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SUM OF VALUES OF THE IDEAL CLASS ZETA-FUNCTION OVER NONTRIVIAL ZEROS OF THE RIEMANN ZETA-FUNCTION

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Abstract

We prove an upper bound for the sum of values of the ideal class zeta-function over nontrivial zeros of the Riemann zeta-function. The same result for the Dedekind zeta-function is also obtained. This may shed light on some unproved cases of the general Dedekind conjecture.

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1. Introduction

Let K be a number field of degree n with ring of integers O_K. Let $\mathfrak{C}$ be an ideal class of K. The *ideal class zeta-function* $\zeta_K(\mathfrak{C}; s)$ is defined by

$$\zeta_K(\mathfrak{C}; s) := \sum_{\substack{\mathfrak{a}\in\mathfrak{C} \\ \text{integral}}} \frac{1}{\mathrm{N}(\mathfrak{a})^s}$$

for $\Re(s) > 1$, where the sum is taken over the nonzero integral ideals from the class $\mathfrak{C}$. The sum of all such ideal class zeta-functions is equal to the *Dedekind zeta-function* ζ_K, a generalisation of the *Riemann zeta-function* $\zeta = \zeta_{\mathbb{Q}}$. It is known that the Riemann zeta-function ζ divides the Dedekind zeta-function ζ_K for any quadratic number field K (in the sense that the quotient ζ_K/ζ is an entire function). This fact is a particular case of the *Dedekind conjecture* [9] which states that if K/L is an extension of number fields, then the quotient ζ_K/ζ_L is entire. This conjecture has been proved when the number field K/L is Galois [2] or solvable [15, 16].

In this article, we shall consider a number field K of degree n. A complex variable is denoted by $s = \sigma + it$, and a nontrivial zero of the Riemann zeta-function by

$\rho = \beta + i\gamma$. If the number field K is Galois or solvable over $\mathbb{Q}$, then

$$\sum_{\rho} \zeta_K(\rho) = 0,$$

where the sum is taken over all nontrivial zeros of the Riemann zeta-function. There are many similar results for the sum of the values of some functions over the nontrivial zeros of the Riemann zeta-function. For example, Steuding [11] studied the sum of values of the Hurwitz zeta-function over the nontrivial zeros of the Riemann zeta-function. Garunkštis and Kalpokas [6] did the same for the periodic zeta-function associated with a rational parameter. Recently, Tongsomporn *et al.* [14] did the same for an irrational parameter. The proof of these results (without assuming the Riemann hypothesis) made use of the method of Conrey *et al.* [5]. The basic idea is to interpret the sum in question as a sum of residues and then apply Cauchy's residue theorem and the method of contour integration in combination with the functional equation of the zeta-function and Gonek's lemma [7, Lemma 2]. The following is our main result.

THEOREM 1.1. *Let K be a number field of degree n. Let r_1 and r_2 be the number of its real embeddings and pairs of complex conjugate embeddings, respectively, and d be the absolute value of its discriminant. Let $\mathfrak{C}$ be any ideal class of the number field K and c'_m be the number of integral ideals of norm m from the ideal class which is the complement of $\mathfrak{C}$. Then, as T tends to infinity,*

$$\sum_{0<\gamma<T} \zeta_K(\mathfrak{C};\rho) = -\frac{i^{r_1+r_2}}{d^{1/2n}n^{1/2}} \exp\Big(-i\pi\frac{n+1}{4}\Big) \times \sum_{k\le d\,(T/2\pi)^n} \frac{1}{k^{(n-1)/2n}} \exp\Big(2\pi i n\Big(\frac{k}{d}\Big)^{1/n}\Big) \sum_{\ell\mid k} c'_\ell \Lambda\Big(\frac{k}{\ell}\Big) + O(\max\{T^{n/2+\varepsilon}, T^{165/146+\varepsilon}\}),$$

where the sum is taken over the nontrivial zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function, Λ denotes the von Mangoldt function and $\varepsilon > 0$ is arbitrary but fixed. In particular,

$$\sum_{0<\gamma<T} \zeta_K(\mathfrak{C};\rho) \ll T^{(n+1)/2+\varepsilon}.$$

A corresponding result for the Dedekind zeta-function is an immediate consequence.

COROLLARY 1.2. *Let K be a number field of degree n. Then, as T tends to infinity,*

$$\sum_{0<\gamma<T} \zeta_K(\rho) \ll T^{(n+1)/2+\varepsilon},$$

where the sum is taken over the nontrivial zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function.

