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PROJECTIVE DISTRIBUTIVE p-ALGEBRAS

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A characterization of finite (weak) projectives in an equational class of distributive pseudocomplemented lattices is given. In the class of all such lattices, a finite lattice is projective if and only if its poset of join-irreducible elements forms a semilattice in which the minimal elements below the join of x and y are exactly the minimal elements below x or y. A similar condition works for any equational subclass.

Grätzer in his book on distributive lattices poses the problem of characterizing the projectives in the class of distributive p-algebras and their equational subclasses ([3], Problems 53, 60, restated in [4] as Problems II 25 and II 26). In this note a partial solution is provided by giving a necessary condition for projectivity which is also sufficient in the finite case.

1. Preliminaries

A distributive p-algebra is an algebra of the form $\langle L, \wedge, \vee, 0, 1, {}^*\rangle$ where $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and * the pseudocomplement operation on L. The non-trivial equational subclasses of distributive p-algebras form a chain $B_0 \subseteq B_1 \subseteq \ldots \subseteq B_n \subseteq \ldots \subseteq B_\omega$, where B_ω is the class of all distributive p-algebras, B_0 the class of Boolean algebras, B_1 the class of Stone algebras. The reader is referred to [3] for details.

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The necessary condition for projectivity is stated in terms of the topological duality theory of Priestley for bounded distributive lattices ([6], [7], [8]). We follow the terminology of Davey [1].

A partially ordered topological space X is said to be totally order-disconnected if for all $x, y \in X$ with $x \nleq y$ there exists a clopen order-ideal U of X with $x \in U$ and $y \notin U$. X is said to be a p-space if it is a compact totally order-disconnected space such that for every clopen order-ideal U, $[U)$ is open. Let $\mathrm{Min}(X)$ be the set of all minimal elements in X; for $z \in X$ let $\mathrm{Min}_X(z)$ be $\mathrm{Min}(X) \cap (z]$ – we shall usually omit the subscript X. A continuous order-preserving map between p-spaces X and Y is defined to be a p-space morphism if for every $x \in X$, $\phi\big(\mathrm{Min}(x)\big) = \mathrm{Min}\big(\phi(x)\big)$, that is, the map ϕ commutes with the Min operation. Let P_ω denote the category of all p-spaces with maps the p-space morphisms; for $1 \le n < \omega$ let P_n be the full subcategory of P_ω consisting of the p-spaces X satisfying $|\mathrm{Min}(x)| \le n$ for all $x \in X$. The basic duality theory states that for $1 \le k \le \omega$, B_k and P_k are dual categories.

An algebra A in B_k is said to be projective in B_k if for any B, C in B_k, $\gamma : B \to C$ and $\beta : A \to C$, if γ is onto then there is a homomorphism $\alpha : A \to B$ satisfying $\alpha\gamma = \beta$. (Note: since in the categories B_k, epimorphisms are not necessarily onto, the definition just given is not the definition of projective in a general category; for this reason, the term 'weak projective' is often employed for the concept defined above.) In the categories B_k (as in any equational class) the projectives are exactly the retracts of the free algebras in the category. By the duality theory, to characterize the projectives in B_k it is sufficient to describe the retracts of the p-spaces of free B_k algebras.

Denote the p-space of the free B_k algebra on α generators by $F_k(\alpha)$. A description of $F_k(\alpha)$ is given by Davey and Goldberg [2], generalizing the description for the finite case in [9]. Let D_α be the dual space of the free bounded distributive lattice on α generators (that

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is, D_α is the product of α copies of the ordered space on $\{0, 1\}$ with $0 < 1$). For X an ordered topological space denote by $\Gamma(X)$ the hyper-space topology defined on the family of all closed subsets in X (see [2], [5], [7]). Then $F_\omega(X)$ is defined as the ordered space $\{\langle x, A\rangle \in X \times \Gamma(X) \mid A \subseteq (x]\}$ with the subspace order and topology; $F_n(X)$, $1 \le n < \omega$, is the subspace of $F_\omega(X)$ defined on $\{\langle x, A\rangle \in F_\omega(X) \mid |A| \le n\}$. Then $F_\omega(D_\alpha)$ and $F_n(D_\alpha)$ are homeomorphic to the spaces $F_\omega(\alpha)$ and $F_n(\alpha)$.

