J. Aust. Math. Soc. **83** (2007), 385–416

CHARACTERIZATION OF LEFT ARTINIAN ALGEBRAS THROUGH PSEUDO PATH ALGEBRAS

FANG LI

(Received 20 March 2004; revised 27 May 2006)

Communicated by J. Du

Abstract

In this paper, using pseudo path algebras, we generalize Gabriel's Theorem on elementary algebras to left Artinian algebras over a field k when the quotient algebra can be lifted by a radical. Our particular interest is when the dimension of the quotient algebra determined by the nth Hochschild cohomology is less than 2 (for example, when k is finite or char $k = 0$). Using generalized path algebras, a generalization of Gabriel's Theorem is given for finite dimensional algebras with 2-nilpotent radicals which is splitting over its radical. As a tool, the so-called pseudo path algebra is introduced as a new generalization of path algebras, whose quotient by $\ker \iota$ is a generalized path algebra (see Fact 2.6).

The main result is that

(i) for a left Artinian k-algebra A and $r = r(A)$ the radical of A, if the quotient algebra A/r can be lifted then $A \cong PSE_k(\Delta, \mathcal{A}, \rho)$ with $J^s \subset \langle\rho\rangle \subset J$ for some s (Theorem 3.2);

(ii) If A is a finite dimensional k-algebra with 2-nilpotent radical and the quotient by radical can be lifted, then $A \cong k(\Delta, \mathcal{A}, \rho)$ with $\widetilde{J}^2 \subset \langle\rho\rangle \subset \widetilde{J}^2 + \widetilde{J} \cap \ker\widetilde{\varphi}$ (Theorem 4.2),

where Δ is the quiver of A and ρ is a set of relations.

For all the cases we discuss in this paper, we prove the uniqueness of such quivers Δ and the generalized path algebras/pseudo path algebras satisfying the isomorphisms when the ideals generated by the relations are admissible (see Theorem 3.5 and 4.4).

2000 *Mathematics subject classification*: primary 16G10.

1. Introduction

In this paper, k will always denote a field and all modules will be unital. An algebra is said to be *left Artinian* if it satisfies the descending chain condition on left ideals.

It is well-known that for a finite dimensional algebra A over an algebraically closed field k and the nilpotent radical $N = J(A)$, the quotient algebra A/N is semisimple, that is, there are uniquely determined positive integers $n_1 \leq n_2 \leq \cdots \leq n_r$ such that $A/N \cong M_{n_1}(k) \oplus \cdots \oplus M_{n_r}(k)$, where $M_{n_i}(k)$ denotes the algebra of $n_i \times n_i$ matrices with entries in k, which is trivially a k-simple algebra. In the special case that A is an *elementary* algebra [1], every $n_i = 1$, that is $M_{n_i} \cong k$, so that A/N, as a k algebra, is a direct sum of some copies of k and we can write $A/N = \coprod_r(k)$.

Obviously, every finite dimensional path algebra is elementary. Conversely, by Gabriel's famous theorem [1], for each elementary algebra Λ one can construct the corresponding quiver $\Gamma(\Lambda)$ of Λ such that Λ is isomorphic to a quotient algebra of the path algebra $k\Gamma(\Lambda)$. On the other hand, the module category of any algebra A is always Morita-equivalent to that of some elementary algebra [3]. Therefore, from the point of view of representation theory, it should be enough to consider representations of elementary algebras, or equivalently, quotient algebras of path algebras. In particular, this approach has provided the description of finitely generated modules over some given algebras (see for instance [1, 5]).

However, from the point of view of the structure of algebras, finite dimensional algebras cannot be replaced by elementary algebras. This applies, for example, if one wishes to make a classification of finite dimensional algebras.

For this reason, Shao-xue Liu, one of the authors of [2], raised an interesting problem, that is, how to find a generalization of path algebras so as to obtain a generalization of Gabriel's Theorem to arbitrary finite dimensional algebras which would allow these algebras to be represented as quotient algebras of generalized path algebras. The first step in this direction was taken in [2], where an appropriate concept of generalized path algebra was introduced (see Section 2), but results of the desired type could not be found.

In this paper, we hope to solve the Liu's problem by using pseudo path algebras and generalized path algebras in the sense of [2].

Some preparation is given in Section 2. In fact, we find that generalized path algebras are not sufficient to characterize finite-dimensional algebras other than those with 2-nilpotent radicals. For this reason, so-called pseudo path algebras are introduced as a new generalization of path algebras, which can cover generalized path algebras (see Fact 2.6). In Section 3, using pseudo path algebras, we generalize Gabriel's Theorem on elementary algebras to cover left Artinian algebras over a field k in the case that the quotient algebra is lifted by a radical, in particular, when the dimension of the quotient algebra determined by the nth Hochschild cohomology is less than 2 (for example, when k is finite or char $k = 0$). On the other hand, in Section 4, relying on generalized path algebras, a Generalized Gabriel's Theorem is given for finite dimensional algebras with 2-nilpotent radicals in the case where the quotient algebra is lifted. In all the cases we discuss, we prove the uniqueness of the relevant quivers Δ and generalized path algebras/pseudo path algebras if the ideals generated by the relations are admissible (see Theorems 3.5 and 4.4).

Under some conditions, the generalized forms of Gabriel's Theorems are not dependent on the ground field and this offers the possibility of an approach to modular representations of algebras and groups.

Note that when $A \cong k(\Delta, \mathscr{A})/\langle \rho \rangle$ or $A \cong PSE_k(\Delta, \mathscr{A})/\langle \rho \rangle$, the structure of A is determined by the ideal $\langle \rho \rangle$ generated by a set of relations ρ. From this, one can try to classify those associative algebras satisfying the theorems, including many important kinds of algebras. We intend to address these questions in future papers which will shed further light on the significance of the present work.

2. On generalized path algebras and pseudo path algebras

In this section, we first introduce the definitions of generalized path algebra [2] and pseudo path algebra and then discuss their properties and relationship.

A *quiver* Δ is given by two sets Δ_0 and Δ_1 together with two maps $s, e : \Delta_1 \to \Delta_0$. The elements of Δ_0 are called *vertices*, while the elements of Δ_1 are called *arrows*. For an arrow $\alpha \in \Delta_1$, the vertex $s(\alpha)$ is the *start vertex* of α and the vertex $e(\alpha)$ is the *end vertex* of α, and we write $s(\alpha) \xrightarrow{\alpha} e(\alpha)$. A *path* p in Δ is $(a|\alpha_1 \cdots \alpha_n|b)$, where $\alpha_i \in \Delta_1$, for $i = 1, \dots, n$, and $s(\alpha_1) = a$, $e(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \dots, n+1$, and $e(\alpha_n) = b$. $s(\alpha_1)$ and $e(\alpha_n)$ are also called respectively the start vertex and the end vertex of p. Write $s(p) = s(\alpha_1)$ and $e(p) = e(\alpha_n)$. The *length* of a path is the number of arrows in it. To each arrow α, one can assign an edge $\overline{\alpha}$ where the orientation is forgotten. A *walk* between two vertices a and b is given by $(a|\overline{\alpha_1} \cdots \overline{\alpha_n}|b)$, where $a \in \{s(\alpha_1), e(\alpha_1)\}$, $b \in \{s(\alpha_n), e(\alpha_n)\}$, and for each $i = 1, \dots, n-1$, $\{s(\alpha_i), e(\alpha_i)\} \cap \{s(\alpha_{i+1}), e(\alpha_{i+1})\} \neq \emptyset$. A quiver is said to be *connected* if there exists a walk between any two vertices a and b.

In this paper, we will always assume the quiver Δ is finite, that is, the number $|\Delta_0|$ of vertices and the number $|\Delta_1|$ of arrows are both finite.

DEFINITION 2.1. For two algebras A and B, the *rank* of a finitely generated A-B-bimodule M is defined as the least cardinal number of a set of generators. In particular, if $M = 0$, it is said to have rank 0 as a finitely generated A-B-bimodule.

Clearly, every finitely generated A-B-bimodule has a uniquely determined rank.

2.1. Generalized path algebra and tensor algebra Let $\Delta = (\Delta_0, \Delta_1)$ be a quiver and $\mathscr{A} = \{A_i : i \in \Delta_0\}$ be a family of k-algebras A_i with identity e_i, indexed by the vertices of Δ. The elements a_i of $\bigcup_{i \in \Delta_0} A_i$ are called the *$\mathscr{A}$-paths of length zero*, whose start vertex $s(a_i)$ and end vertex $e(a_i)$ are both i. For each $n \geq 1$, an *$\mathscr{A}$-path P of length n* is given by $a_1\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1}$, where $(s(\beta_1)|\beta_1 \cdots \beta_n|e(\beta_n))$ is a path in Δ of length n and $a_i \in A_{s(\beta_i)}$ for $i = 1, \dots, n$ and $a_{n+1} \in A_{e(\beta_n)}$. The terms $s(\beta_1)$

and $e(\beta_n)$ are also called respectively the start vertex and the end vertex of P. Write $s(P) = s(\alpha_1)$ and $e(P) = e(\alpha_n)$. Now consider the quotient R of the k-linear space with basis the set of all $\mathscr{A}$-paths of Δ by the subspace generated by all the elements of the form

$$a_1\beta_1 \cdots \beta_{j-1}(a_j^1 + \cdots + a_j^m)\beta_j a_{j+1} \cdots \beta_n a_{n+1} - \sum_{l=1}^{m} a_1\beta_1 \cdots \beta_{j-1} a_j^l \beta_j a_{j+1} \cdots \beta_n a_{n+1}$$

where $(s(\beta_1)|\beta_1 \cdots \beta_n|e(\beta_n))$ is a path in Δ of length n, $a_i \in A_{s(\beta_i)}$ for each $i = 1, \ldots, n$ and $a_{n+1} \in A_{e(\beta_n)}$ and $a_j^l \in A_{s(\beta_j)}$ for $l = 1, \ldots, m$.

