CONSTRUCTION OF PRINCIPAL FUNCTIONS BY ORTHOGONAL PROJECTION

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1. Normal operators. Given a point set E on an open Riemann surface V we denote by $H(E)$ the space of functions u harmonic in open sets $O(u)$ containing E. Let V_0 be a regular region of V with border α, and consider restrictions f to α of functions in $H(\alpha)$. For $V_1 = V - \bar{V}_0$, an operator L from $H(\alpha)$ to $H(\bar{V}_1)$ is, by definition, *normal* if

$$Lf = f \quad \text{on } \alpha, \tag{1}$$

$$L(c_1 f_1 + c_2 f_2) = c_1 L(f_1) + c_2 L(f_2), \tag{2}$$

$$L1 = 1, \tag{3}$$

$$Lf \geqslant 0 \qquad \text{for } f \geqslant 0, \tag{4}$$

$$\int_\alpha {}^*dLf = 0. \tag{5}$$

For general properties of normal operators we refer to Ahlfors **(1)**, Ahlfors and Sario**(2)**, Oikawa **(5, 6)**, Rodin **(7)**, Sario **(8, 9, 10)**, Sario, Schiffer, and Glasner **(11)**, Sario and Weill **(12)**, and Weill **(13)**.

Let Ω be a regular region with border β_Ω such that $\bar{V}_0 \subset \Omega$. For a given f denote by u_Ω the harmonic function in $\bar{\Omega} \cap \bar{V}_1$ with $u_\Omega|\alpha = f$, $u_\Omega|\beta_\Omega = \text{const}$, $\int_\alpha {}^*du_\Omega = 0$. As Ω exhausts V, u_Ω tends to a harmonic limit $u = L_1 f$ on $\bar{V}_1$, where L_1 is a normal operator. Using Royden's compactification we shall first show (Theorem 1) that a normal operator L is L_1 if and only if, in a sense, Lf is constant on the ideal boundary of V.

2. Principal functions. The *principal function problem* consists in constructing, for a given $s \in H(\bar{V}_1)$ and given L, a function $p \in H(V)$ such that

$$p|\bar{V}_1 = s + L(p - s|\alpha). \tag{6}$$

It is known that the condition

$$\int_\alpha {}^*ds = 0 \tag{7}$$

is necessary and sufficient for the solvability of the problem **(9)**. The solution, called the *principal function*, is unique up to an additive constant. The function s is interpreted as having a singularity on the ideal boundary of V and is called the *singularity function*.

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We shall show that for any given V and s with (7) the existence of the principal function corresponding to L_1 can be proved by the method of orthogonal projection (Theorem 2). It should be noted here that an inequality of the Poincaré type (Lemma 3) takes the place of the Harnack inequality in the existence proof. It is hoped that our study, methodological in nature, will also pave the road to the solution of the main problem, the construction of principal forms in Riemannian spaces.

Reference is made here to the recent interesting Research Announcement by Browder **(3)**. Although there were some technical difficulties in applying his approach to prove earlier results or to extend them in the original direction, his Announcement threw new light on the entire principal function problem and was indeed the immediate incentive to the present study.

3. Weyl's lemma. We denote by $\Gamma = \Gamma(V)$ the space of real measurable 1-forms ω on V with

$$\int_V \omega \wedge {}^*\omega < \infty .$$

With the inner product

$$(\omega_1, \omega_2) = \int_V \omega_1 \wedge {}^*\omega_2$$

and the norm $||\omega|| = \sqrt{(\omega, \omega)}$, Γ becomes a Hilbert space. Let $\Gamma_e{}^1$ be the subspace of Γ of continuous exact differentials:

$$\Gamma_e{}^1 = \{df | f \in C^1(V), df \in \Gamma\}.$$

The closure of $\Gamma_e{}^1$ in Γ is denoted by Γ_e. We also consider the space $\Gamma_{e0}{}^1$ of continuous exact differentials with compact supports in V, i.e.

$$\Gamma_{e0}{}^1 = \{df | f \in C_0{}^1(V)\}.$$

Then $\Gamma_{e0}{}^1 \subset \Gamma$, and we denote by Γ_{e0} the closure of $\Gamma_{e0}{}^1$ in Γ. We shall use Weyl's lemma in the following form **(2)**:

LEMMA 1. *If an element α in Γ_e is orthogonal to $\Gamma_{e0}{}^1$, then there exists a function u in $HD(V)$ such that $\alpha = du$, and vice versa.*