In Section 2, we review some background and useful facts which are related to the Riemann zeta-function and the ideal class zeta-function. The proof of the main result

(Theorem 1.1) is provided in Section 3. Finally, Section 4 contains a few concluding remarks.

2. Preliminaries

We first recall some useful facts about the Riemann zeta-function. Then we do the same for the ideal class zeta-function. Finally, we recall some techniques and state some lemmas that are useful in Section 2.3.

2.1. Riemann zeta-function. The Riemann zeta-function in the half-plane $\sigma > 1$ is defined by

$$\zeta(s) := \sum_{m\geq 1} \frac{1}{m^s} = \prod_p (1 - p^{-s})^{-1},$$

where the product runs through all primes. In this half-plane, the logarithmic derivative of the Riemann zeta-function can be written as a Dirichlet series,

$$\frac{\zeta'}{\zeta}(s) = -\sum_{j\geq 2} \frac{\Lambda(j)}{j^s},$$

where the *von Mangoldt function* is defined by

$$\Lambda(j) := \begin{cases} \log p & \text{if } j = p^k \text{ for some prime } p \text{ and positive integer } k, \\ 0 & \text{otherwise.} \end{cases}$$

The Riemann zeta-function can be continued analytically to a meromorphic function on the whole complex plane with a single singularity at $s = 1$ which is a simple pole. This continuation satisfies the functional equation

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\Big(\frac{\pi s}{2}\Big)\zeta(1-s) =: \Delta(s)\zeta(1-s).$$

This implies that the logarithmic derivative of the Riemann zeta-function satisfies

$$\frac{\zeta'}{\zeta}(s) = \frac{\Delta'}{\Delta}(s) - \frac{\zeta'}{\zeta}(1-s),$$

where by Stirling's formula,

$$\frac{\Delta'}{\Delta}(1-s) = \frac{\Delta'}{\Delta}(s) = -\log\frac{t}{2\pi} + O\Big(\frac{1}{t}\Big)$$

for $t > 1$. Since the Riemann zeta-function has a simple pole at $s = 1$, so does its logarithmic derivative and in the neighbourhood of $s = 1$,

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + O(1).$$

It follows from the functional equation that any negative even integer is a zero for the Riemann zeta-function; these zeros are called *trivial zeros*. All further zeros lie in the strip $0 \leq t \leq 1$, and are called *nontrivial zeros*. Let $N(T)$ be the number of nontrivial zeros of the Riemann zeta-function whose positive imaginary part does not exceed the number T, that is,

$$N(T) := \{\rho = \beta + i\gamma \mid 0 < \gamma \le T\}.$$

The *Riemann–von Mangoldt formula* provides asymptotic formulae:

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$$

and

$$N(T+h) - N(T) \ll \log T$$

for any fixed positive real number h. (Here, every multiple zero is counted according to its multiplicity. For more details, see [13, Ch. 9]). By an approximation formula for the logarithmic derivative of the Riemann zeta-function [13, Theorem 9.6] together with the Riemann–von Mangoldt formula, in the strip $-1 \le \sigma \le 2$,

$$\frac{\zeta'}{\zeta}(\sigma + it) \ll (\log t)^2 \tag{2.1}$$

for $|t - \gamma| \ge c/\log t$, where c is a constant independent of t.

2.2. Ideal class zeta-function. Recall that the ideal class zeta-function associated to the ideal class $\mathfrak{C}$ in the half-plane $\sigma > 1$ can be written as

$$\zeta_K(\mathfrak{C}; s) = \sum_{m \ge 1} \frac{c_m}{m^s},$$

where c_m is the number of integral ideals of norm m from the class $\mathfrak{C}$. According to [10], one can show that $c_m \ll m^{\varepsilon}$ and

$$\sum_{m \le x} c_m = \kappa x + E(\mathfrak{C}; x),$$

where

$$E(\mathfrak{C}; x) \ll \begin{cases} x^{23/73}(\log x)^{315/146} & \text{if } n = 2, \\ x^{1-2/n+8/n(5n+2)}(\log x)^{10/(5n+2)} & \text{if } 3 \le n \le 6, \\ x^{1-2/n+3/2n^2}(\log x)^{2/n} & \text{if } n \ge 7, \end{cases} \tag{2.2}$$

and

$$\kappa := \frac{2^{r_1+r_2}\pi^{r_2}R}{wd^{1/2}}.$$

Here, r_1, r_2, R, w and d are the number of real embeddings, the number of pairs of complex conjugate embeddings, the regulator, the number of roots of unity and the absolute value of the discriminant of the number field K, respectively.

