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THE DIRICHLET PROBLEM ON THE HEISENBERG GROUP III: HARMONIC MEASURE OF A CERTAIN HALF-SPACE

BERNARD GAVEAU AND JACQUES VAUTHIER

0. Introduction. In this short note we give an explicit computation of the harmonic measure of a half space $x > 0$ in the 3-dimensional Heisenberg group in terms of a degenerate hypergeometric function. A probabilistic argument reduces the whole problem to a Hermite-type equation on a half line, that we can solve in terms of the function $G(1/4, 1/2; x^2)$.

A preliminary attempt to compute this kernel was done in [**1**] p. 107 and, cited by Huber [**4**]. Unfortunately a small mistake was made in [**1**] and the problem was still open until now. The first author is very grateful to Prof. Huber for having pointed out the weak argument of [**1**]. Since that time, other harmonic measures and even Green functions have been explicitly computed (see [**2**]).

1. Notations. a) As usual, H_3 is the Heisenberg group of dimension 3 with the coordinates $g = (z, t) \in \mathbf{C} \times \mathbf{R}$, $z = x + iy$ and the product law

(1) $\quad (z, t) \cdot (z', t') = (z + z', t + t' + 2 \operatorname{Im} z\bar{z}').$

The left invariant vector fields are

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}$$

and the subelliptic laplacian is

$$(2) \quad \Delta = X^2 + Y^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4y\frac{\partial^2}{\partial x \partial t} - 4x\frac{\partial^2}{\partial y \partial t} + 4|z|^2\frac{\partial^2}{\partial t^2}$$

(see [**1**])

b) the diffusion process starting time $s = 0$ from $g = 0$ with generator $\frac{1}{2}\Delta$ is

$$(3) \quad g_\omega(s) = \left(X_\omega(s) + iY_\omega(s), \quad 2\int_0^s (YdX - XdY)\right)$$

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where X, Y are independent brownian notions and the last term is a stochastic integral.

The diffusion process starting from $g^0 = (z^0, t^0)$ at time $s = 0$ is just

$$(4) \quad g^0 \cdot g_\omega(s) = (x^0 + X_\omega(s) + i(y^0 + Y_\omega(s)),$$
$$t^0 + 2\int_0^s (YdX - XdY) + 2y^0X(s) - 2x^0Y(s))$$

(see also [**1**] for details).

2. The harmonic measure of $x > 0$. Let D be the half space $x > 0$. We want to solve the Dirichlet problem

$$(5) \quad \begin{cases} \Delta f = 0 & \text{on } D \\ f = \varphi & \text{on } \partial D = \{x = 0\}. \end{cases}$$

Now, it is obvious that no point on ∂D is characteristic for the operator Δ, so that each point on ∂D is very regular in the senseof Bony ([**3**]) and also regular in the usual sense of stochastic processes ([**1**], [**3**]). If g^0 is a point in D, the solution of (5) is given by

$$(6) \quad f(g^0) = E(\varphi(g^0 \cdot g_\omega(S))$$

where the expectation is the expectation in the probability space of (3) and S is the first exit time from D of $g^0 \cdot g_\omega(s)$.

We shall prove that (6) has an explicit expression

$$(7) \quad f(g^0) = \int_{\partial D} k(g^0, g)\varphi(g)dydt$$

where $k(g^0, g)$ is a certain kernel defined on $D \times \partial D$ which is called the harmonic measure of g^0 in D.

3. The Fourier transform of the kernel k. Because g^0 is in D, x^0 is positive. Now S is exactly the first time such that

$$(8) \quad x^0 + X(S) = 0$$

and we have to compute the law of the process at the boundary, which is, using (4)

$$(9) \quad \left(y^0 + Y(S), t^0 + 2\int_0^S (YdX - XdY) + 2y^0X(S) - 2x^0Y(S)\right).$$

But $X(S) = -x^0$ and

$$\int_0^S YdX = -\int_0^S XdY + X(S)Y(S)$$
$$= -\int_0^S XdY - x^0Y(S)$$

so that (9) is

$$(10)\quad \left(y^0 + Y(S), t^0 - 2y^0x^0 - 4\int_0^S XdY - 4x^0Y(S)\right).$$

We compute the characteristic function of (10), namely

$$(11)\quad \Phi(g^0; \eta, \tau) = E\Big[\exp i\Big\{\eta(y^0 + Y(S)) + \tau\Big(t^0 - 2y^0x^0 - 4\int_0^S XdY - 4x^0Y(S)\Big)\Big\}\Big]$$
$$= \exp i(\eta y^0 + \tau(t^0 - 2y^0x^0)) \times E\Big[\exp i\Big\{(\eta - 4x^0\tau)Y(S) - 4\tau\int_0^S XdY\Big\}\Big].$$

It is clear that on the other hand

$$(12)\quad \Phi(g^0; \eta, \tau) = \int_{\partial D} k(g^0; y, t)e^{i(\eta y + \tau t)}dydt.$$

