)\to e_{jl}(t),\quad Q(t+s_n)\to Q(t),$$

as $n\to\infty$ uniformly on $\mathbb{R}$. Then, $Q\in S(E)$ by Theorem 2.4. Let $\bar{Q}$ be another almost-periodic solution in $S(E)$. Since $Q\in S(E)$ is RWUAS in $\Omega(E)$, as was shown in the first paragraph of the proof of the theorem, we obtain $\rho(Q^t,\bar{Q}^t)\to 0$ as $t\to\infty$ and hence $|Q(t)-\bar{Q}(t)|\to 0$ as $t\to\infty$. Hence, $Q\equiv\bar{Q}$ on $\mathbb{R}$, by the almost-periodicity of Q and $\bar{Q}$. Thus, system (E) has Q as a unique almost-periodic solution in $S(E)$. This completes the proof of Theorem 3.8. □

Corollary 3.9. *Under the assumptions* (H$_2$)–(H$_4$) *and supposing that* b_i, r_j, a_{ik}, c_{il}, d_{jk} *and* e_{jl} *are all* ω*-periodic, system (E) then has a unique* ω*-periodic solution in* $S(E)$.

Proof. By Theorem 3.8, let F be the unique positive almost-periodic solution of system (E), but in the periodic case, b_i, r_j, a_{ik}, c_{il}, d_{jk} and e_{jl} are all ω-periodic. Therefore, $F(t+\omega)$ is also an almost-periodic solution of system (E). By the uniqueness of almost-periodic solutions, it follows that $F(t)=F(t+\omega)$ for all $t\in\mathbb{R}$. This completes the proof of Corollary 3.9. □

When $M=0$, system (E) degenerates to the Lotka–Volterra competition system

$$x_i(t)=x_i(t)\biggl[b_i(t)-a_{ii}(t)x_i(t)-\sum_{k=1,k\neq i}^{N}a_{ik}(t)\int_{-\infty}^{t}H_{ik}(t-\sigma)x_k(\sigma)\,\mathrm{d}\sigma\biggr],\quad i=1,\dots,N, \tag{3.9}$$

which was considered by Gopalsamy [**11**].

Corollary 3.10. *If system (3.9) satisfies the conditions*

(i) b_i, a_{ik} *are all periodic with* $b_i^{\iota}>0$ *and* $a_{ii}^{\iota}>0$,

(ii) $\displaystyle\int_0^{\infty}H_{ik}(s)\,\mathrm{d}s=1,\ \int_0^{\infty}sH_{ik}(s)\,\mathrm{d}s<\infty,\ i\neq k,$

(iii) $\displaystyle b_i^{\iota}>\sum_{k=1,k\neq i}^{N}a_{ik}^{\mu}\biggl(\frac{b_k^{\iota}}{a_{kk}^{\mu}}\biggr),$

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

(iv) *there exist strictly positive constants s_i such that*

$$s_i \min_{t\in[0,\omega]} a_{ii}(t) > \sum_{k=1,k\neq i}^{N} \Big(\max_{t\in[0,\omega]} s_k a_{ki}(t)\Big),$$

then system (3.9) has a unique ω-periodic solution.

Remark 3.11. Gopalsamy [**11**] has studied system (3.9). It has been proved that if (i)–(iii) hold and

$$\min_{t\in[0,\omega]} a_{ii}(t) > \sum_{k=1,k\neq i}^{N} \Big(\max_{t\in[0,\omega]} a_{ki}(t)\Big), \tag{3.10}$$

then system (3.9) has a unique ω-periodic solution. Obviously, the results in [**11**] are a special case of Corollary 3.10. In fact, we take $s_i \equiv 1$; the result in [**11**] follows. Therefore, this result generalizes and improves the main results in [**11**]. In particular, when $N = 1$, $M = 0$, we also generalize the results in [**19**].

4. Some examples

Example 4.1. We consider the following system:

$$\left.\begin{aligned}
\dot{x}_1(t) &= x_1(t)\bigg[5 + \sin 2t - 2x_1(t) - \frac{1}{2}\int_{-\infty}^{t} \exp\{-(t-u)\}x_2(u)\,\mathrm{d}u \\
&\qquad - \tfrac{1}{4}(1+\sin\sqrt{3}t)\int_{-\infty}^{t} \exp\{-(t-u)\}y_1(u)\,\mathrm{d}u\bigg], \\
\dot{x}_2(t) &= x_2(t)\bigg[5 + \cos 2t - \frac{1}{2}\int_{-\infty}^{t} \exp\{-(t-u)\}x_1(u)\,\mathrm{d}u - 2x_2(t) \\
&\qquad - \tfrac{1}{4}(1+\sin\sqrt{3}t)\int_{-\infty}^{t} \exp\{-(t-u)\}y_1(u)\,\mathrm{d}u\bigg], \\
\dot{y}_1(t) &= y_1(t)\bigg[-\frac{1+\sin 3t}{20} + \frac{1}{2}\int_{-\infty}^{t} \exp\{-(t-u)\}x_1(u)\,\mathrm{d}u \\
&\qquad + \frac{1}{2}\int_{-\infty}^{t} \exp\{-(t-u)\}x_2(u)\,\mathrm{d}u - y_1(t)\bigg].
\end{aligned}\right\} \tag{4.1}$$