The ideal class zeta-function can also be continued analytically to a meromorphic function on the whole complex plane with only a simple pole at $s = 1$. This continuation satisfies the functional equation

$$\zeta_K(\mathfrak{C}; s) = Z_K(s)\zeta_K(\mathfrak{C}'; 1 - s), \tag{2.3}$$

where

$$Z_K(s) = d^{1/2-s}(\sqrt{2})^n\Big(\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\Big)^n\Big(\sqrt{2}\sin\Big(\pi\frac{s}{2}\Big)\Big)^{r_1}(\sin(\pi s))^{r_2}$$

and $\mathfrak{C}'$ is the ideal class of K which is the complement of $\mathfrak{C}$ (with respect to the trace) (see [8, Ch. 13] for more details). In addition, in a neighbourhood of $s = 1$,

$$\zeta_K(\mathfrak{C}; s) = \frac{\kappa}{s-1} + O(1).$$

The functional equation, together with Stirling's formula and the Phragmén–Lindelöf principle [12, Section 5.65] implies that

$$\zeta_K(\mathfrak{C}; \sigma + it) \ll \begin{cases} t^{\varepsilon} & \text{if } \sigma > 1, \\ t^{n(1-\sigma)/2+\varepsilon} & \text{if } -1/\log t \le \sigma \le 1 + 1/\log t, \\ t^{n(1/2-\sigma)+\varepsilon} & \text{if } \sigma < 0, \end{cases} \tag{2.4}$$

as t tends to infinity.

2.3. Some useful lemmas. As a preparation for the proof of the main result (Theorem 1.1), we state three useful lemmas.

LEMMA 2.1 (Abel's summation formula). *Let $a_1, a_2, \ldots$ be a sequence of real (or complex) numbers and suppose $f(x)$ has a continuous derivative on the interval $[y, x]$, where $0 < y < x$. Define*

$$A(x) = \sum_{m \le x} a_m,$$

where $A(x) = 0$ if $x < 1$. Then,

$$\sum_{y<m\le x} a_m f(m) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\,dt.$$

PROOF. See [1, Theorem 4.2]. □

LEMMA 2.2 (The first derivative test). *Let $f(x)$ and $g(x)$ be real functions such that $g(x)/f'(x)$ is monotonic and $f'(x)/g(x) \ge m > 0$, or $f'(x)/g(x) \le -m < 0$. Then,*

$$\Big|\int_a^b g(t)\exp(if(t))\,dt\Big| \le \frac{4}{m}.$$

PROOF. See [13, Lemma 4.3]. □

The following lemma is generalised from Gonek's lemma [7, Lemma 2].

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LEMMA 2.3 (Generalised Gonek lemma). *Let a be a fixed real number. For large T, let $\mathcal{J}$ be the unique positive integer such that $2^{-\mathcal{J}}T < 1 \le 2^{1-\mathcal{J}}T$. Then,*

$$\int_1^T \left(\left(\frac{t}{2\pi}\right)^{a-1/2} \exp\left(it\log\frac{t}{re}\right)\right)^n dt$$
$$= \frac{2\pi}{\sqrt{n}}\left(\frac{2\pi}{r}\right)^{(n-1)/2-an} \exp\left(-i\left(nr-\frac{\pi}{4}\right)\right)\chi_{(2^{-\mathcal{J}}T,T]}(r) + \sum_{j=1}^{\mathcal{J}} E(r; 2^{-j}T, 2^{1-j}T)$$

with the characteristic function

$$\chi_I(x) = \begin{cases} 1 & \textit{if } x \in I, \\ 0 & \textit{otherwise}, \end{cases}$$

and the error term

$$E(r; A, B) \ll A^{n(a-1/2)} + \frac{A^{n(a-1/2)+1}}{|A-r| + A^{1/2}} + \frac{B^{n(a-1/2)+1}}{|B-r| + B^{1/2}}.$$

PROOF. Substituting $t \mapsto t/n$ together with Gonek's lemma [7, Lemma 2] yields

$$\int_A^B \left(\left(\frac{t}{2\pi}\right)^{a-1/2} \exp\left(it\log\frac{t}{re}\right)\right)^n dt$$
$$= \frac{1}{n^{n(a-1/2)+1}} \int_{nA}^{nB} \left(\frac{t}{2\pi}\right)^{n(a-1/2)} \exp\left(it\log\frac{t}{nre}\right) dt$$
$$= \frac{2\pi}{\sqrt{n}}\left(\frac{2\pi}{r}\right)^{(n-1)/2-an} \exp\left(-i\left(nr-\frac{\pi}{4}\right)\right)\chi_{(A,B]}(r) + E(r; A, B)$$