Given two elements $[a_1\beta_1a_2\beta_2 \cdots a_n\beta_n a_{n+1}]$ and $[b_1\gamma_1 b_2\gamma_2 \cdots b_n\gamma_n b_{n+1}]$ in R, define the product $[a_1\beta_1a_2\beta_2 \cdots a_n\beta_n a_{n+1}] \cdot [b_1\gamma_1 b_2\gamma_2 \cdots b_n\gamma_n b_{n+1}]$ to be equal to $[a_1\beta_1a_2\beta_2 \cdots a_n\beta_n(a_{n+1}b_1)\gamma_1 b_2\gamma_2 \cdots b_n\gamma_n b_{n+1}]$ if a_{n+1} and b_1 are in the same A_i, and 0 otherwise.

It is easy to check that the above multiplication is well-defined and makes R into a k-algebra, called the *$\mathscr{A}$-path algebra* of Δ. Denote it by $R = k(\Delta, \mathscr{A})$. Clearly, R is an A-bimodule, where $A = \oplus_{i\in\Delta_0} A_{i_0}$. All such algebras are said to be *generalized path algebras*.

We note the following facts.

(i) $R = k(\Delta, \mathscr{A})$ has an identity if and only if Δ_0 is finite.

(ii) Any path $(s(\beta_1)|\beta_1 \cdots \beta_n|e(\beta_n))$ in Δ can be considered as an $\mathscr{A}$-path with $a_i = e_i$. Hence the usual path algebra $k\Delta$ can be embedded into the $\mathscr{A}$-path algebra $k(\Delta, \mathscr{A})$. If $A_i = k$ for all $i \in \Delta_0$ then $k(\Delta, \mathscr{A}) = k\Delta$.

(iii) For $R = k(\Delta, \mathscr{A})$, $\dim_k R < \infty$ if and only if $\dim_k A_i < \infty$ for each $i \in \Delta_0$ and Δ is a finite quiver without oriented cycles.

Associated with the pair $(A, {}_A M_A)$ for a k-algebra A and an A-bimodule M, we write the n-fold A-tensor product $M \otimes_A M \otimes \cdots \otimes_A M$ as M^n. Then

$$T(A, M) = A \oplus M \oplus M^2 \oplus \cdots \oplus M^n \oplus \cdots$$

is an abelian group. Writing $M^0 = A$, $T(A, M)$ becomes a k-algebra with multiplication induced by the natural A-bilinear maps $M^i \times M^j \to M^{i+j}$ for $i \geq 0$ and $j \geq 0$. $T(A, M)$ is called the *tensor algebra* of M over A.

We now define a special class of tensor algebras so as to characterize generalized path algebras. An *$\mathscr{A}$-path-type tensor algebra* is defined to be a tensor algebra $T(A, M)$ satisfying

(i) $A = \bigoplus_{i\in\Delta_0} A_i$ for a family of k-algebras $\mathscr{A} = \{A_i : i \in \Delta_0\}$,

(ii) $M = \bigoplus_{i,j\in I} {}_iM_j$ for finitely generated A_i-A_j-bimodules ${}_iM_j$ for all i and j in I and $A_k \cdot {}_iM_j = 0$ if $k \neq i$ and ${}_iM_j \cdot A_k = 0$ if $k \neq j$.

A *free $\mathscr{A}$-path-type tensor algebra* is an $\mathscr{A}$-path-type tensor algebra $T(A, M)$ in which each finitely generated A_i-A_j-bimodule ${}_iM_j$ for i and j in I is a free bimodule with a basis and the rank of this basis is equal to the rank of ${}_iM_j$ as a finitely generated A_i-A_j-bimodule.

$\mathscr{A}$-path-type tensor algebras and generalized path algebras can be constructed from each other as follows.

For an $\mathscr{A}$-path algebra $k(\Delta, \mathscr{A})$, let $A = \bigoplus_{i \in \Delta_0} A_i$. For any i and j, let ${}_iM_j^F$ be the free A_i-A_j-bimodule with basis given by the arrows from i to j. It is easy to see that the number of free generators in the basis is the rank of ${}_iM_j^F$ as a finitely generated bimodule. Define $A_k \cdot {}_iM_j^F = 0$ if $k \neq i$ and ${}_iM_j^F \cdot A_k = 0$ if $k \neq j$. Let $M^F = \bigoplus_{i \to j} {}_iM_j^F$, which is clearly an A-bimodule. Then we get uniquely the free $\mathscr{A}$-path-type tensor algebras $T(A, M^F)$.

Conversely, assume that $T(A, M)$ is an $\mathscr{A}$-path-type tensor algebra with a family of k-algebras $\mathscr{A} = \{A_i : i \in I\}$ and finitely generated A_i-A_j-bimodules ${}_iM_j$ for $i, j \in I$ such that $A = \bigoplus_{i \in I} A_i$ and $M = \bigoplus_{i,j \in I} {}_iM_j$ and $A_k \cdot {}_iM_j = 0$ if $k \neq i$ and ${}_iM_j \cdot A_k = 0$ if $k \neq j$. Trivially, ${}_iM_j = A_i M A_j$. Let the rank of ${}_iM_j$ be r_{ij}. Now we can associate with $T(A, M)$ a quiver $\Delta = (\Delta_0, \Delta_1)$ and its generalized path algebra $R = k(\Delta, \mathscr{A})$ in the following way. Let $\Delta_0 = I$ as the set of vertices. For $i, j \in I$, let the number of arrows from i to j in Δ be the rank r_{ij} of the finitely generated A_i-A_j-bimodules ${}_iM_j$. Obviously, if ${}_iM_j = 0$ then there are no arrows from i to j. Thus we get a quiver $\Delta = (\Delta_0, \Delta_1)$ which is called *the quiver of* $T(A, M)$, and its $\mathscr{A}$-path algebra $R = k(\Delta, \mathscr{A})$ which is called *the corresponding $\mathscr{A}$-path algebra of* $T(A, M)$.

One can find two nonisomorphic finitely generated bimodules which possess the same rank, therefore there exist two $\mathscr{A}$-path-type tensor algebras $T(A, M_1)$ and $T(A, M_2)$, with nonisomorphic bimodules M_1 and M_2, such that their induced quivers and $\mathscr{A}$-path algebras are the same in the above way.

From the above discussion, every $\mathscr{A}$-path-type tensor algebra $T(A, M)$ can be used to construct its corresponding $\mathscr{A}$-path algebra $k(\Delta, \mathscr{A})$; but, from this $\mathscr{A}$-path algebra $k(\Delta, \mathscr{A})$, we can get uniquely the free $\mathscr{A}$-path-type tensor algebra $T(A, M^F)$. Thus, we have the following lemma.

LEMMA 2.2. *Every $\mathscr{A}$-path-type tensor algebra $T(A, M)$ can be used to construct uniquely the free $\mathscr{A}$-path-type tensor algebra $T(A, M^F)$. There is a surjective k-algebra morphism $\pi: T(A, M^F) \to T(A, M)$ such that $\pi({}_iM_j^F) = {}_iM_j$ for any $i, j \in I$.*

PROOF. We need only prove the second conclusion. For $T(A, M)$, let the rank of ${}_iM_j$ be r_{ij}. Thus, for the corresponding $\mathscr{A}$-path algebra $k(\Delta, \mathscr{A})$, the number of arrows from i to j is r_{ij}, and then, in $T(A, M^F)$, the rank of the free generators of ${}_iM_j^F$ given by the arrows is also r_{ij}. Define $\pi: T(A, M^F) \to T(A, M)$ by giving a bijection between the set of the free generators of ${}_iM_j^F$ and the set of the chosen generators of ${}_iM_j$ with cardinal number equal to the rank. Then π can be expanded to become a surjective k-algebra morphism with $\pi({}_iM_j^F) = {}_iM_j$ for any $i, j \in I$. $\square$

Next, we will show in the following Proposition 2.10 that every $\mathscr{A}$-path-type tensor algebra is a homomorphic image of its corresponding $\mathscr{A}$-path algebra.

The following criterion (see [1, Lemma III.1.2]) is useful for constructing algebra morphisms from tensor algebras to other algebras.

LEMMA 2.3. *Let A be a k-algebra and M an A-bimodule. Let Λ be a k-algebra and $f : A \oplus M \to \Lambda$ a map such that the following two conditions are satisfied:*

(i) *$f|_A : A \to \Lambda$ is an algebra morphism;*

(ii) *viewing $f(M)$ as an A-bimodule via $f|_A : A \to \Lambda$, $f|_M : M \to f(M) \subset \Lambda$ is an A-bimodule map.*

Then there is a unique algebra morphism $\widetilde{f} : T(A, M) \to \Lambda$ such that $\widetilde{f}|_{A\oplus M} = f$ and generally, $\widetilde{f}\left(\sum_{n=0}^{\infty} m_1^n \otimes \cdots \otimes m_n^n\right) = \sum_{n=0}^{\infty} f(m_1^n) \cdots f(m_n^n)$ for $m_1^n \otimes \cdots \otimes m_n^n \in M^n$.

Note that the condition that $f(M)$ is an A-bimodule via $f|_A : A \to \Lambda$ is sufficient for the proof of (ii) in [1].

Clearly, all $\mathscr{A}$-paths of length zero, that is, the elements of $\bigcup_{i\in\Delta_0} A_i$, can generate a subalgebra of $k(\Delta, \mathscr{A})$, which is denoted by $k(\Delta_0, \mathscr{A})$. Also, denote by $k(\Delta_1, \mathscr{A})$ the k-linear space consisting of all $\mathscr{A}$-paths of length 1 and by J the ideal in an $\mathscr{A}$-path algebra $k(\Delta, \mathscr{A})$ generated by all elements in $k(\Delta_1, \mathscr{A})$. It is easy to see that $k(\Delta_1, \mathscr{A})$ is an A-subbimodule of $k(\Delta, \mathscr{A})$.

