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ON THE CAPABILITY OF FINITELY GENERATED NON-TORSION GROUPS OF NILPOTENCY CLASS 2

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Abstract. A group is called capable if it is a central factor group. In this paper, we establish a necessary condition for a finitely generated non-torsion group of nilpotency class 2 to be capable. Using the classification of two-generator non-torsion groups of nilpotency class 2, we determine which of them are capable and which are not and give a necessary and sufficient condition for a two-generator non-torsion group of class 2 to be capable in terms of the torsion-free rank of its factor commutator group.

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1. Introduction. A group G is said to be capable if there exists a group H such that $G \cong H/Z(H)$, or equivalently, G is isomorphic to the inner automorphism group of a group H. Capability of groups was first studied by Baer [**4**] in 1938 and his characterisation of finitely generated abelian groups that are capable is given in the following theorem.

THEOREM 1.1 ([**4**]). *Let A be a finitely generated abelian group written as*

$$A = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k},$$

such that each integer n_{i+1} is divisible by n_i, where $\mathbb{Z}_0 = \mathbb{Z}$, the infinite cyclic group. Then A is capable if and only if $k \geq 2$ and $n_{k-1} = n_k$.

In 1979, Beyl et al. [**6**] established a necessary and sufficient condition for a group to be a central quotient in terms of the epicentre defined as follows.

DEFINITION 1.2 ([**6**]). The epicentre $Z^*(G)$ of a group G is defined as

$$\bigcap \{\phi Z(E) \mid (E, \phi) \text{ is a central extension of } G\}.$$

It can be easily seen that the epicentre is a characteristic subgroup of G contained in its centre. Thus, $Z^*(G) \subseteq Z(G)$. The following criterion now characterises capable groups.

THEOREM 1.3 ([**6**]). *A group G is capable if and only if $Z^*(G) = 1$.*

The above criterion for capability is easily formulated but hard to implement. As in all cases before, this still requires the cumbersome process of evaluating factor sets in its implementation. Ellis **[9]** was able to characterise the epicentre with the help of the non-abelian tensor square, and thus made the determination of the epicentre computationally accessible. The non-abelian tensor square is defined as follows.

DEFINITION 1.4 ([**7**]). For a group G, the non-abelian tensor square $G \otimes G$ is generated by the symbols $g \otimes h$, $g, h \in G$, subject to the relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h), \tag{1}$$

$$g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h') \tag{2}$$

for all $g, g', h, h' \in G$, where ${}^{h}g = hgh^{-1}$ denotes the conjugate of g by h.

What we really need for our characterisation is a quotient of the tensor square, the exterior square, that is defined as follows.

DEFINITION 1.5 ([**7**]). For any group G, the exterior square of G is defined as $G \wedge G = (G \otimes G)/\nabla(G)$,where $\nabla(G) = \langle x \otimes x | x \in G\rangle$ and $\nabla(G)$ is a central subgroup of $G \otimes G$.

Similar to the tensor centre $Z^{\otimes}(G) = \{a \in G | a \otimes g = 1_{\otimes}$, for all $g \in G\}$ **[8]**, we can define the exterior centre of a group as $Z^{\wedge}(G) = \{a \in G | a \wedge g = 1_{\wedge}$, for all $g \in G\}$. Here $1_{\otimes}$ and $1_{\wedge}$ denote the identities in $G \otimes G$ and $G \wedge G$, respectively. It can be easily shown that $Z^{\otimes}(G)$ and $Z^{\wedge}(G)$ are characteristic and central subgroups of G with $Z^{\otimes}(G) \subseteq Z^{\wedge}(G)$. With the help of the exterior centre, Ellis in **[9]** obtained the desired external characterisation of the epicentre as follows.

THEOREM 1.6 ([**9**]). *For any group G, the epicentre coincides with the exterior centre, that is,* $Z^{*}(G) = Z^{\wedge}(G)$.