2. A necessary condition for projectivity

If $\langle X, +\rangle$ is a join-semilattice, we say that it satisfies the condition (*) if for any $x, y \in X$, $\mathrm{Min}(x+y) = \mathrm{Min}(x) \cup \mathrm{Min}(y)$. (Note that if $\mathrm{Min}(X) = \emptyset$ or X has a least element, then X automatically satisfies (*).) If P is a poset, we say P satisfies the condition (*n) for $1 \le n < \omega$ if for any $x, y \in P$ such that $|\mathrm{Min}(x) \cup \mathrm{Min}(y)| \le n$ there is a least upper bound $x + y$ and $\mathrm{Min}(x+y) = \mathrm{Min}(x) \cup \mathrm{Min}(y)$.

LEMMA. (1) *The p-spaces of projective algebras in B_ω satisfy* (*).

(2) *The p-spaces of projective algebras in B_n $(1 \le n < \omega)$ satisfy* (*n) .

Proof. We prove (2) by showing that the p-spaces $F_n(\alpha)$ satisfy (*n) and that (*n) is preserved under retracts. The proof of (1) is a simplified version of the same argument.

We first note that for a Hausdorff space P , $1 \le k \le \omega$, the minimal elements in $F_k(P)$ can be characterized as the pairs $\langle x, \{x\}\rangle$ for $x \in P$. It follows that for any $\langle x, A\rangle$ in $F_k(P)$,

$$\mathrm{Min}(\langle x, A\rangle) = \{\langle z, \{z\}\rangle \mid z \in A\} .$$

Now let P be any p-space whose order-relation is a semilattice. We claim that $F_n(P)$ satisfies (*n). Thus let $\langle x, A\rangle, \langle y, B\rangle \in F_n(P)$ and

$$|\mathrm{Min}(\langle x, A\rangle) \cup \mathrm{Min}(\langle y, B\rangle)| \le n .$$

Now if $\langle z, \{z\}\rangle$ is a minimal element in $\mathrm{Min}(\langle x, A\rangle) \cup \mathrm{Min}(\langle y, B\rangle)$, then

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$z \in A \cup B$, whence $|A \cup B| \le n$. Thus if $x + y$ is the supremum of x and y in P then $\langle x+y, A \cup B\rangle \in F_n(P)$, and is the supremum of $\langle x, A\rangle$ and $\langle y, B\rangle$. Finally,

$$\mathrm{Min}(\langle x+y, A \cup B\rangle) = \{\langle z, \{z\}\rangle \mid z \in A \cup B\} = \mathrm{Min}(\langle x, A\rangle) \cup \mathrm{Min}(\langle y, B\rangle) .$$

Now, the ordering of D_α is isomorphic to the power set of α, so $F_n(\alpha)$ satisfies (*n).

Let P, Q be p-spaces in P_n where P satisfies (*n), and Q is a retract of R, so that there are p-space morphisms $f : P \to Q$ and $g : Q \to P$ where f is surjective, g is injective and fg is the identity map on Q. Now for any $x, y \in Q$ such that

$$|\mathrm{Min}(x) \cup \mathrm{Min}(y)| \le n ,$$

$$|\mathrm{Min}(gx) \cup \mathrm{Min}(gy)| = |g(\mathrm{Min}(x)) \cup g(\mathrm{Min}(y))| = |g(\mathrm{Min}(x) \cup \mathrm{Min}(y))| \le n .$$

Thus $gx + gy$, the supremum of gx, gy in P, exists and

$$\mathrm{Min}(gx+gy) = \mathrm{Min}(gx) \cup \mathrm{Min}(gy) .$$

We claim that $f(gx+gy)$ is the supremum of x and y in Q. For $x = fgx \le f(gx+gy)$, similarly $y \le f(gx+gy)$. If $z \in G$, $x, y \le z$ then $gx, gy \le gz$, $gx + gy \le gz$ so $f(gx+gy) \le fgz = z$. Lastly,

$$\mathrm{Min}(f(gx+gy)) = \mathrm{Min}(x) \cup \mathrm{Min}(y)$$

follows from the properties of p-maps.

3. Characterisation of finite projectives

In this section we show that the conditions (*) and (*n) are also sufficient for projectivity in the finite case.

THEOREM. (1) *A finite algebra in* B_ω *is projective if and only if its dual space satisfies* (*).

(2) *A finite algebra in* B_n $(1 \le n < \omega)$ *is projective if and only if its dual space satisfies* (*n).