2.2. Pseudo path algebra and pseudo tensor algebra Let $\Delta = (\Delta_0, \Delta_1)$ be a quiver and $\mathscr{A} = \{A_i : i \in \Delta_0\}$ be a family of k-algebras A_i with identity e_i, indexed by the vertices of Δ. The elements a_i of $\bigcup_{i\in\Delta_0} A_i$ are called the *$\mathscr{A}$-pseudo-paths of length zero*, whose start vertex $s(a_i)$ and the end vertex $e(a_i)$ both are i. For each $n \geq 1$, a *pure $\mathscr{A}$-pseudo-path P of length n* is given by $a_1\beta_1 b_1 \cdot a_2\beta_2 b_2 \cdot \ldots \cdot a_n\beta_n b_n$, where $(s(\beta_1)|\beta_1 \cdots \beta_n|e(\beta_n))$ is a path in Δ of length n and for each $i = 1, \ldots, n$, $b_{i-1} \in A_{e(\beta_{i-1})}$ and $a_i \in A_{s(\beta_i)}$ with $s(\beta_i) = e(\beta_{i-1})$. $s(\beta_1)$ and $e(\beta_n)$ are also called respectively the *start vertex* and the *end vertex* of P. Write $s(P) = s(\beta_1)$ and $e(P) = e(\beta_n)$. A *general $\mathscr{A}$-pseudo-path Q of length n* is given in the form

$$\alpha_1 \cdot c_1 \cdot \alpha_2 \cdot c_2 \cdot \ldots \cdot c_k \cdot \alpha_k \quad \text{or} \quad c_0 \cdot \alpha_1 \cdot c_1 \cdot \alpha_2 \cdot c_2 \cdot \ldots \cdot c_k \cdot \alpha_k \quad \text{or}$$
$$\alpha_1 \cdot c_1 \cdot \alpha_2 \cdot c_2 \cdot \ldots \cdot c_k \cdot \alpha_k \cdot c_{k+1} \quad \text{or} \quad c_0 \cdot \alpha_1 \cdot c_1 \cdot \alpha_2 \cdot c_2 \cdot \ldots \cdot c_k \cdot \alpha_k \cdot c_{k+1}$$

where α_i is a pure $\mathscr{A}$-pseudo-path of length n_i and $\sum_{i=1}^{k} n_i = n$, and the start vertex of α_{i+1} is just the end vertex of α_i, that is, $e(\alpha_i) = s(\alpha_{i+1})$ and $c_i \in A_{e(\alpha_i)}$.

Let V be the k-linear space with basis the set of all general $\mathscr{A}$-paths of Δ.

Consider the quotient R of the k-linear space V by the subspace generated by all the elements of the form

$$(2.1)\qquad a_1\beta_1 b_1\cdots a_j\beta_j\left(b_j^1+\cdots+b_j^m\right)\cdot\gamma-\sum_{l=1}^{m}a_1\beta_1 b_1\cdots a_j\beta_j b_j^l\cdot\gamma$$

$$(2.2)\qquad \alpha\cdot\left(a_1^1+\cdots+a_1^m\right)\beta_1 b_1\cdots a_n\beta_n b_n-\sum_{l=1}^{m}\alpha\cdot a_1^l\beta_1 b_1\cdots a_n\beta_n b_n$$

$$(2.3)\qquad (ab)\cdot c\beta d-a\cdot(b\cdot c\beta d),\quad a\beta b\cdot(cd)-(a\beta b\cdot c)\cdot d$$

$$(2.4)\qquad a\beta b\cdot 1-a\beta b,\quad 1\cdot a\beta b-a\beta b$$

where $a, b, c, d, b_j^l, a_1^l\in\bigcup_{i\in\Delta_0}A_i$ and 1 is the identity of $A=\oplus_{i\in\Delta_0}A_i$.

In R, define the following multiplication. Given two elements

$$[a_1\beta_1 b_1\cdot a_2\beta_2 b_2\cdots a_n\beta_n b_n]\quad\text{and}\quad[c_1\gamma_1 d_1\cdot c_2\gamma_2 d_2\cdots c_n\gamma_m d_m]$$

in which at least one is of length $n\geq 1$, define $[a_1\beta_1 b_1\cdot a_2\beta_2 b_2\cdots a_n\beta_n b_n]\cdot[c_1\gamma_1 d_1\cdot c_2\gamma_2 d_2\cdots c_n\gamma_m d_m]$ to be equal to $[a_1\beta_1 b_1\cdot a_2\beta_2 b_2\cdots a_n\beta_n b_n\cdot c_1\gamma_1 d_1\cdot c_2\gamma_2 d_2\cdots c_n\gamma_m d_m]$ if b_n and c_1 are in the same A_i, and 0 otherwise.

Given two elements a, b of length zero, that is, $a, b\in\bigcup_{i\in\Delta_0}A_i$, define

$$a\cdot b=\begin{cases}ab, & \text{if } a, b \text{ are in the same } A_i, \text{ where } ab \text{ means the product of } a, b \text{ in } A_i,\\ 0, & \text{otherwise.}\end{cases}$$

It is easy to check that the above multiplication in R is well-defined and makes R into a k-algebra, called the *$\mathscr{A}$-pseudo path algebra* of Δ. Denote it by $R=PSE_k(\Delta,\mathscr{A})$. Clearly, R is an A-bimodule.

Note the following facts.

(i) $R=PSE_k(\Delta,\mathscr{A})$ has identity if and only if Δ_0 is finite.

(ii) Any path $(s(\beta_1)|\beta_1\cdots\beta_n|e(\beta_n))$ in Δ can be considered as an $\mathscr{A}$-path with $a_i=e_i$ the identity of A_i. Hence the usual path algebra $k\Delta$ can be embedded into the $\mathscr{A}$-pseudo path algebra $PSE_k(\Delta,\mathscr{A})$. If $A_i=k$ for each $i\in\Delta_0$ then $PSE_k(\Delta,\mathscr{A})=k\Delta$.

(iii) For $R=PSE_k(\Delta,\mathscr{A})$, $\dim_k R<\infty$ if and only if $\dim_k A_i$ is finite for each $i\in\Delta_0$ and Δ is a finite quiver without oriented cycles.

Associated with the pair $(A, {}_AM_A)$ for a k-algebra A and an A-bimodule M, we write the n-fold k-tensor product $M\otimes_k M\otimes\cdots\otimes_k M$ as M^n and we denote by $M(n)$ the sum $\sum_{M_1,M_2,\cdots,M_n}M_1\otimes_k M_2\otimes_k\cdots\otimes_k M_n$ where each M_i is either M or A but no two As are neighbouring and at least one M_i is equal to M. Then we define $\mathscr{P}\mathscr{T}(A,M)=A\oplus M(1)\oplus M(2)\oplus\cdots\oplus M(n)\oplus\cdots$ as an abelian group. Denote

https://doi.org/10.1017/S144678870003799X Published online by Cambridge University Press

by $M(n, l)$ the sum of these items $M_1 \otimes_k M_2 \otimes_k \cdots \otimes_k M_n$ of $M(n)$ in which there are l M_is equal to M. Clearly, $(n-1)/2 \leq l \leq n$ and $M(n) = \sum_{(n-1)/2 \leq l \leq n} M(n, l)$. Writing $M^0 = A$, $\mathscr{P}\mathscr{T}(A, M)$ becomes a k-algebra with multiplication induced by the natural k-bilinear maps:

$$\begin{aligned} M^i \times M^j &\to M^{i+j} && \text{for } i \geq 1,\ j \geq 1; \\ M^i \times A &\to M^i \otimes_k A && \text{for } i \geq 1; \\ A \times M^j &\to A \otimes_k M^j && \text{for } j \geq 1 \end{aligned}$$

and the natural A-bilinear map:

$$A \times A \to A \otimes_A A = A.$$

The associative law of $\mathscr{P}\mathscr{T}(A, M)$ follows from $(A \otimes_A A) \otimes_k M \cong A \otimes_A (A \otimes_k M)$. We call $\mathscr{P}\mathscr{T}(A, M)$ a *pseudo tensor algebra*.

Now, we define a special class of pseudo tensor algebras so as to characterize pseudo path algebras. An *$\mathscr{A}$-path-type pseudo tensor algebra* is defined to be the pseudo tensor algebra $\mathscr{P}\mathscr{T}(A, M)$ satisfying

(i) $A = \bigoplus_{i \in \Delta_0} A_i$ for a family of k-algebras $\mathscr{A} = \{A_i : i \in \Delta_0\}$,

(ii) $M = \bigoplus_{i,j \in I} {}_iM_j$ for finitely generated A_i-A_j-bimodules ${}_iM_j$ for all i and j in I and $A_k \cdot {}_iM_j = 0$ if $k \neq i$ and ${}_iM_j \cdot A_k = 0$ if $k \neq j$.

A *free $\mathscr{A}$-path-type pseudo tensor algebra* is the $\mathscr{A}$-path-type pseudo tensor algebra $\mathscr{P}\mathscr{T}(A, M)$ in which each finitely generated A_i-A_j-bimodule ${}_iM_j$ for i and j in I is a free bimodule with a basis and the rank of this basis is equal to the rank of ${}_iM_j$ as a finitely generated A_i-A_j-bimodule.

$\mathscr{A}$-path-type pseudo tensor algebras and pseudo path algebras can be constructed from each other as follows.

Given an $\mathscr{A}$-pseudo path algebra $PSE_k(\Delta, \mathscr{A})$, let $A = \bigoplus_{i \in \Delta_0} A_i$. For any i and j, let ${}_iM_j^F$ be the free A_i-A_j-bimodule with basis given by the arrows from i to j. It is easy to see that the number of free generators in the basis is the rank of ${}_iM_j^F$ as a finitely generated bimodule. Define $A_k \cdot {}_iM_j^F = 0$ if $k \neq i$ and ${}_iM_j^F \cdot A_k = 0$ if $k \neq j$. Let $M^F = \bigoplus_{i \to j} {}_iM_j^F$, which is clearly an A-bimodule. This gives a uniquely defined free $\mathscr{A}$-path-type pseudo tensor algebra denoted $\mathscr{P}\mathscr{T}(A, M^F)$.