In **[5]**, tensor methods were used for the first time to determine the infinite metacyclic groups that are capable. Explicit knowledge of the tensor square of two-generator p-groups of class 2, p an odd prime, enabled the authors of **[3]**, to investigate the capability of such groups. In **[12]**, using different methods, Magidin investigated the same class of groups and determined the capable ones among them. In addition, he generalised a necessary condition for the capability of finite p-groups established by Hall in **[10]**. Interestingly enough, for small class, that is, the class c is smaller than p, this criterion reduces to what occurs in Baer's result for finitely generated abelian groups, namely that for a capable group in any generating set, there exist two elements of maximal order. In **[13]**, Magidin extended his investigations to the class of two-generator two groups of nilpotency class 2.

In **[1]**, a new classification for two-generator p-groups of nilpotency class 2 is given that corrects and simplifies previous classifications published in **[2, 11, 16]**. Based on this new classification, Magidin and Morse give a complete determination of all capable two-generator p-groups of class 2, using a unified approach and modified tensor methods **[14]**. Thus with the exception of the non-torsion groups among them, the question of capability for two-generator groups of class 2 has been settled, since it suffices to consider the Sylow p-subgroups in case of finite two-generator groups of nilpotency class 2. The purpose of this paper is to determine exactly which two-generator non-torsion groups of class 2 are capable.

In [**15**], the two-generator non-torsion groups were classified and their non-abelian tensor squares were determined. This turned out to be an easier task than in the case of two-generator p-groups of class 2, since in the non-torsion case we only have to deal with split extensions. Also the capability question can be settled easier in this case, using a necessary criterion for the capability of finitely generated non-torsion groups of nilpotency class 2 (Theorem 3.2). The criterion is in terms of the torsion-free rank of G/G', the commutator factor group of the group. According to the necessary condition, all groups with torsion-free rank 1 are not capable and no further calculations of the epicentre are necessary. Thus, our necessary criterion for the non-torsion case is very similar to the one given by Hall [**10**] and Magidin [**12**] for capable p-groups of class 2, guaranteeing the existence of at least two elements of maximal order in every generating set.

In case the torsion-free rank of the factor commutator subgroup is greater than 1, that is, 2 in the case of two-generator groups, a simple calculation shows that the epicentre is trivial in all these cases. This leads to an interesting characterisation of capable two-generator non-torsion groups (Corollary 4.3). The groups are capable if and only if their abelianisations are capable. In case the group has more than two generators, we will show that there exists a group with a factor commutator subgroup of rank 2, which is not capable (Example 3.3).

2. Some preparatory results. This section contains some results to be used throughout the rest of the paper. We start with some tensor expansion formulas for groups of nilpotency class 2, which can be found in Proposition 3.2 and Lemma 3.4 of [**2**].

LEMMA 2.1. *Let G be a group of nilpotency class 2 with $g, g', h, h' \in G$ and $m, n \in \mathbb{Z}$. Then*

$$gg' \otimes h = (g \otimes h)(g' \otimes h)([g, g'] \otimes h)(g' \otimes [g, h]), \tag{3}$$

$$g \otimes hh' = (g \otimes h)(g \otimes h')(g \otimes [h, h'])([h, g] \otimes h'), \tag{4}$$

$$g^n \otimes h^m = (g \otimes h)^{nm}(h \otimes [g, h])^{n\binom{m}{2}}(g \otimes [g, h])^{m\binom{n}{2}}. \tag{5}$$

For the non-abelian tensor square of a direct product, the following formula can be found in [**7**]. Here $\otimes_{\mathbb{Z}}$ denotes the usual tensor product of abelian groups.