Proof. As above, we prove (2) only, as (1) is obtained by a simplification. Let Q be any finite p-space in P_n satisfying (*n). Since finitely generated algebras in B_n are finite, there is a p-space

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map embedding Q in a finite p-space $F_n(D_m)$. Thus Q is a subspace of a finite p-space R satisfying (*n), where $\mathrm{Min}_Q(x) = \mathrm{Min}_R(x)$ for $x \in Q$. For $Z \subseteq Q$, denote the supremum of Z in Q (where it exists) by $\sum Z$. For $x \in R$, let g be defined as follows:

$$g(x) = \begin{cases} \sum \{z \in Q \mid z \leq x\} & \text{if } \mathrm{Min}_R(x) \subseteq Q , \\ \sum \{z \in Q \mid z \leq x\} \cup \{w\} & \text{otherwise,} \end{cases}$$

where w is a fixed element of $\mathrm{Min}(Q)$. We note first that g is well-defined. If $\mathrm{Min}_R(x) \nsubseteq Q$ then $|\mathrm{Min}_R(x)| < n$. Thus

$$|\mathrm{Min}(\{z \in Q \mid z \leq x\} \cup \{w\})| \leq n ,$$

so the supremum of $\{z \in G \mid z \leq x\} \cup \{w\}$ exists by (*n). In the other case, $g(x)$ is well-defined by (*n). We must show that g is a p-map, that is, that $\mathrm{Min}_Q(gx) = g(\mathrm{Min}_R(x))$. Assuming $\mathrm{Min}_R(x) \subseteq Q$, we have

$$\begin{aligned} \mathrm{Min}_Q(gx) &= \mathrm{Min}_Q\left(\sum \{z \in Q \mid z \leq x\}\right) \\ &= \bigcup\{\mathrm{Min}_Q(z) \mid z \in Q,\ z \leq x\} \\ &= \bigcup\{\mathrm{Min}_R(z) \mid z \in Q,\ z \leq x\} \\ &= \mathrm{Min}_R(x) \\ &= g(\mathrm{Min}_R(x)) \quad \text{(since } g \text{ is the identity on } Q \text{).} \end{aligned}$$

Assuming $\mathrm{Min}_R(x) \nsubseteq Q$, we have

$$\begin{aligned} \mathrm{Min}_Q(gx) &= \mathrm{Min}_Q\left(\sum \{z \in Q \mid z \leq x\} \cup \{w\}\right) \\ &= \bigcup\{\mathrm{Min}_Q(z) \mid z \in Q,\ z \leq x\} \cup \{w\} \\ &= \bigcup\{\mathrm{Min}_R(z) \mid z \in Q,\ z \leq x\} \cup \{w\} \\ &= \left(\mathrm{Min}_R(x) \cap Q\right) \cup \{w\} \\ &= g\left(\mathrm{Min}_R(x) \cap Q\right) \cup g\left(\mathrm{Min}_R(x) - Q\right) \\ &= g(\mathrm{Min}_R(x)) . \end{aligned}$$

Thus g is a p-map from R onto Q. Since g is the identity on Q, this shows that Q is a retract of R.

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Finite projective Stone algebras have already been characterized by Balbes and Grätzer. The above theorem characterizes the spaces of these algebras as posets which are disjoint unions of a family of finite lattices. Dualising, this shows that the corresponding algebra is a product of distributive lattices in which 1 is join-, 0 meet-irreducible and the meet of join-irreducible elements is join-irreducible. This is the result of Balbes and Grätzer.

In the infinite case, the p-spaces of projectives can perhaps be characterized by conditions relating the (partial) semilattice structure to the topology. As an example of such a condition, it can be shown that in the p-space of a projective in B_ω, the join operation on the semi-lattice of prime ideals is continuous. Necessary and sufficient conditions, though, do not appear easy to find.

References

[1] Brian A. Davey, "Subdirectly irreducible distributive double p-algebras", *Algebra Universalis* 8 (1978), 73-88.

[2] Brian A. Davey and Moshe S. Goldberg, "The free p-algebra generated by a distributive lattice", *Algebra Universalis* (to appear).

[3] George Grätzer, *Lattice theory. First concepts and distributive lattices* (Freeman, San Francisco, 1971).

[4] George Grätzer, *General lattice theory* (Mathematische Reihe, 52. Birkhauser Verlag, Basel, Stuttgard, 1978).

[5] Kazimierz Kuratowski, *Topology*, Volume I, new edition (Academic Press, New York, London; Panstwowe Wydawnictwo Naukowe, Warsaw; 1966).

[6] H.A. Priestley, "Representation of distributive lattices by means of ordered Stone spaces", *Bull. London Math. Soc.* 2 (1970), 186-190.

[7] H.A. Priestley, "Stone lattices: a topological approach", *Fund. Math.* 84 (1974), 127-143.

[8] H.A. Priestley, "The construction of spaces dual to pseudocomplemented distributive lattices", *Quart. J. Math. Oxford Ser.* (2) 26 (1975), 215-228.

https://doi.org/10.1017/S0004972700007632 Published online by Cambridge University Press

[9] A. Urquhart, "Free distributive pseudo-complemented lattices", *Algebra Universalis* 3 (1973), 13-15.

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