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EVERY ARITHMETIC PROGRESSION CONTAINS INFINITELY MANY *b*-NIVEN NUMBERS

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Abstract

For an integer $b \geq 2$, a positive integer is called a b-Niven number if it is a multiple of the sum of the digits in its base-b representation. In this article, we show that every arithmetic progression contains infinitely many b-Niven numbers.

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1. Introduction

Let $\mathbb{N}$ denote the set of positive integers and let $b \geq 2$ be an integer. For all $n \in \mathbb{N}$ and $0 \leq i \leq \lfloor \log_b n \rfloor$, let $v_b(n,i)$ be nonnegative integers such that $v_b(n,i) \leq b-1$ and $n = \sum_{i=0}^{\lfloor \log_b n \rfloor} v_b(n,i) b^i$. In other words, $v_b(n,i)$ is the $(i+1)$st digit from the right in the base-b representation of n. Furthermore, define $s_b : \mathbb{N} \to \mathbb{N}$ by $s_b(n) = \sum_{i=0}^{\lfloor \log_b n \rfloor} v_b(n,i)$. A positive integer n is *b-Niven* if $s_b(n) \mid n$.

It was shown in 1993 by Cooper and Kennedy [1] that there are no 21 consecutive 10-Niven numbers. Their result was generalised in 1994 by Grundman [4], who showed that there are no $2b+1$ consecutive b-Niven numbers. In 1994, Wilson [6] proved that for each b, there are infinitely many occurrences of $2b$ consecutive b-Niven numbers. These results were recently extended by Grundman *et al.* [5], who investigated the maximum lengths of arithmetic progressions of b-Niven numbers. Furthermore, an asymptotic estimate for the number of b-Niven numbers not exceeding x was found in 2003 by De Koninck *et al.* [2] and in 2008, they [3] showed that given any $r \in \{2, 3, \ldots, 2b\}$, there exists a constant $c = c(b,r)$ such that the number of r-tuples of consecutive b-Niven numbers not exceeding x is asymptotic to $cx/(\log x)^r$ as x tends to infinity.

In this article, we prove that every arithmetic progression contains infinitely many b-Niven numbers.

2. Main results

The following lemma is sometimes referred to as the 'postage stamp theorem', the 'chicken McNugget theorem' or the 'Frobenius coin theorem'.

LEMMA 2.1. *Let u and v be integers with $uv \geq 0$ and $\gcd(u, v) = 1$. Then every integer w such that w shares the same sign with u and v and satisfies $|w| \geq (|u| - 1)(|v| - 1)$ can be written in the form $w = gu + hv$ for some nonnegative integers g and h.*

The following two lemmas, which will be useful in our proof, are easy exercises in elementary number theory.

LEMMA 2.2. *If $d \mid b - 1$, then for all $u \in \mathbb{N}$, we have $d \mid u$ if and only if $d \mid s_b(u)$.*

LEMMA 2.3. *For all integers n and n' with $2 \leq n' \leq n$, $s_b(n') \leq (b - 1)\lceil \log_b(n) \rceil$.*

For positive integers m and r, let

$$\mathcal{S}_{m,r} = \{mx + r : x \in \mathbb{N}\}.$$

PROPOSITION 2.4. *Let $d = \gcd(s_b(m), s_b(r), b - 1)$. If $\gcd(s_b(m), s_b(r)) = d$, then $\mathcal{S}_{m,r}$ contains at least one b-Niven number.*

PROOF. Let $k_0(b, m, r) \in \mathbb{N}$ be such that for all integers $k \geq k_0$,

$$k \geq (b-1)\left\lceil \log_b\left(\frac{s_b(m)}{d} \cdot k + \frac{s_b(r)}{d}\right)\right\rceil + (b-2)\left((b-1)\left\lceil \log_b\left(\frac{s_b(m)}{d} \cdot k + \frac{s_b(r)}{d}\right)\right\rceil - 1\right). \tag{2.1}$$

Note that k_0 is well defined since b, m, r are constants and the right-hand side of (2.1) is of order $O(\log k)$. Using Dirichlet's theorem on primes in arithmetic progressions, let $k \in \mathbb{N}$ be such that $k \geq \max\{k_0, b, m\}$ and

$$p = \frac{s_b(m)}{d} \cdot k + \frac{s_b(r)}{d}$$

is a prime. Since $p > k \geq \max\{b, m\}$, we have $p \nmid bm$. Furthermore, let $\tilde{x}$ be the smallest positive integer such that $\tilde{x} \equiv -m^{-1}r \pmod{p}$. From Lemma 2.3 and (2.1),