4. Royden's boundary. A real-valued continuous function f on V is said to be a *continuous Dirichlet function* if there exists an ω in $\Gamma(V)$ such that

$$\int_V \omega \wedge \omega_0 = -\int_V f d\omega_0$$

for any C^2-form ω_0 on V with compact support; we set $\omega = df$. Denote by $R(V)$ the family of continuous Dirichlet functions on V and by $R_0(V)$ the subfamily of functions with compact supports in V. For $f, g \in R(V)$ we set

$$\rho(f, g) = ||df - dg|| + \sum_{n=1}^{\infty} 2^{-n} \sup_{K_n} |f - g| \cdot (1 + |f - g|)^{-1},$$

where $\{K_n\}$ is an exhaustion of V. Endowed with ρ, $R(V)$ is a complete metric space. Let $R_\delta(V)$ be the closure of $R_0(V)$ in $R(V)$ in terms of this metric ρ.

The *Royden compactification* V^* of V is the compact Hausdorff space with the following two properties: (a) it contains V as its open and dense subspace such that every function in $R(V)$ can be extended continuously to V^* with infinite values admitted; (b) $R(V)$, considered as a family of functions on V^*, separates points in V^*. The compact set $\beta = V^* - V$ is the *Royden boundary* of V. The set

$$\delta = \{p \in V^* | f(p) = 0 \text{ for every } f \in R_\delta(V)\}$$

is a compact subset of β. We call δ the *Royden harmonic boundary* of V. For details and fundamental properties of these concepts we refer to **(4)**.

LEMMA 2. *Let u be a function in $HD(V_1)$ such that du can be continued to all of V so as to be a 1-form ω in $\Gamma_{e0}(V)$. Then u is finitely continuous on*

$$V_1 \cup \beta = V^* - \bar{V}_0$$

and a constant on δ.

Proof. We take a smaller boundary neighbourhood V'_1 with $\bar{V}'_1 \subset V_1$, if necessary, to assume that $u \in HD(\bar{V}_1)$. Then we can continue u to V as a function u_0 in $C^1(V)$. Clearly $u_0 \in R(V)$; thus u_0 is continuous on V^* and a fortiori u is continuous on $V_1 \cup \beta$, with infinite values admitted. If $V \in O_G$, then $\beta = \emptyset$, $HD(\bar{V}_1) = HB(\bar{V}_1)$, and the assertion is trivial. Therefore we may assume that $V \notin O_G$.

Since $R(V) = HD(V) + R_\delta(V)$, there exists a function $v \in HD(V)$ such that $u_0 - v \in R_\delta(V)$. Then $d(u_0 - v) \in \Gamma_{e0}$ and consequently $dv \in \Gamma_{e0}$. By Lemma 1, dv is orthogonal to $\Gamma_{e0}{}^1$ and hence to Γ_{e0}. In particular,

$$||dv||^2 = (dv, dv) = 0,$$

which means that $u_0 - \text{const} \in R_\delta(V)$, or $u_0 = u = \text{const}$ on δ. By the maximum principle, u is finitely continuous at β.

5. A characterization of the L_1-operator. We can now give a characterization of the operator L_1 in terms of the Royden compactification: it is a normal operator such that $L_1 f$ is finitely continuous at the Royden boundary β and a constant c_f on the Royden harmonic boundary δ for every f in $H(\alpha)$. Explicitly, we have for a given $u \in H(\bar{V}_1)$:

THEOREM 1. *Necessary and sufficient for $L_1 u = u$ is that u satisfies the following conditions:*

(8) *u is finitely continuous on $V_1 \cup \beta$,*

(9) *u is a constant on δ,*

(10) $$\int_\alpha {}^*du = 0.$$

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Proof. If $V \in O_G$, the properties $L_1 u = u$ and (8) are each equivalent to $u \in HB(\bar{V}_1) = HD(\bar{V}_1)$, which includes (10). Condition (9) is trivially satisfied by every u in $H(\bar{V}_1)$ because $\delta = \emptyset$. Thus, we have only to consider the case $V \notin O_G$.