Conversely, assume that $\mathscr{P}\mathscr{T}(A, M)$ is an $\mathscr{A}$-path-type pseudo tensor algebra with a family of k-algebras $\mathscr{A} = \{A_i : i \in I\}$ and finitely generated A_i-A_j-bimodules ${}_iM_j$ for all i and j in I such that $A = \bigoplus_{i \in I} A_i$ and $M = \bigoplus_{i,j \in I} {}_iM_j$ and $A_k \cdot {}_iM_j = 0$ if $k \neq i$ and ${}_iM_j \cdot A_k = 0$ if $k \neq j$. Trivially, ${}_iM_j = A_iMA_j$. Let the rank of ${}_iM_j$ be r_{ij}. Now we can associate with $\mathscr{P}\mathscr{T}(A, M)$ a quiver $\Delta = (\Delta_0, \Delta_1)$ and its pseudo path algebra $R = PSE_k(\Delta, \mathscr{A})$ in the following way. Let $\Delta_0 = I$ as the set of vertices. For $i, j \in I$, let the number of arrows from i to j in Δ be the rank r_{ij} of the finitely generated A_i-A_j-bimodules ${}_iM_j$. Obviously, if ${}_iM_j = 0$ then there are no arrows from i to j. Thus, we get a quiver $\Delta = (\Delta_0, \Delta_1)$ which is called *the quiver of*

$\mathscr{P}\mathscr{T}(A, M)$, and its $\mathscr{A}$-pseudo path algebra $R = PSE_k(\Delta, \mathscr{A})$ which is called *the corresponding $\mathscr{A}$-pseudo path algebra of $\mathscr{P}\mathscr{T}(A, M)$*.

One can find two non-isomorphic finitely generated bimodules which possess the same rank, therefore there exist two $\mathscr{A}$-path-type pseudo tensor algebras $\mathscr{P}\mathscr{T}(A, M_1)$ and $\mathscr{P}\mathscr{T}(A, M_2)$, with non-isomorphic M_1 and M_2, such that their induced quivers and $\mathscr{A}$-pseudo path algebras are the same.

From the above discussion, every $\mathscr{A}$-path-type pseudo tensor algebra $\mathscr{P}\mathscr{T}(A, M)$ can be used to construct its corresponding $\mathscr{A}$-pseudo path algebra $PSE_k(\Delta, \mathscr{A})$; but, from this $\mathscr{A}$-pseudo path algebra $PSE_k(\Delta, \mathscr{A})$, we can get uniquely the free $\mathscr{A}$-path-type pseudo tensor algebra $\mathscr{P}\mathscr{T}(A, M^F)$. Thus, we have the following lemma.

LEMMA 2.4. *Every $\mathscr{A}$-path-type pseudo tensor algebra $\mathscr{P}\mathscr{T}(A, M)$ can be used to construct uniquely the free $\mathscr{A}$-path-type pseudo tensor algebra $\mathscr{P}\mathscr{T}(A, M^F)$. There is a surjective k-algebra morphism π: $\mathscr{P}\mathscr{T}(A, M^F) \to \mathscr{P}\mathscr{T}(A, M)$ such that $\pi({}_iM_j^F) = {}_iM_j$ for any $i, j \in I$.*

PROOF. We need only prove the second conclusion. For $\mathscr{P}\mathscr{T}(A, M)$, let the rank of ${}_iM_j$ be r_{ij}. Thus, for the corresponding $\mathscr{A}$-pseudo path algebra $PSE_k(\Delta, \mathscr{A})$, the number of the arrows from i to j is r_{ij}, and then, in $\mathscr{P}\mathscr{T}(A, M^F)$, the rank of the free generators of ${}_iM_j^F$ given by the arrows is also r_{ij}. Define π: $\mathscr{P}\mathscr{T}(A, M^F) \to \mathscr{P}\mathscr{T}(A, M)$ by giving a bijection between the set of the free generators of ${}_iM_j^F$ and the set of the chosen generators of ${}_iM_j$ with cardinal number equal to the rank. Then π can be expanded to become a surjective k-algebra morphism with $\pi({}_iM_j^F) = {}_iM_j$ for any $i, j \in I$. □

Next, we will show (in Proposition 2.9) that every $\mathscr{A}$-path-type pseudo tensor algebra is a homomorphic image of its corresponding $\mathscr{A}$-pseudo path algebra.

The following criterion for constructing algebra morphisms from pseudo tensor algebras to other algebras is useful, which is modified from [1, Lemma III.1.2]. Contrast it with Lemma 2.3.

LEMMA 2.5. *Let A be a k-algebra and M an A-bimodule. Let Λ be a k-algebra and $f : A \oplus M \to \Lambda$ a k-linear map such that $f|_A : A \to \Lambda$ is an algebra morphism. Then there is a unique algebra homomorphism $\tilde{f} : \mathscr{P}\mathscr{T}(A, M) \to \Lambda$ such that $\tilde{f}|_{A\oplus M} = f$ and generally, $\tilde{f}\left(\sum_{n=0}^{\infty} m_1^n \otimes_k \cdots \otimes_k m_n^n\right) = \sum_{n=0}^{\infty} f(m_1^n) \cdots f(m_n^n)$ for $m_1^n \otimes_k \cdots \otimes_k m_n^n \in M(n)$.*

PROOF. Consider the map $\phi : M \times M \to \Lambda$ defined by $\phi(m_1, m_2) = f(m_1)f(m_2)$ for m_1 and m_2 in M. We have for $\alpha \in k$ that

$$\phi(m_1\alpha, m_2) = f(m_1\alpha)f(m_2) = f(m_1)f(\alpha m_2) = \phi(m_1, \alpha m_2).$$

https://doi.org/10.1017/S144678870003799X Published online by Cambridge University Press

Hence there is a unique group morphism $f_2 : M \otimes_k M \to \Lambda$ such that

$$f_2(m_1 \otimes_k m_2) = f(m_1)f(m_2).$$

Moreover, f_2 is a k-linear map. Similarly, for the map $\phi : M \times A \to \Lambda$ defined by $\phi(m, a) = f(m)f(a)$ for $m \in M$ and $a \in A$, one can induce the k-linear map $f_2 : M \otimes_k A \to \Lambda$ satisfying $f_2(m \otimes_k a) = f(m)f(a)$.

By induction, we can obtain the unique k-linear map $f_n : M(n) \to \Lambda$ satisfying $f_n(v_1 \otimes_k \cdots \otimes_k v_n) = f(v_1)\cdots f(v_n)$. Since $f|_A$ is a k-algebra homomorphism, we define $\widetilde{f} : \mathscr{P}\mathscr{T}(A, M) \to \Lambda$ by $\widetilde{f}|_{A\oplus M} = f$ and

$$\widetilde{f}\left(\sum_{n=0}^{\infty} m_1^n \otimes_k \cdots \otimes_k m_n^n\right) = \sum_{n=0}^{\infty} f(m_1^n)\cdots f(m_n^n)$$

for $m_1^n \otimes_k \cdots \otimes_k m_n^n \in M(n)$, which can easily be seen to be a k-algebra homomorphism uniquely determined by f.

In fact, for $m_1 \otimes_k \cdots \otimes_k m_n \in M(n)$ and $\bar{m}_1 \otimes_k \cdots \otimes_k \bar{m}_l \in M(l)$, if $m_n, \bar{m}_1 \in A$, then

$$\begin{aligned}
&\widetilde{f}\big((m_1 \otimes_k \cdots \otimes_k m_n) \cdot (\bar{m}_1 \otimes_k \cdots \otimes_k \bar{m}_l)\big) \\
&\quad = \widetilde{f}(m_1 \otimes_k \cdots \otimes_k m_{n-1} \otimes_k m_n \otimes_A \bar{m}_1 \otimes_k \bar{m}_2 \otimes_k \cdots \otimes_k \bar{m}_l) \\
&\quad = \widetilde{f}(m_1 \otimes_k \cdots \otimes_k m_{n-1} \otimes_k m_n\bar{m}_1 \otimes_k \bar{m}_2 \otimes_k \cdots \otimes_k \bar{m}_l) \\
&\quad = f(m_1)\cdots f(m_{n-1})f(m_n\bar{m}_1)f(\bar{m}_2)\cdots f(\bar{m}_l) \\
&\quad = f(m_1)\cdot f(m_{n-1})f(m_n)f(\bar{m}_1)f(\bar{m}_2)\cdots f(\bar{m}_l) \\
&\quad = \widetilde{f}(m_1 \otimes_k \cdots \otimes_k m_n)\widetilde{f}(\bar{m}_1 \otimes_k \cdots \otimes_k \bar{m}_l).
\end{aligned}$$

In the other cases, it can be proved similarly. □

Comparing the definitions of generalized path algebra, tensor algebra and pseudo path algebra, pseudo tensor algebra, the following facts hold:

FACT 2.6. (1) There is a natural surjective homomorphism

$$\iota : PSE_k(\Delta, \mathscr{A}) \longrightarrow k(\Delta, \mathscr{A}) \quad \text{with}$$
$$\ker \iota = \langle a\beta b \cdot c - a\beta bc,\ c \cdot a\beta b - ca\beta b,\ a\alpha b \cdot c\beta d - a\alpha 1 \cdot bc \cdot 1\beta d\rangle$$

for any $a, b, c, d \in A = \oplus_i A_i$, $\alpha, \beta \in \Delta_1$, where 1 is the identity of A. It follows that

$$PSE_k(\Delta, \mathscr{A})/\ker \iota \cong k(\Delta, \mathscr{A})$$

as algebras.

(2) There is a natural surjective homomorphism

$$\tau : \mathcal{PT}(A, M) \longrightarrow T(A, M) \quad \text{with}$$
$$\ker\tau = \langle m \otimes c - mc \otimes 1,\ c \otimes m - 1 \otimes cm,\ mb \otimes cn - m \otimes bc \otimes n\rangle$$

for any $b, c \in A, m, n \in M$, where 1 is the identity of A. It follows that

$$\mathcal{PT}(A, M)/\ker\tau \cong T(A, M)$$

as algebras.

Clearly, all $\mathcal{A}$-pseudo-paths of length zero (equivalently, $\mathcal{A}$-paths of length zero), that is, the elements of $\bigcup_{i\in\Delta_0} A_i$, can generate a subalgebra of $PSE_k(\Delta, \mathcal{A})$ (respectively, $k(\Delta, \mathcal{A})$). Denote this subalgebra by $PSE_k(\Delta_0, \mathcal{A})$ (respectively, $k(\Delta_0, \mathcal{A})$). Then, $PSE_k(\Delta_0, \mathcal{A}) = k(\Delta_0, \mathcal{A})$, or say, $\iota_{PSE_k(\Delta_0, \mathcal{A})} = id$. Denote by $PSE_k(\Delta_1, \mathcal{A})$ (respectively, $k(\Delta_1, \mathcal{A})$) the k-linear space consisting of all pure $\mathcal{A}$-pseudo-paths (respectively, all $\mathcal{A}$-paths) of length 1 and by J (respectively, $\widetilde{J}$) the ideal in $PSE_k(\Delta, \mathcal{A})$ (respectively, $k(\Delta, \mathcal{A})$) generated by all elements in $PSE_k(\Delta_1, \mathcal{A})$ (respectively, $k(\Delta_1, \mathcal{A})$).