for large A and $A < B \le 2A$. Then,

$$\int_1^T \left(\left(\frac{t}{2\pi}\right)^{a-1/2} \exp\left(it\log\frac{t}{re}\right)\right)^n dt$$
$$= \sum_{j=1}^{\mathcal{J}} \int_{2^{-j}T}^{2^{1-j}T} \left(\left(\frac{t}{2\pi}\right)^{a-1/2} \exp\left(it\log\frac{t}{re}\right)\right)^n dt + O(1)$$
$$= \frac{2\pi}{\sqrt{n}}\left(\frac{2\pi}{r}\right)^{(n-1)/2-an} \exp\left(-i\left(nr-\frac{\pi}{4}\right)\right)\chi_{(2^{-\mathcal{J}}T,T]}(r) + \sum_{j=1}^{\mathcal{J}} E(r; 2^{-j}T, 2^{1-j}T). \qquad \square$$

3. Proof of Theorem 1.1

In this section, let $\mathfrak{C}$ be an ideal class of the class group, $\mathfrak{C}'$ the class containing the complements of the members of the class $\mathfrak{C}$, and c_m and c'_m the number of integral ideals of norm m from the class $\mathfrak{C}$ and $\mathfrak{C}'$, respectively.

By the Riemann–von Mangoldt formula, for a given number $t_o \ge 3$, there is a positive integer $t \in [t_o, t_o + 1)$ such that

$$|t - \gamma| \ge \frac{c}{\log t},$$

where γ is any imaginary value of a nontrivial zero $\rho = \beta + i\gamma$ of the Riemann zeta-function and c is a constant (independent of t). Now, let $\varepsilon > 0$ and $T \geq 3$ be such that

$$|T - \gamma| \geq \frac{c}{\log T}.$$

Note that the least imaginary value γ of a nontrivial zero of the Riemann zeta-function $\rho = \beta + i\gamma$ in the upper half-plane is a little larger than 14 and the logarithmic derivative ζ'/ζ of the Riemann zeta-function has simple poles at the zeros of the Riemann zeta-function and is analytic elsewhere except for a simple pole at $s = 1$. By Cauchy's theorem,

$$\sum_{0<\gamma<T} \zeta_K(\mathfrak{C}; \rho) = \frac{1}{2\pi i} \int_C \frac{\zeta'}{\zeta}(s)\zeta_K(\mathfrak{C}; s)\, ds,$$

where the counter-clockwise oriented contour C is a rectangle with vertices $a + i$, $a + iT$, $1 - a + iT$, $1 - a + i$ with $a := 1 + 1/\log T$. We rewrite the contour integral as

$$\frac{1}{2\pi i}\Big\{ \int_{a+i}^{a+iT} + \int_{a+iT}^{1-a+iT} + \int_{1-a+iT}^{1-a+i} + \int_{1-a+i}^{a+i} \Big\} \frac{\zeta'}{\zeta}(s)\zeta_K(\mathfrak{C}; s)\, ds =: \sum_{j=1}^{4} \mathcal{I}_j.$$

First, we start with the lower horizontal integral. We obtain

$$\mathcal{I}_4 = \frac{1}{2\pi i} \int_{1-a+i}^{a+i} \frac{\zeta'}{\zeta}(s)\zeta_K(\mathfrak{C}; s)\, ds \ll 1 \tag{3.1}$$

since this integral is independent of T.

Next, we consider the vertical line segment on the right, which lies inside the half-plane of absolute convergence for the Dirichlet series. By interchanging integration and summation, it follows that

$$\begin{aligned}
\mathcal{I}_1 &= \frac{1}{2\pi i} \int_{a+i}^{a+iT} \frac{\zeta'}{\zeta}(s)\zeta_K(\mathfrak{C}; s)\, ds \\
&= \frac{1}{2\pi} \int_1^T \Big(- \sum_{j\geq 2} \frac{\Lambda(j)}{j^{a+it}} \Big)\Big(\sum_{m\geq 1} \frac{c_m}{m^{a+it}} \Big)\, dt \\
&= -\frac{1}{2\pi} \sum_{j\geq 2} \frac{\Lambda(j)}{j^a} \sum_{m\geq 1} \frac{c_m}{m^a} \int_1^T (jm)^{-it}\, dt.
\end{aligned}$$

It is easy to see that the latter integral is bounded. By the Laurent expansion of the ideal class zeta-function and the logarithmic derivative of the Riemann zeta-function at $s = 1$,

$$\mathcal{I}_1 \ll \frac{\zeta'}{\zeta}(a)\zeta_K(\mathfrak{C}; a) \ll (\log T)^2. \tag{3.2}$$

In view of the estimates for the logarithmic derivative of the Riemann zeta-function and the ideal class zeta-function, (2.1) and (2.4), we obtain the upper horizontal integral

$$\mathcal{I}_2 = \frac{1}{2\pi i}\int_{a+iT}^{1-a+iT}\frac{\zeta'}{\zeta}(s)\zeta_K(\mathfrak{C};s)\,ds \ll T^{n/2+\varepsilon} \tag{3.3}$$

by the trivial estimation.