First assume that $L_1 u = u$. Let $\{U_n\}_{n=0}^{\infty}$ be an exhaustion of V by regular regions, with $U_0 = V_0$. For f in $H(\alpha)$, let $S_n f$ be the harmonic function in $\bar{U}_n - U_0$ with continuous boundary value 0 on ∂U_n and f on $\alpha = \partial V_0 = \partial U_0$. Let $f_0 \in C^1(\bar{U}_0)$ such that $f_0 = f$ on α. We set $S'_n f = S_n f$ on $\bar{U}_n - U_0$, $S'_n f = f_0$ on $\bar{U}_0$, and $S'_n f = 0$ on $V - U_n$. Then $S'_n f \in R_0(V)$ and since

$$||d(S'_n f) - d(S'_{n+m} f)||^2 = ||d(S'_n f)||^2 - ||d(S'_{n+m} f)||^2,$$

we see that $\{S'_n f\}_{n=1}^{\infty}$ converges to a function, say $S'f$, on V in the ρ-metric. Hence $S'f \in HD(\bar{V}_1)$, $Sf|\alpha = f$, Sf is continuous on $V_1 \cup \beta$, and $Sf = 0$ on δ. Moreover $||d(S'_n f) - d(Sf)||_{V_1} \to 0$ as $n \to \infty$, and $S'_n f$ converges to Sf uniformly in compact sets of $\bar{V}_1 = V_1 \cup \alpha$. It follows that

$$\lim_{n\to\infty} \int_\alpha {}^*d(S_n f) = \int_\alpha {}^*d(Sf). \tag{11}$$

Let w_n be harmonic in $\bar{U}_n - U_0$ with $w_n|\partial U_0 = 1$, $w_n|\alpha = 0$. Similarly let w be continuous on $V_1 \cup \beta$ and harmonic in V_1 with boundary value 1 at δ and 0 at α. By the same argument as above, $||dw'_n - dw||_{V_1} \to 0$ as $n \to \infty$, and w'_n converges to w uniformly in compact sets of $\bar{V}_1 = V_1 \cup \alpha$; here we set $w'_n = w_n$ on $\bar{U}_n - U_0$ and $w'_n = 1$ on $V - U_n$. We have

$$\int_\alpha {}^*dw_n = - \int_{\partial U_n} {}^*dw_n = -||dw'_n||_{V_1}^2.$$

Hence

$$\lim_{n\to\infty} \int_\alpha {}^*dw_n = \int_\alpha {}^*dw = -\lim_{n\to\infty} ||dw'_n||_{V_1}^2 = -||dw||^2 < 0. \tag{12}$$

If we put

$$L_1^{(n)} f = S_n f - \left(\int_\alpha {}^*d(S_n f) \Big/ \int_\alpha {}^*dw_n \right) w_n,$$

then $L_1^{(n)} f$ is constant on ∂U_n and

$$\int_\alpha {}^*d(L_1^{(n)} f) = 0.$$

Thus by the definition of $L_1 f$, $L_1 f = \lim_n L_1^{(n)} f$ on V. Using (11) and (12), we obtain the representation

$$L_1 f = Sf - \left(\int_\alpha {}^*d(Sf) \Big/ \int_\alpha {}^*dw \right) w. \tag{13}$$

By the properties of Sf and w, $L_1 f$ is continuous on $V_1 \cup \beta$ and a constant on δ. Thus, in particular, $u = L_1 u$ satisfies (8), (9), and (10).

Conversely assume that u satisfies these three conditions. As above, the same is true of $L_1 u$ and of $v = u - L_1 u$. Let $v = c$ on δ. Then $v = cw$ and, by (12), we must have $c = 0$. Therefore $v \equiv 0$ on V_1.

These arguments are based on the fact that every function in $HD(\bar{V}_1)$ takes its maximum and minimum on $\alpha \cup \delta$.

6. Fundamental inequality. Before embarking on the existence proof by the method of orthogonal projection we need the following lemma; it plays a role in our proof of equal importance to the q-lemma in **(9)**.

LEMMA 3. *Let ω be a fixed continuous 1-form defined on α such that*

$$\int_\alpha \omega = 0.$$

Then there exists a constant c depending only on V_0 and ω such that

$$\left| \int_\alpha \phi\omega \right|^2 \leqslant c \int_{V_0} d\phi \wedge {}^*d\phi \tag{14}$$

for every $\phi \in C^1(V_0) \cap C(\bar{V}_0)$.

