CONVEXITY OF THE FIELD OF A LINEAR TRANSFORMATION

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Let U_n be an n-dimensional unitary space with inner product $(x, y) = \overline{(y, x)}$. In U_n let S_{n-1} denote the unit sphere:

$$S_{n-1} = \{x \mid (x, x) = 1\} .$$

Let A be an arbitrary linear transformation of U_n. The subset

$$F(A) = \{\zeta \mid \zeta = (Ax, x),\ x \text{ in } S_{n-1}\}$$

of the ζ-plane ($\zeta = \xi + i\eta$) is called the field of A.

As the image of S_{n-1} under the continuous mapping $x \to (Ax, x)$, F(A) must be compact and connected. Toeplitz proved in [4] that the boundary of F(A) is a convex curve. Hausdorff then showed [2] that F(A) actually fills the interior of this curve (i.e., that F(A) is convex). Proofs of the convexity of F(A) also appear in [3] and [5].

The purpose of this note is to provide a simple inductive proof for the convexity of F(A) which reduces the essential computation to the single case n = 2. We then dispose of this case by verifying directly that F(A) satisfies the definition of a convex set.

THEOREM. F(A) is convex.

Proof. (a) If n=1, then F(A) is a single point.

(b) Deferring the case n=2, we suppose $n \geq 3$ and consider the inductive step from n-1 to n. Let x and y be any two vectors of S_{n-1}; we must show that F(A) contains the segment joining the points (Ax, x) and (Ay, y) in the ζ-plane. Since $n \geq 3$, we can find a vector u in U_n such that (u, x) = (u, y) = 0. The unitary-orthogonal complement in U_n of the line L spanned by u

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is a subspace U_{n-1} of U_n whose unit sphere S_{n-2} is contained in S_{n-1}; furthermore, x and y lie in S_{n-2}. Any vector w in U_n admits a unique decomposition w = v + z, with v in L and z in U_{n-1}; the unitary-orthogonal projection P of U_n onto U_{n-1} is defined by Pw = z. Obviously A_o = PAP = P(AP) is a linear transformation of U_{n-1} into itself. For any z in S_{n-2} (and thus in S_{n-1}) we have Pz = z and thus, decomposing $Az = v_1 + z_1$,

$$(Az, z) = (v_1+z_1, z) = (z_1, z) = (PAz, z) = (PAPz, z) = (A_oz, z);$$

since $(A_oz, z) = (Az, z)$, $F(A_o)$ is a subset of F(A). Also, taking z = x and z = y, we see that (Ax, x) and (Ay, y) are in $F(A_o)$; $F(A_o)$ is convex by hypothesis, and so the segment joining (Ax, x) and (Ay, y) lies in $F(A_o)$ and thus in F(A), as desired.

(c) We turn now to the case n=2. It is well known (see [1], for example) that there exists a coordinate system (or equivalently, a basis) in U_2 with respect to which the matrix of A takes a "superdiagonal" form

$$A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$$

so that for any vector x in the "unit circle" S_1 of U_2, with co-ordinates x_1, x_2 relative to the system, we have

$$(Ax, x) = a|x_1|^2 + b|x_2|^2 + c\overline{x_1}x_2 \qquad (|x_1|^2+|x_2|^2=1)$$

$$= b + (a-b)|x_1|^2 + c\overline{x_1}x_2.$$

If, using the convention arg (0) = 0, we let

$$\alpha = |a-b| \qquad (\alpha \geqslant 0)$$

$$t = \arg(a-b)$$

$$s = |x_1|^2 \qquad (0 \leqslant s \leqslant 1)$$

$$\theta = \arg x_2 - \arg x_1 - t,$$

and consider the set S = [F(A)-b]exp(-it), we find that

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$$S = \{\zeta \mid \zeta = \alpha s + c(s(1-s))^{\frac{1}{2}} \exp(i\theta);\ 0 \leqslant s \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi\}\ .$$

Since S is congruent to F(A), it suffices to prove that S is convex.

If $c = 0$, then S is a line segment and therefore convex. If $c \neq 0$ then we can assume $c = 1$, since F(A) is convex if and only if $c^{-1}F(A) = F(c^{-1}A)$ is convex. Thus we can take S to be the union of the circles

$$C(s): \quad |\zeta - \alpha s| = (s(1-s))^{\frac{1}{2}} = f(s) \qquad (0 \leqslant s \leqslant 1).$$

Let ζ_1 and ζ_2 be any points of S and let ζ_0 be any point on the line joining them: we must show that ζ_0 lies in S. Let $C(s_1)$ and $C(s_2)$ be circles on which ζ_1, ζ_2 lie, and use the fact that ζ_0 can be written in the form

$$\zeta_0 = r\zeta_1 + (1-r)\zeta_2 \qquad (0 \leqslant r \leqslant 1)$$

to define $\qquad s_0 = rs_1 + (1-r)s_2.$

Consider $G(s) = |\zeta_0 - \alpha s| - f(s)$. Obviously $G(0) = |\zeta_0| \geqslant 0$ (i.e., ζ_0 lies outside or on C(0)). We will show that $G(s_0) \leqslant 0$ (i.e., that ζ_0 lies inside or on $C(s_0)$). It follows that $G(s^*) = 0$ (i.e., that ζ_0 lies on $C(s^*)$) for some s^* with $0 \leqslant s^* \leqslant s_0 \leqslant 1$, so that ζ_0 lies in S and the convexity of S will be proved.

To show that $G(s_0) \leqslant 0$, we apply the triangle inequality:

$$|\zeta_0 - \alpha s_0| \leqslant r|\zeta_0 - \alpha s_1| + (1-r)|\zeta_0 - \alpha s_2| = rf(s_1) + (1-r)f(s_2).$$

Since $f''(s) \leqslant 0$ for $0 < s < 1$, we have

$$rf(s_1) + (1-r)f(s_2) \leqslant f(s_0)$$

and so $|\zeta_0 - \alpha s_0| \leqslant f(s_0)$ (i.e., $G(s_0) \leqslant 0$). This completes the proof.

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