It remains to consider the vertical integral on the left. By the functional equation (2.3) and substituting $s \mapsto 1-s$,

$$\begin{aligned}\mathcal{I}_3 &= \frac{1}{2\pi i}\int_{1-a+iT}^{1-a+i}\frac{\zeta'}{\zeta}(s)\zeta_K(\mathfrak{C};s)\,ds\\ &= \frac{1}{2\pi i}\int_{1-a+iT}^{1-a+i}\frac{\zeta'}{\zeta}(s)Z_K(s)\zeta_K(\mathfrak{C}';1-s)\,ds\\ &= \frac{1}{2\pi i}\int_{a-i}^{a-iT}\frac{\zeta'}{\zeta}(1-s)Z_K(1-s)\zeta_K(\mathfrak{C}';s)\,ds.\end{aligned}$$

By the Schwarz reflection principle, the conjugate of this integral is

$$\begin{aligned}\overline{\mathcal{I}_3} &= -\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\zeta'}{\zeta}(1-s)Z_K(1-s)\zeta_K(\mathfrak{C}';s)\,ds\\ &= -\frac{1}{2\pi i^{r_1+r_2+1}}\int_{a+i}^{a+iT}\Big(\frac{\Delta'}{\Delta}(s)-\frac{\zeta'}{\zeta}(s)\Big)\Big(\frac{\Gamma(s)}{(2\pi)^s}\Big)^n d^{s-1/2}E_K(s)\zeta_K(\mathfrak{C}';s)\,ds,\end{aligned}$$

where the function $E_K(s)$ is defined by

$$E_K(s) := (2i)^{r_1+r_2}\Big(\sin\Big(\pi\frac{1-s}{2}\Big)\Big)^{r_1}(\sin(\pi(1-s)))^{r_2} = \sum_{k=-n}^{n} e_k\exp\Big(i\pi k\frac{1-s}{2}\Big) \tag{3.4}$$

for some integers e_k with $e_n = 1$ and $e_{-n} = (-1)^{r_1+r_2}$. Observe that the ideal class zeta-function and logarithmic derivative of the Riemann zeta-function can be represented as absolutely convergent Dirichlet series. By Stirling's formula, for $s = \sigma + it$ and a real number ℓ,

$$\frac{\Gamma(s)}{(2\pi)^s}\exp\Big(i\pi\ell\frac{s}{2}\Big) = \Big(\frac{t}{2\pi}\Big)^{\sigma-1/2+it}\exp\Big(-\frac{\pi t(\ell+1)}{2} - i\Big(t - \frac{\pi(\ell+1)\sigma}{2}+\frac{\pi}{4}\Big)\Big)\Big(1+O\Big(\frac{1}{t}\Big)\Big).$$

Now, based on (3.4), we split $\overline{\mathcal{I}_3}$ into a sum of integrals and estimate each integral as follows.

Case I: $-n \le k < n$. The integrals with $\exp(i\pi k(1-s)/2)$ contribute to the error term. More precisely,

$$\begin{aligned}&\int_{a+i}^{a+iT}\frac{\Delta'}{\Delta}(s)\Big(\frac{\Gamma(s)}{(2\pi)^s}\Big)^n d^{s-1/2}e_k\exp\Big(i\pi k\frac{1-s}{2}\Big)\zeta_K(\mathfrak{C}';s)\,ds\\ &\ll \sum_{m\ge1}\frac{c'_m}{m^a}\int_1^T\Big(-\log\frac{t}{2\pi}+O\Big(\frac{1}{t}\Big)\Big)t^{n(a-1/2)}\exp\Big(-\frac{\pi t(n-k)}{2n}\Big)dt \ll \log T\end{aligned}$$

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since the integral is bounded and the last asymptotic formula follows from the Laurent expansion of the ideal class zeta-function at $s = 1$. In a similar way,

$$\int_{a+i}^{a+iT} \frac{\zeta'}{\zeta}(s)\Big(\frac{\Gamma(s)}{(2\pi)^s}\Big)^n d^{s-1/2} e_k \exp\Big(i\pi k\frac{1-s}{2}\Big)\zeta_K(\mathfrak{C}'; s)\, ds \ll (\log T)^2$$

since the ideal class zeta-function and the logarithmic derivative of the Riemann zeta-function have a simple pole at $s = 1$.