Corresponding to system (E), we have $b_1(t) = 5 + \sin 2t$, $b_2(t) = 5 + \cos 2t$, $r_1(t) = \frac{1}{20}(1 + \sin 3t)$, $a_{11}(t) = a_{22}(t) = 2$, $d_{11}(t) = d_{12}(t) = \frac{1}{2}$, $c_{11}(t) = c_{12}(t) = \frac{1}{4}(1 + \sin\sqrt{3}t)$ and $e_{11}(t) = 1$. It is easy to verify that our result applies to (4.1). In fact,

(i) $\alpha_1 = \alpha_2 = \frac{1}{2} > 0$, $q_1 = 3 > 0$, $\beta_1 = \frac{2}{5} > 0$,

(ii) letting $s_1 = s_2 = 1$, $\theta_1 = \frac{3}{2}$, we can obtain

$$\begin{aligned}
s_1 a_{11}^{\iota} &= 1\times 2 > s_2 a_{21}^{\mu} + \theta_1 d_{11}^{\mu} = 1\times\tfrac{1}{2} + \tfrac{3}{2}\times\tfrac{1}{2}, \\
s_2 a_{22}^{\iota} &= 1\times 2 > s_1 a_{12}^{\mu} + \theta_1 d_{12}^{\mu} = 1\times\tfrac{1}{2} + \tfrac{3}{2}\times\tfrac{1}{2}, \\
\theta_1 e_{11}^{\iota} &= \tfrac{3}{2}\times 1 > s_1 c_{11}^{\mu} + s_2 c_{12}^{\mu} = 1\times\tfrac{1}{2} + 1\times\tfrac{1}{2}.
\end{aligned}$$

https://doi.org/10.1017/S0013091504001233 Published online by Cambridge University Press

Thus, Theorem 3.8 applies, and shows that there is a unique almost-periodic solution of (4.1).

Example 4.2. Consider the following system:

$$\left.\begin{aligned}\dot{x}_1(t) &= x_1(t)\left[5+\sin t - 2x_1(t) - \tfrac{1}{8}(1+\sin t)\int_{-\infty}^{t}\exp\{-(t-u)\}x_2(u)\,\mathrm{d}u\right],\\ \dot{x}_2(t) &= x_2(t)\left[9+\cos t - 2\int_{-\infty}^{t}\exp\{-(t-u)\}x_1(u)\,\mathrm{d}u - 2x_2(t)\right].\end{aligned}\right\} \tag{4.2}$$

Corresponding to system (E), we have $b_1(t) = 5+\sin t$, $b_2(t) = 9+\cos t$, $a_{11}(t) = a_{22}(t) = 2$, $a_{12}(t) = \frac{1}{8}(1+\sin t)$, $a_{21}(t) = 2$ and $H_{12}(t) = H_{21}(t) = \exp(-t)$.

Since $\inf_{t\in\mathbb{R}} a_{11}(t) = 2 = \sup_{t\in\mathbb{R}} a_{21} = 2$, we cannot apply [**11**, Theorem 2.1] to system (4.2). However, it is easy to verify that our result applies to (4.2). In fact, putting $s_1 = 1$ and $s_2 = \frac{1}{2}$ yields

$$\begin{aligned} s_1 a_{11}^{\iota}(t) &= 1\times 2 > s_2 a_{21}^{\mu}(t) = \tfrac{1}{2}\times 2,\\ s_2 a_{22}^{\iota}(t) &= \tfrac{1}{2}\times 2 > s_1 a_{12}^{\mu}(t) = 1\times\tfrac{1}{4}.\end{aligned}$$

Thus, from Corollary 3.10, the system (4.2) has a unique periodic solution.

Acknowledgements. The work described here was jointly supported by the National Natural Science Foundation of China under Grant no. 60574043, the Natural Science Foundation of Jiangsu Province, China, under Grant no. BK2006093, the Qing-Lan Engineering Project of Jiangsu Province, China, the Foundation of Developing Science and Technology of Fuzhou University under Grant no. 0030824594, the Foundation of Fujian Education Bureau, under Grant no. JB05048, and the Natural Science Foundation of Fujian Province, China.