$$k \geq s_b(\tilde{x}) + (b-2)(s_b(p) - 1).$$

By Lemma 2.1, there exist nonnegative integers g and h such that

$$k = s_b(\tilde{x}) + g(b-1) + h \cdot s_b(p).$$

Let $\omega \in \mathbb{N}$ be a multiple of $p-1$ such that $b^\omega > \max\{m, r\}$. Note that $b^\omega \equiv 1 \pmod{p}$ by Fermat's little theorem since $p \nmid b$. We now define a function $\tau_b : \mathbb{N} \to \mathbb{N}$ as follows. For each fixed $n \in \mathbb{N}$, let $\sigma_{-1} = 0$ and $\sigma_i = \sum_{j=0}^{i} \nu_b(n, j)$ for $0 \leq i \leq \lfloor \log_b n \rfloor$. Then,

$$\tau_b(n) = \sum_{j=1}^{\sigma_{\lfloor \log_b n \rfloor}} b^{j\omega + \ell_j},$$

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where $\ell_j = i$ for the unique $i \in \{0, 1, 2, \ldots, \lfloor \log_b n \rfloor\}$ satisfying $\sigma_{i-1} < j \leq \sigma_i$. It is important to notice that the construction of $\tau_b(n)$ guarantees $s_b(\tau_b(n)) = \sigma_{\lfloor \log_b n \rfloor} = s_b(n)$ and $\tau_b(n) \equiv \sum_{j=1}^{\sigma_{\lfloor \log_b n \rfloor}} b^{\ell_j} \equiv \sum_{i=0}^{\lfloor \log_b n \rfloor} \nu_b(n, i) b^i \equiv n \pmod p$.

Let $x_0 = \tau_b(\tilde{x})$ and, for each positive integer $t \leq g$, let

$$x_t = x_{t-1} - b^{\lfloor \log_b x_{t-1} \rfloor} + \sum_{\iota=1}^{b} b^{\iota\omega + \lfloor \log_b x_{t-1} \rfloor - 1}.$$

From this construction, $s_b(x_t) = s_b(x_{t-1}) + b - 1$ and

$$x_t \equiv x_{t-1} - b^{\lfloor \log_b x_{t-1} \rfloor} + b \cdot b^{\lfloor \log_b x_{t-1} \rfloor - 1} \equiv x_{t-1} \pmod p$$

for all $t \leq g$. It follows that $s_b(x_g) = s_b(x_0) + g(b-1) = s_b(\tilde{x}) + g(b-1)$ and $x_g \equiv x_0 \equiv \tilde{x} \pmod p$. Lastly, let α and β be integers such that $b^{\alpha\omega} > x_g$ and $b^{\beta\omega} > \tau_b(p)$. We define

$$x = x_g + \sum_{\iota=0}^{h-1} \tau_b(p) \cdot b^{(\iota\beta+\alpha)\omega}.$$

Now, $s_b(x) = s_b(x_g) + h \cdot s_b(\tau_b(p)) = k$, $x \equiv x_g + \sum_{\iota=0}^{h-1} p \cdot b^{(\iota\beta+\alpha)\omega} \equiv -m^{-1} r \pmod p$ and since every summand of x is a distinct power of b where the powers differ by at least ω, we have $s_b(mx + r) = s_b(m) \cdot s_b(x) + s_b(r) = s_b(m) \cdot k + s_b(r) = dp$. Therefore, $mx + r$ is a b-Niven number based on the following observations:

- $mx + r \equiv m(-m^{-1} r) + r \equiv 0 \pmod p$;
- $d \mid (mx + r)$ since $d \mid m$ and $d \mid r$ by Lemma 2.2;
- $\gcd(p, d) = 1$ since $p > b$ and $d \mid b - 1$. □

LEMMA 2.5. *Let n be a nonnegative integer. For all nonnegative integers y, $s_b(yn) = ys_b(n) + z(b-1)$ for some integer z.*

PROOF. Note that for all nonnegative integers n, if $n = \sum_{i=0}^{\lfloor \log_b n \rfloor} \nu_b(n, i) b^i$, then $s_b(n) = \sum_{i=0}^{\lfloor \log_b n \rfloor} \nu_b(n, i) \equiv n \pmod{b-1}$. Hence, $s_b(yn) \equiv yn \equiv ys_b(n) \pmod{b-1}$. □

PROPOSITION 2.6. *Let $d = \gcd(s_b(m), s_b(r), b-1)$. Then there exists a positive multiple $\overline{m}$ of m such that $\gcd(s_b(\overline{m}), s_b(r)) = d$.*

