ON THE IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRIC GROUP

MASARU OSIMA

Introduction. Let T be a Young diagram of n nodes:

$$(1) \qquad T = [a_i]: \qquad a_1 \geqslant a_2 \geqslant \ldots \geqslant a_h, \qquad \sum_{i=1}^{h} a_i = n,$$

a_i being the length of its ith row. With respect to a prime p, we denote by T_0 the p-core of T. If T_0 consists of m nodes, then

$$(2) \qquad m = n - lp,$$

where l is the number of successive p-hooks [**3**] removable from T to yield its p-core T_0. We have stated in [**4**] the following theorem:

If T_0 is a p-core, diagrams T with T_0 as p-core are in one-to-one correspondence with systems $(D_1, D_2, \ldots, D_p)$ of p diagrams.

As an application of this theorem, in §1 the properties of self-associated diagrams will be studied. In §2 we shall give a recurrence formula for the number of irreducible representations and the number of self-associated irreducible representations of a symmetric group.

1. If the rows and columns of a diagram T are interchanged, we obtain another diagram. This is called the *associated diagram* of T, and is denoted by $\tilde{T}$. If $T = \tilde{T}$, then T is called the *self-associated diagram*.

Since a diagram T with T_0 as p-core is completely defined by a system $(D_1, D_2, \ldots, D_p)$ of p diagrams, we set

$$(3) \qquad T = \{T_0; D_1, D_2, \ldots, D_p\}.$$

Let D_i contain l_i nodes; $\quad l_i = 0$ when D_i is void. Then

$$(4) \qquad l = \sum_{i=1}^{p} l_i.$$

From Robinson's fundamental theorem [**5**, p. 287; **4**], we obtain readily

LEMMA 1. *Two diagrams $\{T_0; D_1, D_2, \ldots, D_p\}$ and $\{T'_0; D'_1, D'_2, \ldots, D'_p\}$ are associated if and only if $\tilde{T}_0 = T'_0$ and $\tilde{D}_i = D'_{p-i+1}$ for $i = 1, 2, \ldots, p$.*

Received November 20, 1951.

From Lemma 1 we have

THEOREM 1. *A diagram* $\{T_0; D_1, D_2, \ldots, D_p\}$ *is self-associated if and only if* T_0 *is self-associated and* $\tilde{D}_i = D_{p-i+1}$ *for* $i = 1, 2, \ldots, p$.

We denote by $B(T_0)$ the p-block of the symmetric group S_n of degree n corresponding [**1**, **4**] to the p-core T_0. If T_0 is self-associated, then $B(T_0)$ is called the *self-associated block* of S_n. From Theorem 1 we obtain

THEOREM 2. *Let* T_0 *be a* p*-core containing* m *nodes. The number of self-associated irreducible representations belonging to the self-associated block* $B(T_0)$ *of* S_n *is determined by* l *and is independent of* n *and* m.

We denote by $a(n)$ and $u(n)$ the number of diagrams and the number of self-associated diagrams containing n nodes. Then the number of irreducible representations and the number of self-associated irreducible representations of S_n are equal to $a(n)$ and $u(n)$ respectively. Denote by $v(n)$ the number of pairs of associated irreducible representations of S_n. Then

(5) $$a(n) = u(n) + 2v(n).$$

Let $b(n)$ be the number of irreducible representations of the alternating group A_n. Then we have [**2**, p. 171]

(6) $$b(n) = 2u(n) + v(n), \qquad n > 1.$$

2. We consider in this section the particular case when $p = 2$. Let $\{T_0; D_1, D_2\}$ be a diagram containing n nodes. Then we have from (2), $n = m + 2l$. We denote by $c(l)$ the number of irreducible representations belonging to the 2-block $B(T_0)$ of S_n. Then we see that

(7) $$c(l) = \sum_{t=0}^{l} a(t)a(l-t).$$

LEMMA 2. *A diagram* $T = [\alpha_i]$ *is a* 2*-core if and only if* $\alpha_i = h - i + 1$ *for* $i = 1, 2, \ldots, h$.