Proof. First we note that we have only to prove (14) for ϕ in $H(V_0) \cap C(\bar{V}_0)$. In fact, for $\phi \in C^1(\bar{V}_0)$ we let h_ϕ be the harmonic function in V_0 with continuous boundary value ϕ at $\alpha = \partial V_0$. Since

$$\int_{V_0} dh_\phi \wedge {}^*dh_\phi \leqslant \int_{V_0} d\phi \wedge {}^*d\phi$$

and

$$\int_\alpha h_\phi\, \omega = \int_\alpha \phi\omega,$$

the validity of (14) for h_ϕ gives that for ϕ. We therefore may and will assume in the following that $\phi \in H(V_0) \cap C(\bar{V}_0)$.

Let z_0 be a fixed point in V_0 and let $g_0(z, z_0)$ be Green's function on V_0. We put

$$r(z) = \exp(-g_0(z, z_0)), \qquad d\theta(z) = -{}^*dg_0(z, z_0),$$

and form Green's star domain V'_0 of $g_0(z, z_0)$ on V_0. Explicitly V'_0 is obtained from V_0 by removing all closures of Green's lines issuing from the branch point of $g_0(z, z_0)$ in V_0. Then $r(z)e^{i\theta(z)}$ maps V'_0 onto a unit disk with a finite number of radial slits issuing from some point in the disk different from the origin and terminating at the unit circumference in a one-to-one and conformal fashion, α corresponding to the unit circumference. Fix a positive number a such that $0 < 2a < 1$ and $\{z | z \in V_0, r(z) \leqslant 2a\}$ is the disk in V'_0. We write $\phi_r(\theta) = \phi(re^{i\theta})$, which can be considered as an element in $L^2(0, 2\pi)$ with norm

$$|\phi_r|_2 = \sqrt{\int_0^{2\pi} |\phi_r(\theta)|^2 d\theta}.$$

First assume that $\phi(z_0) = 0$. Except for a finite number of values θ,

$$\phi(e^{i\theta}) - \phi(ae^{i\theta}) = \int_a^1 \frac{\partial}{\partial r} \phi(re^{i\theta})dr.$$

By Schwarz's inequality,

$$|\phi_1(\theta) - \phi_a(\theta)|^2 \leqslant \int_a^1 \left| \frac{\partial}{\partial r} \phi(re^{i\theta}) \right|^2 rdr \cdot \int_a^1 r^{-1}dr.$$

Hence on integrating both sides over $(0, 2\pi)$ and on observing that

$$\left| \frac{\partial}{\partial r} \phi(re^{i\theta}) \right|^2 \leqslant \left| \frac{\partial}{\partial r} \phi(re^{i\theta}) \right|^2 + r^{-2} \left| \frac{\partial}{\partial \theta} \phi(re^{i\theta}) \right|^2 = |\text{grad } \phi|^2,$$

we obtain

$$|\phi_1 - \phi_a|_2 \leqslant \sqrt{\log a^{-1}} \cdot \sqrt{\int_{V_0} d\phi \wedge {}^*d\phi}\,. \tag{15}$$

Let ${}^*\phi(z)$ be the conjugate harmonic function of $\phi(z)$ in $|z - z_0| < 2a$ such that ${}^*\phi(z_0) = 0$. Consider the analytic function $f(z) = \phi(z) + i{}^*\phi(z)$ in $|z - z_0| < 2a$. Since $f(z_0) = 0$, we have

$$f(z) = \int_{z_0}^z f'(z)dz$$

and thus

$$|\phi(z)| \leqslant |f(z)| \leqslant |z - z_0| \max_{|z-z_0| \leqslant a} |f'(z)|$$

for $|z - z_0| \leqslant a$. In particular

$$|\phi_a(\theta)|^2 \leqslant a^2 \max_{|z-z_0| \leqslant a} |f'(z)|^2. \tag{16}$$

As $|f'(z)|^2$ is subharmonic in $|z - z_0| < 2a$,

$$|f'(z)|^2 \leqslant (\pi a^2)^{-1} \int_{|\zeta - z| \leqslant a} |f'(\zeta)|^2 d\xi d\eta$$

for z in $|z - z_0| \leqslant a$, with $\zeta = \xi + i\eta$. Since $|f'(\zeta)|^2 d\xi d\eta = d\phi \wedge {}^*d\phi$ in $|z - \zeta| \leqslant a$, we conclude that