It is easy to see that $PSE_k(\Delta_1, \mathcal{A})$ (respectively, $k(\Delta_1, \mathcal{A})$) is an A-sub-bimodule of $PSE_k(\Delta, \mathcal{A})$ (respectively, $k(\Delta, \mathcal{A})$), and

(i) $\iota\big(PSE_k(\Delta_1, \mathcal{A})\big) = k(\Delta_1, \mathcal{A})$;
(ii) $\iota J = \widetilde{J}$, $\iota^{-1}\widetilde{J} = J$.

We will now show some useful properties of $\mathcal{A}$-pseudo-path algebras which hold similarly for $\mathcal{A}$-path algebras under the relationships in Fact 2.6.

LEMMA 2.7. *Let $\mathcal{PT}(A, M^F)$ be the free $\mathcal{A}$-path-type pseudo tensor algebra built by an $\mathcal{A}$-pseudo path algebra $PSE_k(\Delta, \mathcal{A})$. Then there is a k-algebra isomorphism $\phi : \mathcal{PT}(A, M^F) \to PSE_k(\Delta, \mathcal{A})$ such that for any $t \geq 1$,*

$$\phi\left(\bigoplus_{n,l\geq t} M^F(n, l)\right) = J^t.$$

PROOF. By the multiplication in $PSE_k(\Delta, \mathcal{A})$, $[a_i]\cdot[a_j] = 0$ for $i \neq j$ and $a_i \in A_i$, $a_j \in A_j$. Obviously, we have a k-algebra isomorphism

$$f : A = \bigoplus_{i\in I} A_i \to PSE_k(\Delta_0, \mathcal{A})$$

by $f(a_1 + \cdots + a_n) = [a_1] + \cdots + [a_n]$. Also we can define

$$f : M^F = \bigoplus_{i,j\in I} {}_iM_j^F \to PSE_k(\Delta_1, \mathcal{A})$$

by giving a bijection between a chosen basis for each ${}_iM_j^F$ and the set of arrows from i to j, that is, $f(am_{\alpha_{ij}}b) = a\alpha_{ij}b$ where α_{ij} is an arrow from i to j and $m_{\alpha_{ij}}$ is the corresponding element in the basis of ${}_iM_j^F$, $a, b \in A$. Since $PSE_k(\Delta_0, \mathscr{A})$ is a k-subalgebra of $PSE_k(\Delta, \mathscr{A})$, there is, by Lemma 2.5, a k-algebra morphism $\widetilde{f}: \mathscr{P}\mathscr{T}(A, M^F) \to PSE_k(\Delta, \mathscr{A})$ such that

$$\widetilde{f}|_{A\oplus M^F} = f \quad \text{and} \quad \widetilde{f}\left(\sum_{n=0}^{\infty} m_1^n \otimes \cdots \otimes m_n^n\right) = \sum_{n=0}^{\infty} f(m_1^n) \cdots f(m_n^n)$$

for $m_1^n \otimes \cdots \otimes m_n^n \in M^F(n)$. Thus, $\widetilde{f}\big((A \otimes_k M^F \otimes_k A)^t\big) = \big(A \cdot PSE_k(\Delta_1, \mathscr{A}) \cdot A\big)^t$ and moreover, $\widetilde{f}\big(\bigoplus_{n,l\geq t} M^F(n,l)\big) = J^t$, in particular, $\widetilde{f}\big(\bigoplus_{k\geq 1} M^F(k)\big) = J$. But, $PSE_k(\Delta, \mathscr{A}) = PSE_k(\Delta_0, \mathscr{A}) \cup J \cup \cdots \cup J^t \cup \cdots$. Hence $\widetilde{f}$ is surjective.

Let $\{x_\lambda\}$ denote a k-basis of A. For $M^F(n,l)$, we have a k-basis formed by some elements of the form

$$x_{\lambda_{i_1}} \otimes x_{\lambda_{j_1}} m_1 x_{\lambda_{k_1}} \otimes x_{\lambda_{i_2}} \otimes x_{\lambda_{j_2}} m_2 x_{\lambda_{k_2}} \otimes \cdots \otimes x_{\lambda_{i_l}} \otimes x_{\lambda_{j_l}} m_l x_{\lambda_{k_l}} \otimes \cdots$$

where there is some $\mathscr{A}$-pseudo-path

$$[x_{\lambda_{i_1}} \cdot x_{\lambda_{j_1}} \beta_1 x_{\lambda_{k_1}} \cdot x_{\lambda_{i_2}} \cdot x_{\lambda_{j_2}} \beta_2 x_{\lambda_{k_2}} \cdots x_{\lambda_{i_l}} \cdot x_{\lambda_{j_l}} \beta x_{\lambda_{k_l}} \cdots]$$

in $PSE_k(\Delta, \mathscr{A})$ such that, for $j = 1, \ldots, t$, m_j is amongst the chosen basis elements in ${}_{s(\beta_j)}M^F_{s(\beta_{j+1})}$ for the corresponding arrow β_j. Then

$$\begin{aligned}&\widetilde{f}\big(x_{\lambda_{i_1}} \otimes x_{\lambda_{j_1}} m_1 x_{\lambda_{k_1}} \otimes x_{\lambda_{i_2}} \otimes x_{\lambda_{j_2}} m_2 x_{\lambda_{k_2}} \otimes \cdots \otimes x_{\lambda_{i_l}} \otimes x_{\lambda_{j_l}} m_l x_{\lambda_{k_l}} \otimes \cdots\big) \\ &\quad = \big[x_{\lambda_{i_1}} \cdot x_{\lambda_{j_1}} \beta_1 x_{\lambda_{k_1}} \cdot x_{\lambda_{i_2}} \cdot x_{\lambda_{j_2}} \beta_2 x_{\lambda_{k_2}} \cdots x_{\lambda_{i_l}} \cdot x_{\lambda_{j_l}} \beta_l x_{\lambda_{k_l}} \cdots\big]\end{aligned}$$

This implies that distinct basis elements are mapped to distinct $\mathscr{A}$-pseudo-paths and, for $a_1 + \cdots + a_n \neq 0$ in A, $f(a_1 + \cdots + a_n) = [a_1] + \cdots + [a_n] \neq 0$. Hence $\widetilde{f}$ is injective. Therefore $\phi = \widetilde{f}$ is a k-algebra isomorphism with the desired properties. □

By Lemma 2.7, $PSE_k(\Delta, \mathscr{A}) \overset{\phi^{-1}}{\cong} \mathscr{P}\mathscr{T}(A, M^F)$. Then $\ker \iota \overset{\phi^{-1}}{\cong} \ker \tau$. Thus a natural induced algebra homomorphism $\overline{\phi^{-1}}$ is obtained from ϕ^{-1} so that

$$PSE_k(\Delta, \mathscr{A}) / \ker \iota \overset{\overline{\phi^{-1}}}{\cong} \mathscr{P}\mathscr{T}(A, M^F) / \ker \tau.$$

Moreover, by Fact 2.6, we get the following $\widetilde{\phi}$ from $\overline{\phi^{-1}}$ as above so as to obtain the result on $\mathscr{A}$-path algebras analogous to Lemma 2.7 for $\mathscr{A}$-pseudo-path algebras.

LEMMA 2.8. *Let $T(A, M^F)$ be the free $\mathscr{A}$-path-type tensor algebra built by an $\mathscr{A}$-path algebra $k(\Delta, \mathscr{A})$. There is a k-algebra isomorphism $\widetilde{\phi}$: $T(A, M^F) \to k(\Delta, \mathscr{A})$ such that for any $t \geq 1$,*

$$\widetilde{\phi}\left(\bigoplus_{j\geq t} M^{Fj}\right) = \widetilde{J}^t.$$

From this, we obtain the commutative diagram

$$\begin{tikzcd} \mathscr{P}\mathscr{T}(A, M^F) \arrow[r, "\cong,\ \phi"] \arrow[d, "\tau"] & PSE_k(\Delta, \mathscr{A}) \arrow[d, "\iota"] \\ T(A, M^F) \arrow[r, "\cong,\ \tilde{\phi}"] & k(\Delta, \mathscr{A}). \end{tikzcd} \tag{2.5}$$

PROPOSITION 2.9. *Let $\mathscr{P}\mathscr{T}(A, M)$ be an $\mathscr{A}$-path-type pseudo tensor algebra with the corresponding $\mathscr{A}$-pseudo path algebra $PSE_k(\Delta, \mathscr{A})$. Then there is a surjective k-algebra homomorphism $\varphi : PSE_k(\Delta, \mathscr{A}) \to \mathscr{P}\mathscr{T}(A, M)$ such that for any $t \geq 1$,*

$$\varphi(J^t) = \bigoplus_{n, l \geq t} M(n, l).$$

PROOF. Let $\mathscr{P}\mathscr{T}(A, M^F)$ be the free $\mathscr{A}$-path-type pseudo tensor algebra built by the $\mathscr{A}$-pseudo path algebra $PSE_k(\Delta, \mathscr{A})$. Then, by Lemma 2.7, there is a k-algebra isomorphism $\phi : \mathscr{P}\mathscr{T}(A, M^F) \to PSE_k(\Delta, \mathscr{A})$ such that for any $t \geq 1$, $\phi\left(\bigoplus_{n, l \geq t} M^F(n, l)\right) = J^t$.

On the other hand, by Lemma 2.4, there is a surjective k-algebra morphism $\pi : \mathscr{P}\mathscr{T}(A, M^F) \to \mathscr{P}\mathscr{T}(A, M)$ such that $\pi\left({}_iM_j^F\right) = {}_iM_j$ for all $i, j \in I$, so $\pi(M^F) = M$.