PROPOSITION 2.2. *Let G and H be groups, then*

$$(G \times H) \otimes (G \times H) = (G \otimes G) \times (G^{\mathrm{ab}} \otimes_{\mathbb{Z}} H^{\mathrm{ab}}) \times (H^{\mathrm{ab}} \otimes_{\mathbb{Z}} G^{\mathrm{ab}}) \times (H \otimes H),$$

where G^{ab} and H^{ab} are the abelianisations of G and H, that is, $G^{\mathrm{ab}} = G/G'$ and $H^{\mathrm{ab}} = H/H'$.

In [**3**], for a group G and $h \in G$, the exterior centraliser of h was defined as $C^{\wedge}(h) = \{x \in G | x \wedge h = 1_{\wedge}\}$, or equivalently $C^{\wedge}(h) = \{x \in G | x \otimes h \in \nabla(G)\}$. To effectively compute the epicentre of a group, we will use the following proposition from [**3**].

PROPOSITION 2.3. *Let G be a group with $g \in G$. Then $C^{\wedge}(g)$ is a subgroup of G with $C^{\wedge}(g) \subseteq C_G(g)$, the centraliser of g in G and $\bigcap_{g \in G} C^{\wedge}(g) = Z^{\wedge}(G)$.*

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3. The finitely generated case. The topic of this section is deriving a necessary condition for the capability of a finitely generated non-torsion group of nilpotency class 2. Our main result is a corollary of the following theorem.

THEOREM 3.1. *Let G be a non-torsion group of nilpotency class 2 and $T(G)$ its torsion subgroup. If $G/T(G)$ is infinite cyclic and $T(G)$ has finite exponent, then G is not capable.*

Proof. Suppose G is a non-torsion group of nilpotency class 2. Assume that $G/T(G)$ is infinite cyclic and $T(G)$ has finite exponent. It follows that $G/T(G)$ is abelian, and hence $G' \subseteq T(G)$. Thus G is the semi-direct product of $T(G)$ and $\langle b\rangle$, that is, $G = T(G) \rtimes \langle b\rangle$, where $\langle b\rangle$ is infinite cyclic. Hence every $g \in G$ can be written as $g = b^i h$, where $i \in \mathbb{Z}$ and $h \in T(G)$. We will show that there exists a positive integer n such that $b^n \in Z^*(G)$, or equivalently $b^n \otimes g \in \nabla(G)$ for all $g \in G$. In order to do this, we expand $b^n \otimes g$ by using Lemma 2.1 as follows:

$$b^n \otimes g = b^n \otimes b^i h = (b \otimes b)^{ni}(b \otimes h)^n (b \otimes [b, h])^{ni+\binom{n}{2}}. \qquad (6)$$

By (5) we have

$$b \otimes h^s = (b \otimes h)^s \left(h \otimes [b, h]^{\binom{s}{2}}\right) \text{ for } s \in \mathbb{Z}. \qquad (7)$$

Let $k = |h|$. If $\exp(T(G)) = m$, then $k|m$. Using (7) with $s = 2m$, we obtain

$$1_\otimes = (b \otimes h^{2m}) = (b \otimes h)^{2m} \left(h \otimes [b, h]^{\binom{2m}{2}}\right). \qquad (8)$$

Set $n = 2m$ in (6). Since m divides $2mi + \binom{2m}{2}$, it follows by (8) that $b^{2m} \otimes g = (b \otimes b)^{2mi}(b \otimes h)^{2m}(b \otimes [b, h]^{2mi+\binom{2m}{2}}) = (b \otimes b)^{2mi}$. Therefore, $b^n \otimes g \in \nabla(G)$ for all $g \in G$. Hence, $1 \neq b^n \in Z^*(G)$. Since $Z^*(G)$ is non-trivial, by Theorem 1.3 it follows that G is not capable. □

As a corollary, we obtain the following necessary condition that a finitely generated non-torsion group of nilpotency class 2 is capable.