$$|f'(z)|^2 \leqslant (\pi a^2)^{-1} \int_{V_0} d\phi \wedge {}^*d\phi.$$

This with (16) gives

$$|\phi_a(\theta)|^2 \leqslant \pi^{-1} \int_{V_0} d\phi \wedge {}^*d\phi$$

and therefore

$$|\phi_a|_2 \leqslant \sqrt{2 \int_{V_0} d\phi \wedge {}^*d\phi}\,. \tag{17}$$

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Since $|\phi_1|_2 \leqslant |\phi_1 - \phi_a|_2 + |\phi_a|_2$, we infer from (15) and (17) that

$$\int_0^{2\pi} |\phi(e^{i\theta})|^2 d\theta \leqslant c_1 \int_{V_0} d\phi \wedge {}^*d\phi \tag{18}$$

where $c_1 = 2 + \log a^{-1}$ depends only on V_0. By a piecewise analytic representation of α with parameter θ, ω can be expressed as $\omega = \Omega(\theta)d\theta$ on α. Here $\Omega(\theta)$ is bounded, say $|\Omega(\theta)| \leqslant c_2$, and piecewise continuous on α, with c_2 depending only on ω on V_0. By Schwarz's inequality and (18) we obtain

$$\begin{aligned} \left| \int_\alpha \phi\omega \right|^2 &= \left| \int_0^{2\pi} \phi(e^{i\theta}) \cdot \Omega(\theta) d\theta \right|^2 \\ &\leqslant \int_0^{2\pi} |\phi(e^{i\theta})|^2 d\theta \cdot \int_0^{2\pi} |\Omega(\theta)|^2 d\theta \\ &\leqslant c \int_0^{2\pi} d\phi \wedge {}^*d\phi \end{aligned}$$

where $c = c_1 \cdot c_2^2$ depends on ω and V_0.

Next consider $\phi \in H(V_0) \cap C(\bar{V}_0)$, with not necessarily $\phi(z_0) = 0$; then $\tilde{\phi} = \phi - \phi(z_0)$ satisfies $\tilde{\phi}(z_0) = 0$. By the above reasoning,

$$\left| \int_\alpha \tilde{\phi}\omega \right|^2 \leqslant c \int_{V_0} d\tilde{\phi} \wedge {}^*d\tilde{\phi}. \tag{19}$$

In view of the assumption

$$\int_\alpha \omega = 0$$

we have

$$\int_\alpha \tilde{\phi}\omega = \int_\alpha \phi\omega - \phi(z_0) \int_\alpha \omega = \int_\alpha \phi.$$

Obviously $d\tilde{\phi} = d\phi$ and substitution in (19) gives (14) for ϕ.

7. Existence proof by orthogonal projection. We proceed to the proof by orthogonal projection of the existence of principal functions:

THEOREM 2. *Let s be a harmonic function on $V_1 \cup \alpha$ with property* (7). *Then there exists a harmonic function p on V which satisfies equation* (6).

Proof. First we extend s to all of the surface V as a function $s_0 \in C^2(V)$. For $\omega \in \Gamma_{e0}^1$ we put

$$T(\omega) = -\int_V \omega \wedge {}^*ds_0 = -\int_V ds_0 \wedge {}^*\omega. \tag{20}$$

This is well-defined because ω has compact support and $\omega \wedge {}^*ds_0 = ds_0 \wedge {}^*\omega$ is a continuous 2-form with compact support in V. Clearly T gives rise to a linear operator on Γ_{e0}^1.

We shall show next that T is a bounded linear operator on Γ_{e0}^1. Take a regular region $W \subset V$ which contains $\bar{V}_0$ and the support of $\omega \in \Gamma_{e0}^1$. Then

$$(21) \qquad -T(\omega) = \int_{\bar{V}_0} \omega \wedge {}^*ds_0 + \int_{W-\bar{V}_0} \omega \wedge {}^*ds_0.$$

By Schwarz's inequality

$$(22) \qquad \left| \int_{\bar{V}_0} \omega \wedge {}^*ds_0 \right| \leqslant \sqrt{\int_{\bar{V}_0} ds_0 \wedge {}^*ds_0} \cdot \|\omega\|.$$

Since $\omega \in \Gamma_{e0}^1$, there exists a function ϕ in $C_0^1(V)$ with its support in W and such that $\omega = d\phi$ on W. As

$$\omega \wedge {}^*ds_0 = d\phi \wedge {}^*ds_0 = d(\phi^*ds_0) - \phi d^*ds_0 = d(\phi^*ds_0)$$

in $V - \bar{V}_0$ and thus in $W - \bar{V}_0$, we have by Green's formula

$$(23) \qquad \int_{W-\bar{V}_0} \omega \wedge {}^*ds_0 = \int_{W-\bar{V}_0} d(\phi^*ds_0) = \int_{\partial W+\alpha} \phi^*ds_0 = \int_\alpha \phi^*ds_0.$$