Therefore, $\varphi = \pi\phi^{-1} : PSE_k(\Delta, \mathscr{A}) \to \mathscr{P}\mathscr{T}(A, M)$ is a surjective k-algebra morphism with $\varphi(J^t) = \pi\left(\bigoplus_{n, l \geq t} M^F(n, l)\right) = \bigoplus_{n, l \geq t} M(n, l)$ for any $t \geq 1$. $\square$

From the equation $\varphi = \pi\phi^{-1}$ and the description of $\ker\iota$ and $\ker\tau$ in Fact 2.6, we have $\varphi(\ker\iota) = \ker\tau$. Then, by Proposition 2.9, we naturally induce a surjective k-algebra homomorphism

$$\widetilde{\varphi} : PSE_k(\Delta, \mathscr{A})/\ker\iota \to \varphi\left(PSE_k(\Delta, \mathscr{A})\right)/\varphi(\ker\iota) = \mathscr{P}\mathscr{T}(A, M)/\ker\tau.$$

Thus the following analogue of Proposition 2.9 holds for $\mathscr{A}$-path-type tensor algebras.

PROPOSITION 2.10. *Let $T(A, M)$ be an $\mathscr{A}$-path-type tensor algebra with the corresponding $\mathscr{A}$-path algebra $k(\Delta, \mathscr{A})$. Then there is a surjective k-algebra homomorphism $\widetilde{\varphi} : k(\Delta, \mathscr{A}) \to T(A, M)$ such that for any $t \geq 1$,*

$$\widetilde{\varphi}\left(\widetilde{J}^t\right) = \bigoplus_{j \geq t} M^j.$$

Also, we obtain the commutative diagram

$$\begin{tikzcd} PSE_k(\Delta, \mathscr{A}) \arrow[r, "\varphi"] \arrow[d, "\iota"] & \mathscr{P}\mathscr{T}(A, M) \arrow[d, "\tau"] \\ k(\Delta, \mathscr{A}) \arrow[r, "\widetilde{\varphi}"] & T(A, M). \end{tikzcd} \tag{2.6}$$

A *relation* σ on an $\mathscr{A}$-pseudo path algebra $PSE_k(\Delta, \mathscr{A})$ (respectively, $\mathscr{A}$-path algebra $k(\Delta, \mathscr{A})$) is a k-linear combination of some general $\mathscr{A}$-pseudo paths (respectively, some $\mathscr{A}$-paths) P_i with the same start vertex and the same end vertex, that is, $\sigma = k_1 P_1 + \cdots + k_n P_n$ with $k_i \in k$ and $s(P_1) = \cdots = s(P_n)$ and $e(P_1) = \cdots = e(P_n)$. If $\rho = \{\sigma_t\}_{t \in T}$ is a set of relations on $PSE_k(\Delta, \mathscr{A})$ (respectively, $k(\Delta, \mathscr{A})$), the pair $(PSE_k(\Delta, \mathscr{A}), \rho)$ (respectively, $(k(\Delta, \mathscr{A}), \rho)$) is called an *$\mathscr{A}$-pseudo path algebra with relations* (respectively, *$\mathscr{A}$-path algebra with relations*). Associated with $(PSE_k(\Delta, \mathscr{A}), \rho)$ (respectively, $(k(\Delta, \mathscr{A}), \rho)$) is the quotient k-algebra $PSE_k(\Delta, \mathscr{A}, \rho) \overset{\text{def}}{=} PSE_k(\Delta, \mathscr{A})/\langle \rho \rangle$ (respectively, $k(\Delta, \mathscr{A}, \rho) \overset{\text{def}}{=} k(\Delta, \mathscr{A})/\langle \rho \rangle$), where $\langle \rho \rangle$ denotes the ideal in $PSE_k(\Delta, \mathscr{A})$ (respectively, in $k(\Delta, \mathscr{A})$) generated by the set of relations ρ. When the length $l(P_i)$ of each P_i is at least j, we have $\langle \rho \rangle \subset J^j$ (respectively, $\langle \rho \rangle \subset \widetilde{J}^j$).

For an element $x \in PSE_k(\Delta, \mathscr{A})$ (respectively, $\in k(\Delta, \mathscr{A})$), we denote by $\bar{x}$ the corresponding element in $PSE_k(\Delta, \mathscr{A}, \rho)$ (respectively, $k(\Delta, \mathscr{A}, \rho)$).

FACT 2.11. $\delta \in k(\Delta, \mathscr{A})$ is a relation if and only if all $\sigma \in \iota^{-1}(\delta)$ are relations on $PSE_k(\Delta, \mathscr{A})$.

This fact can easily be seen from the definition of ι. Note that the lengths of paths in a relation are not restricted here, so we have the following.

PROPOSITION 2.12. *Suppose that Δ is a finite quiver. Then*

(i) *each element x in $PSE_k(\Delta, \mathscr{A})$ (respectively, $k(\Delta, \mathscr{A})$) is a sum of some relations;*

(ii) *every ideal I of $PSE_k(\Delta, \mathscr{A})$ (respectively, $k(\Delta, \mathscr{A})$) can be generated by a set of relations.*

PROOF. (i) Let 1 be the identity of A and e_i the identity of A_i for $i \in \Delta_0$. Then $1 = \sum_{i \in \Delta} e_i$ is a decomposition into orthogonal idempotents e_i and we have $x = 1 \cdot x \cdot 1 = \sum_{i,j \in \Delta_0} e_i \cdot x \cdot e_j$. Due to the multiplication of $|\Delta_0| = n < \infty$, $e_i \cdot x \cdot e_j$ can be expanded as a k-linear combination of some such $\mathscr{A}$-paths which have the same start vertex i and the same end vertex j, so $e_i \cdot x \cdot e_j$ is a relation on $PSE_k(\Delta, \mathscr{A})$.

(ii) Assume I is generated by $\{x_\lambda\}_{\lambda \in \Lambda}$. By (i), each x_λ is a sum of some relations $\{\sigma_{\lambda,i}\}$. Then I is generated by all $\{\sigma_{\lambda,i}\}$. □

By the definition of J, we have

$$PSE_k(\Delta, \mathscr{A}, \rho)/\bar{J} = \big(PSE_k(\Delta, \mathscr{A})/\langle \rho \rangle\big)/(J/\langle \rho \rangle) \cong PSE_k(\Delta, \mathscr{A})/J \cong \oplus_{i \in \Delta_0} A_i.$$

Suppose all A_i are k-simple algebras and $J^t \subset \langle \rho \rangle$ for some integer t. Then $PSE_k(\Delta, \mathscr{A}, \rho)/\bar{J} \cong \oplus_{i \in \Delta_0} A_i$ is semisimple and $\bar{J}^t = 0$. It follows that

https://doi.org/10.1017/S144678870003799X Published online by Cambridge University Press

$\bar{J} = \operatorname{rad} PSE_k(\Delta, \mathscr{A}, \rho)$. Similar reasoning holds for $\widetilde{J}$ of $k(\Delta, \mathscr{A})$. Hence we get the following.

PROPOSITION 2.13. (i) *Let $(PSE_k(\Delta, \mathscr{A}), \rho)$ be an $\mathscr{A}$-pseudo path algebra with relations where A_i is simple for all $i \in \Delta_0$. Assume that $J^t \subset \langle\rho\rangle$ for some t. Then the image $\bar{J}$ of J in $PSE_k(\Delta, \mathscr{A}, \rho)$ is* $\operatorname{rad} PSE_k(\Delta, \mathscr{A}, \rho)$, *that is,* $\bar{J} = \operatorname{rad} PSE_k(\Delta, \mathscr{A}, \rho)$;

(ii) *Let $(k(\Delta, \mathscr{A}), \rho)$ be an $\mathscr{A}$-path algebra with relations where A_i is simple for each $i \in \Delta_0$. Assume that $\widetilde{J}^t \subset \langle\rho\rangle$ for some t. Then the image $\bar{\widetilde{J}}$ of $\widetilde{J}$ in $k(\Delta, \mathscr{A}, \rho)$ is* $\operatorname{rad} k(\Delta, \mathscr{A}, \rho)$, *that is,* $\bar{\widetilde{J}} = \operatorname{rad} k(\Delta, \mathscr{A}, \rho)$.

Now, suppose that A is a left Artinian algebra over k and $r = r(A)$ is the radical of A. Then, for all $l \geq 0$, the ring r^l/r^{l+1} is an A-bimodule by $a \cdot (r^l/r^{l+1}) \cdot b = ar^lb/r^{l+1}$ for $a, b \in A$. From $r \cdot r^l/r^{l+1} = 0$ and $r^l/r^{l+1} \cdot r = 0$, we know that r^l/r^{l+1} is a semisimple left and right A-module. For $\bar{x} = x + r \in A/r$, let

$$\bar{x} \cdot (r^l/r^{l+1}) \stackrel{\text{def}}{=} x \cdot (r^l/r^{l+1}) = xr^l/r^{l+1} \quad \text{and}$$
$$(r^l/r^{l+1}) \cdot \bar{x} = (r^l/r^{l+1}) \cdot x = r^lx/r^{l+1}.$$

Then r^l/r^{l+1} is also an A/r-bimodule and a semisimple left and right A/r-module.