Case II: $k = n$. It remains to evaluate the integral with $\exp(i\pi n(1-s)/2)$. We write the integral as the difference of one with $\Delta'/\Delta(s)$ and the other with $\zeta'/\zeta(s)$, and estimate each of those integrals as follows.

Subcase I: the logarithmic derivative $\Delta'/\Delta(s)$. By the fundamental theorem of calculus,

$$\begin{aligned}\mathcal{I}_{3,\Delta} &:= \int_{a+i}^{a+iT} \frac{\Delta'}{\Delta}(s)\Big(\frac{\Gamma(s)}{(2\pi)^s}\Big)^n d^{s-1/2} \exp\Big(i\pi n\frac{1-s}{2}\Big)\zeta_K(\mathfrak{C}'; s)\, ds \\ &= i\exp\Big(i\pi\frac{n}{2}\Big)\int_1^T \frac{\Delta'}{\Delta}(a+i\tau)\, dJ(\tau), \end{aligned} \tag{3.5}$$

where the function $J(\tau)$ is defined by

$$J(\tau) := \int_1^\tau \bigg(\Big(\frac{d^{1/n}t}{2\pi}\Big)^{a-1/2+it} \exp\Big(-i\Big(t+\frac{\pi}{4}\Big)\Big)\Big(1+\frac{b_t}{t}\Big)\bigg)^n \sum_{m\geq 1}\frac{c_m'}{m^{a+it}}\, dt$$

for some real numbers b_t (which are bounded as functions of t). Now, we consider the function $J(\tau)$. It can be rewritten as

$$J(\tau) = d^{a-1/2}\exp\Big(-i\pi\frac{n}{4}\Big)\sum_{m\geq 1}\frac{c_m'}{m^a}\int_1^\tau \bigg(\Big(\frac{t}{2\pi}\Big)^{a-1/2}\exp\Big(it\log\frac{d^{1/n}t}{2\pi m^{1/n}e}\Big)\Big(1+\frac{b_t}{t}\Big)\bigg)^n dt.$$

Applying the generalised Gonek lemma (Lemma 2.3), the function $J(\tau)$ is equal to

$$\begin{aligned} &\frac{2\pi}{d^{1/2n}n^{1/2}}\exp\Big(i\pi\frac{1-n}{4}\Big)\sum_{m\geq 1}\frac{c_m'}{m^{(n-1)/2n}}\exp\Big(-2\pi i n\Big(\frac{m}{d}\Big)^{1/n}\Big)\chi_{(2^{-\mathcal{J}}\tau,\tau]}\Big(2\pi\Big(\frac{m}{d}\Big)^{1/n}\Big) + O \\ &= \frac{2\pi}{d^{1/2n}n^{1/2}}\exp\Big(i\pi\frac{1-n}{4}\Big)\sum_{d((2^{-\mathcal{J}}\tau)/2\pi)^n < m \leq d(\tau/2\pi)^n}\frac{c_m'}{m^{(n-1)/2n}}\exp\Big(-2\pi i n\Big(\frac{m}{d}\Big)^{1/n}\Big) + O, \end{aligned}$$

where $\mathcal{J}$ is the unique positive integer such that $2^{-\mathcal{J}}\tau < 1 \leq 2^{1-\mathcal{J}}\tau$, and the error term $O = O(\tau^{n(a-1/2)})$ can be estimated similarly to [7, Lemma 4].

Now, the method of partial summation and the asymptotic formula for the number of nonzero integral ideals in the class $\mathfrak{C}'$ with norm up to a given number can be applied to evaluate the above series. For convenience, let $x = d(\tau/2\pi)^n$, $y = d(2^{-\mathcal{J}}\tau/2\pi)^n$ and

$$\sum_{m\leq X} c_m' = \kappa X + O(X^\alpha(\log X)^\beta),$$

where α, β are determined in (2.2). By Abel's partial summation (Lemma 2.1),

$$\sum_{y<m\le x} \frac{c'_m}{m^{(n-1)/2n}} \exp\Big(-2\pi i n\Big(\frac{m}{d}\Big)^{1/n}\Big)$$
$$= \frac{\kappa \exp(-2\pi i n(x/d)^{1/n})}{x^{-(n+1)/2n}} - \frac{\kappa \exp(-2\pi i n(y/d)^{1/n})}{y^{-(n+1)/2n}}$$
$$+ \int_y^x \frac{\kappa \exp(-2\pi i n(t/d)^{1/n})}{t^{(n-1)/2n}}\Big(2\pi i\Big(\frac{t}{d}\Big)^{1/n} + \frac{n-1}{2n}\Big)\,dt + O(x^{\alpha-(n-3)/2n}(\log x)^{\beta}).$$