The authors thank the referees and editor for their valuable suggestions and comments, which improved the presentation of this paper.

References

1. C. Alvarez and A. C. Lazer, An application of topological degree to the periodic competing species problem, *J. Austral. Math. Soc.* B **28** (1986), 202–219.
2. M. Araki and M. Kondo, Stability and transient behavior of composite systems, *IEEE Trans. Automat. Contr.* **17** (1972), 537–541.
3. E. Beretta and Y. Kuang, Convergence results in a well-known delayed predator–prey system, *J. Math. Analysis Applic.* **204** (1996), 840–853.
4. T. Burton and V. Hutson, Repellers in systems with infinite delay, *J. Math. Analysis Applic.* **37** (1989), 240–263.
5. L. Chen, *Mathematical models and methods in ecology* (Scientific Press, Beijing, 1988; in Chinese).
6. J. C. Eilbeck and J. Lopez-Gomez, On the periodic Lotka–Volterra competition model, *J. Math. Analysis Applic.* **210** (1997), 58–87.
7. A. M. Fink, *Almost periodic differential equations*, Lecture Notes in Mathematics, Volume 377 (Springer, 1974).

8. H. I. Freedman and S. Ruan, Uniform persistence in functional differential equations, *J. Diff. Eqns* **115** (1995), 173–192.
9. K. Gopalsamy, Exchange of equilibria in two-species Lotka–Volterra competition models, *J. Austral. Math. Soc.* B **24** (1982), 160–170.
10. K. Gopalsamy, Global asymptotic stability in a periodic Lotka–Volterra system, *J. Austral. Math. Soc.* B **27** (1985), 66–72.
11. K. Gopalsamy, Dynamics of an almost periodic integro-differential equation, *Tohoku Math. J.* **37** (1985), 323–332.
12. J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.* **21** (1978), 11–41.
13. Y. Hino, S. Murakami and T. Naito, *Functional differential equations with infinite delay*, Lecture Notes in Mathematics, Volume 1473 (Springer, 1991).
14. P. Korman, Some new results on the periodic competition model, *J. Math. Analysis Applic.* **171** (1992), 131–138.
15. Y. Kuang, *Delay differential equations with application in population dynamics* (Academic Press, New York, 1993).
16. J. Lopez-Gomez, R. Ortega and A. Tineo, The periodic predator–prey Lotka–Volterra model, *Adv. Diff. Eqns* **1** (1996), 403–423.
17. Z. Ma and W. Wang, Asymptotic behavior of predator–prey system with time-dependent coefficients, *Applic. Analysis* **34** (1989), 79–90.
18. S. Murakami, Almost periodic solutions of a system of integro-differential equations, *Tohoku Math. J.* **39** (1987), 71–79.
19. G. Seifert, Almost periodic solutions delay differential equations with infinite delays, *J. Diff. Eqns* **41** (1981), 416–425.
20. Z. Teng, On the positive almost periodic solutions of a class of Lotka–Volterra type systems with delays, *J. Math. Analysis Applic.* **249** (2000), 433–444.
21. X. Wen, Global attractivity of positive solution of multispecies ecological competition–predator delay system, *Acta Math. Sinica* **45** (2002), 83–92.
22. Y. Xia and J. Cao, Almost periodicity in an ecological model with M-predators and N-preys by 'pure-delay type' system, *Nonlin. Dynam.* **39** (2005), 275–304.
23. Y. Xia, F. Chen, A. Chen and J. Cao, Existence and global attractivity of an almost periodic ecological model, *Appl. Math. Computat.* **157** (2004), 449–475.
24. Y. Xia, J. Cao, H. Zhang and F. Chen, Almost periodic solutions of n-species competitive system with feedback controls, *J. Math. Analysis Applic.* **294** (2004), 503–522.
25. P. Yang and R. Xu, Global attractivity of the periodic Lotka–Volterra system, *J. Math. Analysis Applic.* **233** (1999), 221–232.
26. T. Yoshizawa, *Stability theory for the existence of periodic solutions and almost-periodic solutions*, Applied Mathematical Science, Volume 14 (Springer, 1975).
27. J. Zhao and W. Chen, Global asymptotic stability of a periodic ecological model, *Appl. Math. Computat.* **147** (2004), 881–892.
28. X. Zhao, The qualitative analysis of n-species Lotka–Volterra periodic competition systems, *Math. Comput. Modelling* **15** (1991), 3–8.