PROPOSITION 2.14. *Let A be a left Artinian algebra over k and let $r = r(A)$ be the radical of A. Write $A/r = \bigoplus_{i=1}^{s} \overline{A}_i$ where $\overline{A}_i$ is a simple subalgebra for each i. Then, for all $l \geq 0$,*

(i) *r^l/r^{l+1} is finitely generated as an A/r-bimodule;*

(ii) *${}_iM_j^{(l)} \stackrel{\text{def}}{=} \overline{A}_i \cdot r^l/r^{l+1} \cdot \overline{A}_j$ is finitely generated as $\overline{A}_i$-$\overline{A}_j$-bimodule for each (i, j).*

PROOF. (i) Since A is left Artinian, r^l/r^{l+1} is finitely generated as a left A-module by [1, Corollary I.3.2], so we can write $r^l/r^{l+1} = \sum_{p=1}^{w} A\overline{x}_p$ with some $\overline{x}_p \in r^l/r^{l+1}$. But, due to the definitions of actions,

$$A\overline{x}_p = (A/r)\overline{x}_p \quad \text{and} \quad r^l/r^{l+1} = \sum_{p=1}^{w}(A/r)\overline{x}_p.$$

Moreover,

$$r^l/r^{l+1} = r^l/r^{l+1} \cdot A/r = \left(\sum_{p=1}^{w}(A/r)\overline{x}_p\right)(A/r) = \sum_{p=1}^{w}(A/r)\overline{x}_p(A/r),$$

which means that r^l/r^{l+1} is finitely generated as an A/r-bimodule.

https://doi.org/10.1017/S144678870003799X Published online by Cambridge University Press

(ii) We note that

$$ {}_iM_j^{(l)} = \overline{A}_i \cdot r^l/r^{l+1} \cdot \overline{A}_j = \overline{A}_i \cdot \left(\sum_{p=1}^{w} (A/r)\overline{x}_p(A/r) \right) \cdot \overline{A}_j = \sum_{p=1}^{w} \sum_{u,v=1}^{s} \overline{A}_i\overline{A}_u\overline{x}_p\overline{A}_v\overline{A}_j $$
$$ = \sum_{p=1}^{w} \overline{A}_i\overline{x}_p\overline{A}_j. $$

Hence, ${}_iM_j^{(l)}$ is finitely generated as an $\overline{A}_i$-$\overline{A}_j$-bimodule. □

In particular, for $l = 1$, ${}_iM_j \overset{\text{def}}{=} \overline{A}_i \cdot r/r^2 \cdot \overline{A}_j$ is finitely generated as an $\overline{A}_i$-$\overline{A}_j$-bimodule for each pair (i, j). Henceforth the rank of ${}_iM_j$ will be denoted by t_{ij}.

For $k \neq i$, we have

$$ \overline{A}_k \cdot {}_iM_j = \overline{A}_k \cdot \left(\overline{A}_i \cdot r/r^2 \cdot \overline{A}_j\right) = \left(\overline{A}_k\overline{A}_i\right) \cdot \left(r/r^2 \cdot \overline{A}_j\right) = 0 \cdot r/r^2 \cdot \overline{A}_j = 0 $$

and similarly, for $k \neq j$, we have ${}_iM_j \cdot \overline{A}_k = 0$. Thus we obtain the $\mathscr{A}$-path-type pseudo tensor algebra $\mathscr{P}\mathscr{T}(A/r, r/r^2)$, the $\mathscr{A}$-path-type tensor algebra $T(A/r, r/r^2)$ and the corresponding $\mathscr{A}$-pseudo path algebra $PSE_k(\Delta, \mathscr{A})$ and $\mathscr{A}$-path algebra $k(\Delta, \mathscr{A})$, with $\mathscr{A} = \{\overline{A}_i : i \in \Delta_0\}$, where Δ is called *the quiver of the left Artinian algebra* A.

In what follows, A is always a left Artinian algebra. We will firstly show that under some important conditions, a left Artinian algebra A is isomorphic to some $PSE_k(\Delta, \mathscr{A}, \rho)$.

3. When the quotient algebra can be lifted

Firstly, we introduce the concept of the set of primitive orthogonal simple subalgebras of a left Artinian algebra. For a left Artinian algebra A and $A/r = \bigoplus_{i=1}^{s} \overline{A}_i$ with simple subalgebras $\overline{A}_i$ for all i, where $r = r(A)$ is the radical of A, assume that there are simple k-subalgebras $B_1, \cdots, B_s$ of A such that, for all i, $B_i \cong \overline{A}_i$ as k-algebras under the canonical morphism $\eta : A \to A/r$ and

$$ B_iB_j = \begin{cases} B_i, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} $$

Then, $\widehat{B} = \{B_1, \cdots, B_s\}$ is said to be *the set of primitive orthogonal simple subalgebras of* A.

Obviously,

$$ \overline{A}_i\overline{A}_j = \begin{cases} \overline{A}_i, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} $$

By the definition, $\eta(B_i) = \overline{A}_i$ for all i. Every B_i is a simple k-subalgebra of A, so $B = B_1 + \cdots + B_s$ is a semisimple subalgebra of A.

Our original idea is to introduce the concept of primitive orthogonal simple subalgebras as a generalization of primitive orthogonal idempotents and then transplant the method of primitive orthogonal idempotents in elementary algebras into a left Artinian algebras.

In a left Artinian algebra A, we will show the existence of the set of primitive orthogonal simple k-subalgebras when A/r can be lifted.

An algebra morphism ε: $A/r \to A$ satisfying $\eta\varepsilon = 1$ will be called a *lifting* of the quotient algebra A/r. In this case, we say that *A/r can be lifted.* Evidently, a lifting ε is always a monomorphism and $\operatorname{im}\varepsilon = B$ is a subalgebra of A which is isomorphic to A/r. Then B is semisimple. Moreover, $A = B \oplus r$ as a direct sum of k-linear spaces. Hence A/r can be lifted if and only if A is split over its radical r.

Now, we assume that A/r can be lifted such that $A = B \oplus r$ as above. For the canonical morphism η: $A \to A/r$, $\operatorname{im}\eta|_B = (B+r)/r = A/r$, and $\ker\eta|_B = 0$ since $r \cap B = 0$. Thus $\eta(B) = A/r$ and $B \overset{\eta|_B}{\cong} A/r$ as k-algebras. Since B is semisimple, we write $B = \bigoplus_{i=1}^s B_i$ with simple k-subalgebras B_i for all i. Then

$$B_i B_j = \begin{cases} B_i, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

Moreover, $\eta(B) = \sum_{i=1}^s \eta(B_i)$ where $\eta(B_i)$ is a simple k-subalgebra of A/r for all i. Let $\overline{A}_i$ denote $\eta(B_i)$. Then $\widehat{B} = \{B_1, \cdots, B_s\}$ is the set of primitive orthogonal simple subalgebras of A.

LEMMA 3.1. *Assume that A is a left Artinian k-algebra with $r = r(A)$ the radical of A, and that A/r can be lifted so that $A = B \oplus r$ with $\widehat{B} = \{B_1, \cdots, B_s\}$ the set of primitive orthogonal simple subalgebras of A as constructed above. Write $A/r = \bigoplus_{i=1}^s \overline{A}_i$ where $\overline{A}_i$ is a simple algebra for all i. The following statements hold.*

(i) *Let $\{r_\lambda : \lambda \in I\}$ be a set of elements in r with the index set I such that the images $\overline{r}_\lambda$ in r/r^2 for all $\lambda \in I$ generate r/r^2 as an A/r-bimodule. Then $B_1 \cup \cdots \cup B_s \cup \{r_\lambda : \lambda \in I\}$ generates A as a k-algebra.*

(ii) *There is a surjective k-algebra homomorphism $\widetilde{f} : \mathscr{P}\mathscr{T}(A/r, r/r^2) \to A$ with*

$$\bigoplus_{n \geq rl(A)} \bigoplus_{\max\{rl(A),\,(n-1)/2\} \leq l \leq n} M(n, l) \subset \ker \widetilde{f} \subset \bigoplus_{j \geq 2} M(j),$$

where $\mathrm{rl}(A)$ *denotes the Loewy length of A as a left A-module.*

PROOF. (i) Since r is nilpotent, there is a least m such that $r^m = 0$ but $r^{m-1} \neq 0$. It is easy to see that m is just the Loewy length $\mathrm{rl}(A)$.

In what follows, we will prove this result by using induction on m.

When $m = 1$, we have $r = 0$ and A is semisimple. Thus $B_i = \overline{A}_i$. Hence A is generated as a k-algebra by $B_1 \cup \cdots \cup B_s$.

When $m = 2$, we have $r^2 = 0$ and, for the canonical morphism η, we have $\eta(B_i) = \overline{A}_i$. So, as a k-algebra, A/r can be generated by $(B_1 + r) \cup \cdots \cup (B_s + r)$. Write $A/r = \langle B_1 + r, \cdots, B_s + r\rangle / r$. We have

$$\langle B_1 + r, \cdots, B_s + r\rangle / r = (\langle B_1, \cdots, B_s\rangle + r)/r.$$

Thus, $A/r = (\langle B_1, \cdots, B_s\rangle + r)/r$. Hence $A = \langle B_1, \cdots, B_s\rangle + r$. But,

$$r/r^2 = \sum_{\lambda \in I} A/r \cdot \overline{r}_\lambda = \sum_{\lambda \in I} A/r \cdot \left(r_\lambda + r^2\right) = \sum_{\lambda \in I}\left(Ar_\lambda + r^2\right) = \left(\sum_{\lambda \in I} Ar_\lambda\right) + r^2.$$

Then from $r^2 = 0$ we get $r = \sum_{\lambda \in I} Ar_\lambda$. It follows that

$$\begin{aligned} A &= \langle B_1, \cdots, B_s\rangle + r = \langle B_1, \cdots, B_s\rangle + \sum_{\lambda \in I} (\langle B_1, \cdots, B_s\rangle + r)\, r_\lambda \\ &= \langle B_1, \cdots, B_s\rangle + \sum_{\lambda \in I} \langle B_1, \cdots, B_s\rangle r_\lambda = \langle B_1 \cup \cdots \cup B_s \cup \{r_\lambda : \lambda \in I\}\rangle \end{aligned}$$

as a k-algebra.

Assume now that the claim holds for $m = l \geq 2$. Then consider the claim in the case $m = l + 1$.

Let P be the k-subalgebra of A generated by $B_1 \cup \cdots \cup B_s \cup \{r_\lambda : \lambda \in I\}$. Firstly, we will show that $P/(P \cap r^l) = A/r^l$.

Since $(A/r^l)/(r/r^l) \cong A/r$ is semisimple, $r(A/r^l) = r/r^l$ holds. By the induction assumption, $r^{l+1} = 0$ and $r^i \neq 0$ for any $i \leq l$. For any t, $(r/r^l)^t(A/r^l) = r^t A/r^l = r^t/r^l$ since $r^t A = r^t$ due to the existence of the identity of A. Thus $(r/r^l)^t(A/r^l) = 0$ if and only if $t \geq l$. (If there were $t < l$ such that $r^t = r^l$, then $r^{t+1} = r^{l+1} = 0$, which contradicts $\mathrm{rl}(A) = m = l + 1$). Therefore $\mathrm{rl}(A/r^l) = l$.