By virtue of

$$\int_\alpha {}^*ds_0 = \int_\alpha {}^*ds = 0$$

and $\phi \in C^1(V_0) \cap C(\bar{V}_0)$, we can apply Lemma 3 to

$$\int_\alpha \phi^*ds_0$$

so as to obtain a constant c depending only on ${}^*ds_0$ and V_0 and such that

$$\left| \int_\alpha \phi^*ds_0 \right|^2 \leqslant c \int_{V_0} d\phi \wedge {}^*d\phi.$$

Because of $d\phi = \omega$ and (23) we infer that

$$(24) \qquad \left| \int_{W-\bar{V}_0} \omega \wedge {}^*ds_0 \right| \leqslant \sqrt{c}\, \|\omega\|.$$

From (21), (22), and (24), we obtain

$$(25) \qquad |T(\omega)| \leqslant K\|\omega\|$$

for $\omega \in \Gamma_{e0}^1$, where

$$K = \sqrt{c} + \int_{\bar{V}_0} ds_0 \wedge {}^*ds_0$$

depends only on s_0 and V_0.

By the general Hilbert space theory, T can be extended to

$$\Gamma_{e0} = \overline{\Gamma_{e0}^1}$$

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so as to satisfy (25) again on Γ_{e0}. Since Γ_{e0} is self-adjoint, there exists a unique element λ in Γ_{e0} such that

$$(26) \qquad T(\omega) = (\omega, \lambda)$$

for $\omega \in \Gamma_{e0}$. In particular, by (20) and (26),

$$(27) \qquad \int_V (\lambda + ds_0) \wedge {}^*\omega = 0$$

for $\omega \in \Gamma_{e0}{}^1$.

Again let $\{U_n\}_{n=0}^{\infty}$ be an exhaustion of V with $U_0 = V_0$. Although $\lambda + ds_0$ is not an element of $\Gamma_e(V)$ in general, we can conclude that

$$\lambda + ds_0 \in \Gamma_e(U_n),$$

because

$$\lambda \in \Gamma_{e0}(V) \subset \Gamma_e(V) \subset \Gamma_e(U_n)$$

and $s_0 \in C^2(\bar{U}_n)$ and hence $ds_0 \in \Gamma_e(U_n)$. Since (27) holds for

$$\omega \in \Gamma_{e0}{}^1(U_n) \subset \Gamma_{e0}{}^1(V),$$

there exists by Lemma 1 a $q_n \in HD(U_n)$ such that $dq_n = \lambda + ds_0$ on U_n. Clearly $dq_{n+m} = dq_n$ on U_n and therefore $q_{n+m} = q_n +$ const on U_n. Let c_n be a constant such that $q_{n+1} = q_n + c_n$ on U_n and set $p_1 = q_1$ on U_1, $p_n = q_n - c_{n-1}$ on U_n with $n > 1$. Then $p_n \in HD(U_n)$ and $dp_n = \lambda + ds_0$ on U_n and $p_{n+m} = p_n$ on U_n. Thus if we put

$$(28) \qquad p(z) = p_n(z)$$

for z in U_n, then $p(z)$ does not depend on the choice of U_n to which z belongs. Therefore $p \in H(V)$ and

$$(29) \qquad dp = \lambda + ds_0.$$

The function $u = p - s_0$ belongs to $C^2(V)$ and clearly

$$(30) \qquad u \in H(\bar{V}_1)$$

together with p and s_0. Since

$$\int_\alpha {}^*dp = \int_{V_0} d^*dp = \int_{V_0} \Delta p = 0,$$

we have by (7)

$$(31) \qquad \int_\alpha {}^*du = 0.$$

From $du = \lambda \in \Gamma_{e0}$, $u \in HD(V_1)$, and Lemma 2, it follows that

$$(32) \qquad u \in C(V_1 \cup \beta), \qquad u = \text{const on } \delta.$$

On applying Theorem 1 to this u, we conclude by (30), (32), and (31) that $L_1 u = u$ on $\bar{V}_1 = V_1 \cup \alpha$. In view of $u = p - s_0 = p - s$ on $\alpha \cup V_1$, we have $L_1(p - s) = p - s$ on V_1, i.e. p satisfies (6).

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