Integration by parts yields

$$\int_y^x \frac{\kappa \exp(-2\pi i n(t/d)^{1/n})}{t^{(n-1)/2n}} 2\pi i\Big(\frac{t}{d}\Big)^{1/n}\,dt = -\frac{\kappa \exp(-2\pi i n(x/d)^{1/n})}{x^{-(n+1)/2n}}$$
$$+ \frac{\kappa \exp(-2\pi i n(y/d)^{1/n})}{y^{-(n+1)/2n}} + \frac{\kappa(n+1)}{2n}\int_y^x \frac{\exp(-2\pi i n(t/d)^{1/n})}{t^{(n-1)/2n}}\,dt.$$

This implies that

$$\sum_{y<m\le x} \frac{c'_m}{m^{(n-1)/2n}} \exp\Big(-2\pi i n\Big(\frac{m}{d}\Big)^{1/n}\Big)$$
$$= \int_y^x \frac{\kappa \exp(-2\pi i n(t/d)^{1/n})}{t^{(n-1)/2n}}\,dt + O(x^{\alpha-(n-3)/2n}(\log x)^{\beta}) \ll \tau^{n\alpha-(n-3)/2}(\log \tau)^{\beta},$$

where the last asymptotic estimate follows from the integral term by applying the first derivative test (Lemma 2.2). More precisely, let $f(t) = -2\pi n(t/d)^{1/n}$ and $g(t) = 1/t^{(n-1)/2n}$. Then for $y \le t \le x$,

$$\left|\frac{f'(t)}{g(t)}\right| = \Big(\frac{2\pi}{d^{1/n}}\Big) t^{(1-n)/2n} \ge \Big(\frac{2\pi}{d^{1/n}}\Big) x^{(1-n)/2n}.$$

By the first derivative test (Lemma 2.2),

$$\left|\int_y^x g(t)\exp(if(t))\,dt\right| \le \frac{2d^{1/n}}{\pi} x^{(n-1)/2n} \ll x^{(n-1)/2n} \ll x^{\alpha-(n-3)/2n}.$$

Therefore,

$$J(\tau) \ll \max\{\tau^{n(a-1/2)}, \tau^{165/146}(\log \tau)^{315/146}\}.$$

Substituting $J(\tau)$ into (3.5),

$$\mathcal{I}_{3,\Delta} \ll \max\{T^{n/2}\log T, T^{165/146}(\log T)^{461/146}\}.$$

Subcase II: the logarithmic derivative $\zeta'/\zeta(s)$. By Stirling's formula and the generalised Gonek lemma (Lemma 2.3),

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$$
\begin{aligned}
\mathcal{I}_{3,\zeta} &:= \int_{a+i}^{a+iT} \frac{\zeta'}{\zeta}(s)\Big(\frac{\Gamma(s)}{(2\pi)^s}\Big)^n d^{s-1/2} \exp\Big(i\pi n\frac{1-s}{2}\Big)\zeta_K(\mathfrak{C}'; s)\, ds \\
&= -i \exp\Big(i\pi \frac{n}{4}\Big) d^{a-1/2} \sum_{j\geq 2} \frac{\Lambda(j)}{j^a} \sum_{m\geq 1} \frac{c'_m}{m^a} \\
&\quad \times \int_1^T \Big(\Big(\frac{t}{2\pi}\Big)^{a-1/2} \exp\Big(it \log \frac{d^{1/n} t}{2\pi(jm)^{1/n} e}\Big)\Big(1 + O\Big(\frac{1}{t}\Big)\Big)\Big)^n\, dt \\
&= -\frac{2\pi i}{d^{1/2n} n^{1/2}} \exp\Big(i\pi \frac{n+1}{4}\Big) \sum_{d(2^{-\mathcal{J}}T/2\pi)^n < k \leq d(T/2\pi)^n} \frac{1}{k^{(n-1)/2n}} \exp\Big(-2\pi i n\Big(\frac{k}{d}\Big)^{1/n}\Big) \\
&\quad \times \sum_{\ell \mid k} c'_\ell \Lambda\Big(\frac{k}{\ell}\Big) + O(T^{n/2} \log T),
\end{aligned}
\tag{3.6}
$$

where $\mathcal{J}$ is the unique positive integer such that $2^{-\mathcal{J}}T < 1 \leq 2^{1-\mathcal{J}}T$ and the error term follows as in [7, Lemma 4] together with the Laurent expansion of the logarithmic derivative of the Riemann zeta-function and the ideal class zeta-function at $s = 1$. After conjugation, we obtain the first desired result.