PROOF. Let i_0 be the smallest nonnegative integer such that $\nu_b(m, i_0) \neq 0$. Then there exists a nonnegative integer $a \leq b - 1$ such that $\nu_b(am, i_0) = \nu_b(a \cdot \nu_b(m, i_0), 0) \geq b/2$. Next, if $\nu_b(am, i_0 + 1) \neq b - 1$, then let $m' = am$; otherwise, let $m' = (b+1)am$ so that $\nu_b(m', i_0) = \nu_b(am, i_0) \geq b/2$ and

$$\nu_b(m', i_0 + 1) \equiv \nu_b(am, i_0 + 1) + \nu_b(am, i_0) \equiv b - 1 + \nu_b(am, i_0) \not\equiv b - 1 \pmod b.$$

Furthermore, define m'' to be a multiple of m' such that the leading digit of m'' in base-b representation is at least $b/2$, that is, $\nu_b(m'', \lfloor \log_b m'' \rfloor) \geq b/2$. Let $m^* = b^2 m'' + m'$. Then m^* is a multiple of m such that $\nu_b(m^*, i_0) \geq b/2$, $\nu_b(m^*, i_0 + 1) \neq b - 1$ and $\nu_b(m^*, \lfloor \log_b m^* \rfloor) \geq b/2$.

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Let x, y, z be integers such that $xs_b(r) + ys_b(m) + z(b-1) = d$. Define y^* such that $m^* = y^*m$ and let z^* be an integer such that $s_b(m^*) = y^*s_b(m) + z^*(b-1)$ by Lemma 2.5. Letting $m^{**} = (b^{\lfloor \log_b m^* \rfloor - i_0} + 1)m^*$, we see that $\nu_b(m^{**}, \lfloor \log_b m^* \rfloor) = \nu_b(m^*, \lfloor \log_b m^* \rfloor) + \nu_b(m^*, i_0) - b$ and $\nu_b(m^{**}, \lfloor \log_b m^* \rfloor + 1) = \nu_b(m^*, i_0 + 1) + 1 \leq b - 1$. Hence, $s_b(m^{**}) = 2s_b(m^*) - (b-1) = 2y^*s_b(m) + (2z^* - 1)(b-1)$. By Lemma 2.1, there exist nonnegative integers g and h such that $gz^* + h(2z^* - 1) \equiv z \pmod{s_b(r)}$. Let j be a nonnegative integer such that $gy^* + h(2y^*) + j \equiv y \pmod{s_b(r)}$. Consider

$$\begin{aligned}\overline{m} = &\sum_{\iota=0}^{g-1} m^* b^{\iota(\lfloor \log_b m^* \rfloor + 1)} + \sum_{\iota=0}^{h-1} m^{**} b^{\iota(\lfloor \log_b m^{**} \rfloor + 1) + g(\lfloor \log_b m^* \rfloor + 1)} \\ &+ \sum_{\iota=0}^{j-1} m b^{\iota(\lfloor \log_b m \rfloor + 1) + g(\lfloor \log_b m^* \rfloor + 1) + h(\lfloor \log_b m^{**} \rfloor + 1)}.\end{aligned}$$

By construction, $\overline{m}$ is a multiple of m and

$$\begin{aligned} s_b(\overline{m}) &= gs_b(m^*) + hs_b(m^{**}) + js_b(m) \\ &= g(y^*s_b(m) + z^*(b-1)) + h(2y^*s_b(m) + (2z^*-1)(b-1)) + js_b(m) \\ &= (gy^* + h(2y^*) + j)s_b(m) + (gz^* + h(2z^* - 1))(b-1) \\ &\equiv ys_b(m) + z(b-1) \equiv d \pmod{s_b(r)}. \end{aligned}$$

Note that $d \mid s_b(\overline{m})$ since $d \mid s_b(m)$ and $d \mid b - 1$. Therefore, $\gcd(s_b(\overline{m}), s_b(r)) = d$. $\square$

Combining Propositions 2.4 and 2.6, we obtain the following theorem.

THEOREM 2.7. *Let m and r be positive integers. The arithmetic progression $\mathcal{S}_{m,r}$ contains infinitely many b-Niven numbers.*

PROOF. By Proposition 2.6, there exists a multiple $\overline{m}$ of m such that

$$\gcd(s_b(\overline{m}), s_b(r), b-1) = \gcd(s_b(\overline{m}), s_b(r)).$$

Hence, by Proposition 2.4, $\mathcal{S}_{\overline{m},r}$, and thus $\mathcal{S}_{m,r}$, contains at least one b-Niven number since $\mathcal{S}_{\overline{m},r}$ is a subset of $\mathcal{S}_{m,r}$. Let this b-Niven number be $\eta m + r$ for some nonnegative integer η. Applying the same argument on the arithmetic progression, $\mathcal{S}_{m,(\eta+1)m+r}$ yields another b-Niven number and our proof is complete by induction. $\square$

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