Let $d(n)$ be the number of 2-cores containing n nodes. Then from Lemma 2,

(8) $$d(n) = \begin{cases} 1 & n = \frac{1}{2}k(k+1) \\ 0 & n \neq \frac{1}{2}k(k+1) \end{cases} \qquad (k = 0, 1, 2, \ldots).$$

Further we have

(9) $$a(n) = \sum_{l} d(n-2l)c(l).$$

Hence (7), (8), and (9) yield the following

THEOREM 3. *For a given integer* n, *let* l_i $(i = 1, 2, \ldots, r)$ *be solutions of the equations* $n - 2l = \frac{1}{2}k(k+1)$ $(k = 0, 1, 2, \ldots)$ *in non-negative integers. Then*

$$a(n) = \sum_{i=1}^{r} \sum_{t=0}^{l_i} a(t)a(l_i - t),$$

where $a(n)$ *denotes the number of irreducible representations of* S_n.

n	$u(n)$	$v(n)$	$a(n)$	$b(n)$
2	0	1	2	1
3	1	1	3	3
4	1	2	5	4
5	1	3	7	5
6	1	5	11	7
7	1	7	15	9
8	2	10	22	14
9	2	14	30	18
10	2	20	42	24
11	2	27	56	31
12	3	37	77	43
13	3	49	101	55
14	3	66	135	72
15	4	86	176	94
16	5	113	231	123
17	5	146	297	156
18	5	190	385	200
19	6	242	490	254
20	7	310	627	324
21	8	392	792	408
22	8	497	1002	513
23	9	623	1255	641
24	11	782	1575	804
25	12	973	1958	997
26	12	1212	2436	1236
27	14	1498	3010	1526
28	16	1851	3718	1883
29	17	2274	4565	2308
30	18	2793	5604	2829
31	20	3411	6842	3451
32	23	4163	8349	4209
33	25	5059	10143	5109
34	26	6142	12310	6194
35	29	7427	14883	7485
36	33	8972	17977	9038
37	35	10801	21637	10871
38	37	12989	26015	13063
39	41	15572	31185	15654
40	46	18646	37338	18738

Example. For $n = 9$, we have $l_1 = 4$ and $l_2 = 3$. Hence

$$a(9) = \sum_{t=0}^{4} a(t)a(4-t) + \sum_{t=0}^{3} a(t)a(3-t)$$
$$= 20 + 10 = 30.$$

From Lemma 2 we have immediately

LEMMA 3. *A 2-core is self-associated.*

Let $\{T_0; D_1, D_2\}$ be a self-associated diagram containing n nodes. According to Theorem 1, we obtain $\tilde{D}_1 = D_2$. If D_1 contains s nodes, then $l = 2s$ and $n = \frac{1}{2}k(k+1) + 4s$. Hence we obtain

THEOREM 4. *For a given integer n, let s_i $(i = 1, 2, \ldots, q)$ be solutions of the equations $n - 4s = \frac{1}{2}k(k+1)$ $(k = 0, 1, 2, \ldots)$ in non-negative integers. Then*

$$u(n) = \sum_{i=1}^{q} a(s_i)$$

where $u(n)$ denotes the number of self-associated representations of S_n.

Example. For $n = 21$, we have $s_1 = 5$ and $s_2 = 0$. Hence

$$u(21) = a(5) + a(0) = 7 + 1 = 8.$$

Now from (5) and (6) we can easily determine the number $b(n)$ of irreducible representations of the alternating group A_n. The accompanying table gives the values of $u(n)$, $v(n)$, $a(n)$, and $b(n)$ up to $n = 40$.

REFERENCES

1. R. Brauer and G. de B. Robinson, *On a conjecture by Nakayama*, Trans. Royal Soc. Canada, Sec. III, vol. 40 (1947), 11-25.
2. F. D. Murnaghan, *The theory of group representations* (Baltimore, 1938).
3. T. Nakayama, *On some properties of irreducible representations of a symmetric group* I, Jap. J. Math., vol. 17 (1941), 89-108.
4. T. Nakayama and M. Osima, *Note on blocks of symmetric groups*, Nagoya Math. J., vol. 2 (1951), 111-117.
5. G. de B. Robinson, *On the representations of the symmetric group* III, Amer. J. Math., vol. 70 (1948), 277-294.

Okayama University,
Japan

https://doi.org/10.4153/CJM-1952-034-x Published online by Cambridge University Press