Next, we focus on (3.6). Applying the fact that $c'_k \ll k^{\varepsilon}, \Lambda(k) \ll k^{\varepsilon}$ and the number-of-divisors function $\sigma_o(k) \ll k^{\varepsilon}$ for any $\varepsilon > 0$,

$$\sum_{\ell \mid k} c'_\ell \Lambda\Big(\frac{k}{\ell}\Big) \ll k^{\varepsilon},$$

and then

$$\mathcal{I}_{3,\zeta} \ll \sum_{d(2^{-\mathcal{J}}T/2\pi)^n < k \leq d(T/2\pi)^n} \frac{1}{k^{(n-1)/2n-\varepsilon}} + O(T^{n/2} \log T) \ll T^{(n+1)/2+\varepsilon}$$

by applying Abel's summation formula (Lemma 2.1). Hence,

$$\overline{\mathcal{I}_3} \ll T^{(n+1)/2+\varepsilon}. \tag{3.7}$$

The same estimate holds for $\mathcal{I}_3$. Summing up (3.1), (3.2), (3.3) and (3.7), we obtain

$$\sum_{0<\gamma<T} \zeta_K(\mathfrak{C}; \rho) \ll T^{(n+1)/2+\varepsilon}.$$

4. Concluding remarks

The results we have obtained are not trivial. To see this, we first deduce from Theorem 1.1 that

$$\sum_{T<\gamma\leq 2T} \zeta_K(\mathfrak{C}; \rho) \ll T^{(n+1)/2+\varepsilon}. \tag{4.1}$$

In fact, this estimate also implies the one given in Theorem 1.1. Note that the hypothetical zeros of the Riemann zeta-function off the critical line appear in pairs, that is, if $\rho = \beta + i\gamma$ is one, then $1 - \beta + i\gamma$ is another (as follows from the functional

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equation). If now $\frac{1}{2} < \beta < 1$, then, in view of the bound (2.4), the contribution of this zero to the sum in question would be of size

$$\zeta_K(\mathfrak{C};\rho) \ll T^{n\beta/2+\varepsilon}$$

for $\gamma \in (T, 2T]$. Multiplying this by the number of all zeros $\rho = \frac{1}{2} + i\gamma$ with $T < \gamma \leq 2T$ yields

$$\sum_{T<\gamma\leq 2T} \zeta_K(\mathfrak{C};\rho) \ll T^{1+n/2+\varepsilon} \log T.$$

Comparing this with (4.1) shows that Theorem 1.1 gives a better bound. One should mention that there are techniques available to find unconditional bounds for $\zeta_K(\mathfrak{C}; \frac{1}{2} + it)$ which may improve (4.1); however, the authors are not aware of any such result in the literature and their derivation would not be too easy.

The proof of Theorem 1.1 is of interest also for another reason. It should be mentioned that, at least to the best knowledge of the authors, the above reasoning is the first application of the method of Conrey *et al.* to a zeta-function of degree $n > 1$ (in the sense of the extended Selberg class). Note that Conrey *et al.* [3–5] rewrote the first derivative of a Dedekind zeta-function $\zeta_K(s) = \zeta(s)L(s,\psi)$ of a quadratic number field at $s = \rho$ as

$$\zeta_K'(\rho) = \zeta'(\rho)L(1-\rho,\psi),$$

where the symmetry of the zeros of the Dirichlet L-function $L(s,\psi)$ for a real character ψ (by the functional equation) is used. This allowed them to evaluate the sum over the values $\zeta_K'(\rho)$. Their reasoning is then based on an evaluation of the integral

$$\int \zeta'(s)L(1-s,\psi)\, ds,$$

or rather a variation of it after applying the functional equation. On both vertical sides of the path of integration, the integrand is of order $O(t^{1/2+\varepsilon})$ (as for a degree-one element of the Selberg class on the left). The underlying symmetry, however, does not apply to our case in general.

In principle, the method can be extended further, for example, to consider the mean square. It is to be expected that, as in our note, the bound comes from the integral over the left vertical line segment.

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