THEOREM 3.2. *Let G be a finitely generated non-torsion group of nilpotency class 2. If G is capable, then the torsion-free rank of G/G' is greater than 1.*

Proof. Let G be a finitely generated non-torsion group of nilpotency class 2 and assume that G is capable. By our assumption, it follows that G' is a finitely generated abelian group. First assume that G' is finite. Then $G' \subseteq T(G)$ and hence $G/T(G)$, and consequently G/G' have torsion-free rank > 1 by Theorem 3.1, since G is capable. Now assume that G' is infinite. Then there exist two non-commuting elements $a, b \in G$ with a, b and $[a, b]$ of infinite order. It remains to be shown that $\langle \bar{a}, \bar{b}\rangle$ is of torsion-free rank 2, where $\bar{a} = aG'$ and $\bar{b} = bG'$. Assume to the contrary that there exist integers n, m, not both zero such that $a^n = b^m c$ with $c \in G'$. Without loss of generality, we can assume $n > 0$ and then $[a, b]^n = [a^n, b] = [b^m c, b] = 1$. It follows that $[a, b]$ has finite order, a contradiction. We conclude that the torsion-free rank of G/G' is greater than 1. □

That the above condition is only necessary follows from the next example.

EXAMPLE 3.3. There exists a finitely generated non-torsion group G of class 2, which is not capable but has torsion-free rank 2.

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Proof. Let p be an odd prime and $G = H \oplus P$, where $H = \langle c, d\rangle$ is a free abelian group of rank 2 and $P = \langle a, b \mid a^{p^2} = b^p = 1, [a, b] = a^p\rangle$. Since $G' = \langle a^p\rangle$, it follows that G/G' is of torsion-free rank 2. We claim now that $a^p \in Z^*(G)$, which will imply $Z^*(G) \neq 1$ and thus that G is not capable by Theorem 1.3. To verify $a^p \in Z^*(G)$, it suffices to show $a^p \otimes g \in \nabla(G)$ for $g \in \{a, b, c, d\}$, since $G = \langle a, b, c, d\rangle$. Obviously, $a^p \otimes a = (a \otimes a)^p \in \nabla(G)$. If $g = c$ or d, then $a^p \otimes g = [a, b] \otimes g = 1_\otimes$ by Proposition 2.2. Lastly, since p is odd, it follows by (5) that $1_\otimes = a \otimes b^p = (a \otimes b)^p(b \otimes [a, b]^{\binom{p}{2}}) = (a \otimes b)^p$. On the other hand, $a^p \otimes b = (a \otimes b)^p(a \otimes [a, b]^{\binom{p}{2}}) = (a \otimes b)^p$. Hence, $a^p \otimes b = 1_\otimes$ and we conclude $a^p \in Z^*(G)$. □

4. The two-generator case. In this section, we determine the capability of two-generator non-torsion groups of nilpotency class 2 using their classification as given in [**15**]. Contrary to the case of two-generator torsion groups of class 2, which are a direct product of their p-components, in the non-torsion case only the torsion subgroup is a direct product of the p-torsion subgroups, which are abelian of rank not exceeding 2. The following description of two-generator non-torsion groups of nilpotency class 2 is an easy consequence of results in [**15**].

THEOREM 4.1. *Let G be a two-generator non-torsion group of nilpotency class 2. Then G is isomorphic to exactly one group of the following types:*

$$G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle, \tag{9}$$

where $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = \infty$, $|b| = \infty$, $|c| = k$ *with* $k \leq \infty$*; or*

$$G \cong T \rtimes \langle b\rangle, \tag{10}$$

where T is a finite abelian group.

To determine the capability of the groups of Theorem 4.1, we will make use of the criteria developed in the previous section and the explicit knowledge of the non-abelian tensor square of a group of Type (9) as given in [**15**].

THEOREM 4.2. *Let G be a two-generator non-torsion group of nilpotency class 2. Then G is capable if G is of Type* (9) *and not capable if G is of Type* (10).