Now, to compute (11), we remark that S is a random variable independant of Y, and in the expectation in (11), we can integrate out Y: in fact the expectation in (11) is given by

$$E\Big[\exp i\int_0^S (-4\tau X(s) + \eta - 4x^0\tau)dY(s)\Big]$$

and because of the exponential martingal of Mc Kean [**6**] and because S and $X(S)$ are certain variables with respect to Y, this is exactly:

$$\psi(x^0, \tau, \eta) \equiv E\Big[\exp\Big(-\frac{1}{2}\int_0^S (-4\tau X(s) + \eta - 4x^0\tau)^2 ds\Big)\Big]$$

so that

$$(13)\quad \Phi(g^0; \eta, \tau) = \exp i(\eta y^0 + \tau(t^0 - 2y^0x^0)) \times E\Big[\exp\Big(-\frac{1}{2}\int_0^S (4\tau(X(s) + x^0) - \eta)^2 ds\Big)\Big].$$

Now the expectation on the right hand side of (13) is an expectation on the brownian motion $X(s) + x^0$ starting at $s = 0$ from x^0, until its exit time of the right half line $\mathbf{R}^+$ at time S. Call now u the solution of the Dirichlet problem

$$(14)\quad \begin{cases} \dfrac{1}{2}\dfrac{d^2u}{dx^2} - 2\beta^2(x + \alpha)^2u = 0 \text{ on } [-x_0, +\infty[\\ u(+\infty) = 0 \\ u(-x_0) = 1 \end{cases}$$

where

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(15) $2\beta^2 = 8\tau^2, \quad \alpha = -\frac{\eta}{4\tau}.$

We have then the well known lemma:

LEMMA. *We have*

(16) $\psi(x^0, \tau, \eta) = u(0)$

(*which comes from* [**5**]).

The only point is to solve (14). Call $x' = x + \alpha$, $u(x) = v(x')$; then we have

$$(17) \quad \begin{cases} \frac{1}{2}\frac{d^2v}{dx'^2} - 2\beta^2 x'^2 v = 0 \text{ on } [-x_0 + \alpha, +\infty[\\ v(-x_0 + \alpha) = 1 \\ v(+\alpha) = 0. \end{cases}$$

Now we compare the differential equation with the Hermite equation

$$(18) \quad \frac{d^2y}{dx_1^2} - 2x_1\frac{dy}{dx_1} + 2\nu y = 0.$$

If we write

$$z = y \exp\left(-\frac{x_1^2}{2}\right)$$

we obtain

$$\frac{d^2z}{dx_1^2} + z(2\nu + 1 - x_1^2) = 0.$$

Writing $x_1 = \theta x'$ this is transformed in

$$\frac{d^2z}{dx'^2} + \theta^2 z(2\nu + 1 - \theta^2 x'^2) = 0$$

which can be compared to (17) if we write

$$\nu = -\frac{1}{2}, \quad \theta = +\sqrt{2|\beta|}.$$

But (18) has two solutions given in terms of the degenerate hypergeometric G

$$G\left(\frac{1}{4}, \frac{1}{2}, x_1^2\right) \sim x_1^{1/2}$$

if $x_1 \to \infty$ and

$$e^{x_1^2}G\left(\frac{1}{4}, \frac{1}{2}, -x_1^2\right) \sim e^{x_1^2}x_1^{-1/2}$$

(see [7]).

So the corresponding z are

$$\exp\left(-\frac{x_1^2}{2}\right)G\left(\frac{1}{4}, \frac{1}{2}, +x_1^2\right)$$

$$\exp\left(+\frac{x_1^2}{2}\right)G\left(\frac{1}{4}, \frac{1}{2}, -x_1^2\right)$$

and we can only retain the first one because we look for a solution vanishing at infinity.

We then obtain

$$u(x) = v(x') = z(\sqrt{2|\beta|}x')$$
$$= \exp(-|\beta|x'^2)G\left(\frac{1}{4}, \frac{1}{2}; 2|\beta|x'^2\right)$$

and because we want that $u(-x_0) = 1$, we obtain

$$u(x) = \frac{\exp(-|\beta|\,(x + \alpha)^2)G\left(\frac{1}{4}, \frac{1}{2}; 2|\beta|\,(x + \alpha)^2\right)}{\exp(-|\beta|\,(x_0 - \alpha)^2)G\left(\frac{1}{4}, \frac{1}{2}; 2|\beta|\,(x_0 - \alpha)^2\right)}.$$

Because of (16) and (15) we then get

$$(19)\quad \psi(x^0, \tau, \eta) = \frac{\exp\left(-\frac{\eta^2}{8|\tau|}\right)G\left(\frac{1}{4}, \frac{1}{2}; \frac{\eta^2}{4|\tau|}\right)}{\exp\left(-2|\tau|\left(x_0 + \frac{\eta}{4\tau}\right)^2\right)G\left(\frac{1}{4}, \frac{1}{2}; 2|\tau|\left(x_0 + \frac{\eta}{4\tau}\right)^2\right)}.$$

Now because of (12), (13) and (19) we obtain by taking the inverse Fourier transform

$$(20)\quad k(y^0; y, t) = \frac{1}{(2\pi)^2}\int \exp[-i(\eta y + \tau t - \eta y^0 - \tau(t^0 - 2y^0x^0))]$$
$$\times\ \psi(x^0, \tau, \eta)d\tau d\eta.$$

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Université Paris VI,
Paris, France