Proof. By Theorem 4.1, the non-torsion two-generator group of nilpotency class 2 split into two disjoint cases, namely of Type (9), where G/G' has torsion-free rank 2, and of Type (10), where G/G' has torsion-free rank 1. Let G be a group of Type (10). Then T is its torsion subgroup. Thus, G/T is infinite cyclic. Since T is finite, it has finite exponent. Thus, G satisfies the assumptions of Theorem 3.1 and it follows that a group of Type (10) is not capable.

Now let G be a group of Type (9). We observe that G/G' is free abelian of rank 2 and thus satisfies the necessary condition for capability as given in Theorem 3.2. By [**15**], we have $G \otimes G \cong \mathbb{Z}^4 \oplus \mathbb{Z}_k^2$ with $\mathbb{Z}^4 \cong \langle a \otimes a\rangle \oplus \langle b \otimes b\rangle \oplus \langle a \otimes b\rangle \oplus \langle b \otimes a\rangle$ and $\mathbb{Z}_k^2 \cong \langle a \otimes c\rangle \oplus \langle b \otimes c\rangle$. With the help of Proposition 2.3 and our knowledge of $G \otimes G$ as given above, we will show that G has a trivial epicentre. Hence, G is capable by Theorem 1.3. Let $G = \langle a, b\rangle$. We first determine $C^\wedge(a)$ and $C^\wedge(b)$. Let $g = a^m b^n c^l \in C^\wedge(a)$. By Lemma 2.1, we obtain

$$g \otimes a = (a \otimes a)^m (b \otimes a)^n (a \otimes c)^{-mn-l} (b \otimes c)^{-\binom{n}{2}}.$$

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Since $g \otimes a \in \nabla(G)$ and $\langle b \otimes a\rangle \cap \langle a \otimes a, a \otimes c, b \otimes c\rangle = \{1_{\otimes}\}$, it follows that $n = 0$. Similarly, $l = 0$ if $|c| = \infty$, and $l = k$ if $|c| = k < \infty$, respectively. Thus $g = a^m$, $m \in \mathbb{Z}$. This implies $C^{\wedge}(a) = \langle a\rangle$. Likewise, $g \otimes b \in \nabla(G)$ implies $C^{\wedge}(b) = \langle b\rangle$. Thus by Theorem 1.6 and Proposition 2.3, we have that $Z^*(G) = Z^{\wedge}(G) \subseteq C^{\wedge}(a) \cap C^{\wedge}(b) = \langle a\rangle \cap \langle b\rangle = \{1\}$. Therefore, it follows by Theorem 1.3 that a group of Type (9) is capable. □

As an immediate corollary to Theorem 4.2, we obtain that the necessary condition for capability of finitely generated non-torsion groups of class 2, as given in Theorem 3.2, is also sufficient in the two-generator case.

COROLLARY 4.3. *Let G be a two-generator non-torsion group of nilpotency class 2. Then the following conditions are equivalent:*
(i) *G is capable;*
(ii) *G/G' is free abelian of rank 2;*
(iii) *G/G' is capable.*

In conclusion, we want to remark that a similar characterisation holds for two-generator groups of odd order and nilpotency class 2 where (ii) in Corollary 4.3 is replaced by $G/G' \cong \mathbb{Z}_m \oplus \mathbb{Z}_m$. However, this characterisation does not extend to the even case as can be seen from the following examples. The group,

$$G = \langle a, b \mid a^{2^{k+1}} = b^{2^k} = [a, b]^{2^k} = [a, b, a] = [a, b, b] = 1\rangle$$

is capable, but $G/G' \cong \mathbb{Z}_{2^{k+1}} \oplus \mathbb{Z}_{2^k}$ is not. These results follow from [**3**, **12**] in the case of odd primes p and from [**13**] for $p = 2$. Likewise, the condition G/G' capable is not sufficient to imply G capable in the case $p = 2$. To see this, consider $G = Q_8$, the quaternion group, which is not capable. However, its factor commutator group is, since it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

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