Let $\zeta : A \to A/r^l$ be the canonical morphism and $\widetilde{B}_i = \zeta(B_i)$ be simple algebras for all i and π the canonical morphism from A/r^l to $(A/r^l)/(r/r^l) = A/r$. Then $\pi\zeta = \eta$. It follows that $\pi(\widetilde{B}_i) = \overline{A}_i$. This means that $\widehat{\widetilde{B}} = \left\{\widetilde{B}_1, \cdots, \widetilde{B}_s\right\}$ is the set of primitive radical-orthogonal simple algebras of A/r^l. We have that all elements in $\{\overline{r}_\lambda : \lambda \in I\}$ in r/r^2 generate r/r^2 as an A/r-module. But, $A/r \cong (A/r^l)/(r/r^l)$ and $r/r^2 \cong (r/r^l)/(r/r^l)^2$. So, all elements in $\{\overline{r}_\lambda : \lambda \in I\}$ in $(r/r^l)/(r/r^l)^2$ generate $(r/r^l)/(r/r^l)^2$ as an $(A/r^l)/(r/r^l)$-module. Let $\widetilde{r}_\lambda = \zeta(r_\lambda) \in r/r^l$. Then $\pi(\widetilde{r}_\lambda) = \overline{r}_\lambda$. Thus, by the induction assumption, $\widetilde{B}_1 \cup \cdots \cup \widetilde{B}_s \cup \left\{\widetilde{r}_\lambda : \lambda \in I\right\}$ generates the k-algebra A/r^l.

On the other hand, $B_1 \cup \cdots \cup B_s \cup \{r_\lambda : \lambda \in I\}$ generates P. It follows that $\widetilde{B}_1 \cup \cdots \cup \widetilde{B}_s \cup \{\widetilde{r_\lambda} : \lambda \in I\}$ generates the k-algebra $P/(P \cap r^l)$. But $P/(P \cap r^l)$ can be embedded into A/r^l. Therefore, we deduce that $P/(P \cap r^l) = A/r^l$.

It will be proved below that in fact $P = A$, which means that A is generated by $B_1 \cup \cdots \cup B_s \cup \{r_\lambda : \lambda \in I\}$.

Let $x \in A$. Then there exists $y \in P$ such that $x + r^l = y + P \cap r^l$. It follows that $x - y \in r^l$. Thus there are $\alpha_i \in r^{l-1}$ and $\beta_i \in r$ such that $x - y = \sum_{i=1}^q \alpha_i \beta_i$. But $\alpha_i + r^l$ and $\beta_i + r^l$ in A/r^l and $A/r^l = P/(P \cap r^l)$. Then there are a_i and b_i in P such that $\alpha_i + r^l = a_i + P \cap r^l$ and $\beta_i + r^l = b_i + P \cap r^l$. Since $\alpha_i \in r^{l-1}$ and $\beta_i \in r$, we have $a_i \in r^{l-1}$ and $b_i \in r$. Let $a_i' = \alpha_i - a_i$ and $b_i' = \beta_i - b_i$. Then $a_i', b_i' \in r^l$. Hence $\alpha_i\beta_i = (a_i + a_i')(b_i + b_i') = a_ib_i + a_i'b_i + a_ib_i' + a_i'b_i' = a_ib_i \in P$ for all i where $a_i'b_i \in r^{l+1} = 0$, $a_ib_i' \in r^{2l-1} = 0$, $a_i'b_i' \in r^{2l} = 0$. It follows that $x - y \in P$. Hence $x \in P$.

(ii) $r/r^2 = A/r \cdot r/r^2 \cdot A/r = \sum_{i,j=1}^s \overline{A}_i \cdot r/r^2 \cdot \overline{A}_j$ is a direct sum decomposition since $\overline{A}_i^2 = \overline{A}_i$ and $\overline{A}_i\overline{A}_j = 0$ for $i \neq j$. Corresponding to this, in A, we let $W = \sum_{i,j=1}^s B_i r B_j$, where $B_i \overset{\eta}{\cong} \overline{A}_i$. W is a direct sum of $B_i r B_j$ since $B_i^2 = B_i$ and $B_iB_j = 0$ for $i \neq j$. Obviously W is a subalgebra of r and then of A. Also r/r^2 is an (A/r)-bimodule with the action of A/r as above.

$(A/r) \oplus (r/r^2)$ is a k-algebra in which the multiplication is derived from that of A/r and r/r^2 and the A/r-bimodule action of r/r^2.

For each pair of integers i, j with $1 \leq i, j \leq s$, choose elements $\{y_u^{ij}\}_{u \in \Omega_{ij}}$ in $B_i r B_j$ such that $\{\overline{y}_u^{ij}\}_{u \in \Omega_{ij}}$ is a k-basis for $\overline{A}_i \cdot r/r^2 \cdot \overline{A}_j$ where $\overline{y}_u^{ij} = y_u^{ij} + r^2$ is the image in r/r^2. Then $\bigcup_{i,j=1}^s \{\overline{y}_u^{ij}\}_{u \in \Omega_{ij}}$ is a basis for r/r^2. It follows from (i) that $\bigcup_{i,j,u} \{y_u^{i,j}\}_{u \in \Omega_{ij}} \cup B_1 \cup \cdots \cup B_s$ generates A as a k-algebra.

It is easy to see that $\{y_u^{ij}\}_{u \in \Omega_{ij}}$ is k-linear independent in $B_i r B_j$. From the fact that W is a direct sum of $B_i r B_j$, it follows that $\bigcup_{i,j=1}^s \{y_u^{ij}\}_{u \in \Omega_{ij}}$ is a k-linear independent set in W.

Define $f : (A/r) \oplus (r/r^2) \to A$ by $f|_{\overline{A}_i} = \eta^{-1}$ and $f(\overline{y}_u^{ij}) = y_u^{ij}$. Then $f|_{A/r}\colon A/r \to B = f(A/r)$ is a k-algebra isomorphism since $B \overset{\eta|_B}{\cong} A/r$, and $f|_{r/r^2}\colon r/r^2 \to f(r/r^2)$ $(\subset W \subset r)$ is an isomorphism of k-linear spaces. Thus $f : (A/r) \oplus (r/r^2) \to A$ is a k-linear map. Hence, by Lemma 2.5, there is a unique algebra morphism $\widetilde{f} : \mathscr{P}\mathscr{T}(A/r, r/r^2) \to A$ such that $\widetilde{f}|_{(A/r)\oplus(r/r^2)} = f$. As said above, $\bigcup_{i,j,u} \{y_u^{i,j}\}_{u \in \Omega_{ij}} \cup B_1 \cup \cdots \cup B_s$ generates A as a k-algebra. Therefore $\widetilde{f}$ is surjective.

By the definition of $\widetilde{f}$, we have $\widetilde{f}\big((r/r^2)^j\big) = f(r/r^2)^j \subset r^j \subset r^2$ for $j \geq 2$, where $(r/r^2)^j$ denotes $r/r^2 \otimes_k r/r^2 \otimes_k \cdots \otimes_k r/r^2$ with j copies of r/r^2. Also $f|_{A/r}$ and $f|_{r/r^2}$ are monomorphic. By the definition of f on A/r and r/r^2, it is easy to see that $\widetilde{f}|_{(A/r)\oplus(r/r^2)}\colon (A/r) \oplus (r/r^2) \to A$ is a monomorphism with image intersecting r^2

trivially. In the notation of Section 2, $M(n) = \sum_{M_1, M_2, \cdots, M_n} M_1 \otimes_k M_2 \otimes_k \cdots \otimes_k M_n$ where each M_i is either r/r^2 or A/r but no two A/r's are neighbouring and at least one M_i equals M. Then $\mathscr{P}\mathscr{T}(A/r, r/r^2) = A/r \oplus M(1) \oplus M(2) \oplus \cdots \oplus M(n) \oplus \cdots$. It follows that $\ker \widetilde{f} \subset \bigoplus_{j\geq 2} M(j)$.

On the other hand, $M(n, l)$ equals the sum of those items $M_1 \otimes_k M_2 \otimes_k \cdots \otimes_k M_n$ of $M(n)$ in which there are l M_is equal to r/r^2 and $M(n) = \sum_{(n-1)/2 \leq l \leq n} M(n, l)$ as in Section 2. Also $\widetilde{f}((r/r^2)^j) = 0$ for $j \geq rl(A)$ since $r^j = 0$ in this case. It follows that $\widetilde{f}(M(n, l)) = 0$ for any n and any possible $l \geq rl(A)$. Therefore we get

$$\bigoplus_{n \geq rl(A)} \bigoplus_{\max\{rl(A), (n-1)/2\} \leq l \leq n} M(n, l) \subset \ker \widetilde{f}. \qquad \square$$

THEOREM 3.2 (Generalized Gabriel's Theorem Under Lifting). *Assume that A is a left Artinian k-algebra and A/r can be lifted. Then $A \cong PSE_k(\Delta, \mathscr{A}, \rho)$ with $J^s \subset \langle \rho \rangle \subset J$ for some s, where Δ is the quiver of A and ρ is a set of relations on $PSE_k(\Delta, \mathscr{A})$.*

PROOF. Let Δ be the associated quiver of A. By Lemma 3.1(ii), there is a surjective k-algebra morphism $\widetilde{f} : \mathscr{P}\mathscr{T}(A/r, r/r^2) \to A$ with

$$\bigoplus_{n \geq rl(A)} \bigoplus_{\max\{rl(A), (n-1)/2\} \leq l \leq n} M(n, l) \subset \ker \widetilde{f} \subset \bigoplus_{j \geq 2} M(j).$$

By Proposition 2.9, there is the surjective k-algebra homomorphism

$$\varphi : PSE_k(\Delta, \mathscr{A}) \to \mathscr{P}\mathscr{T}(A/r, r/r^2)$$

such that for any $t \geq 1$,

$$\varphi(J^t) = \bigoplus_{n, l \geq t} M(n, l).$$

Then $\widetilde{f}\varphi : PSE_k(\Delta, \mathscr{A}) \to A$ is a surjective k-algebra morphism with the kernel $I = \ker(\widetilde{f}\varphi) = \varphi^{-1}(\ker \widetilde{f})$.

But, $\varphi(J^{rl(A)}) = \bigoplus_{n, l \geq rl(A)} M(n, l)$ and $\varphi(J^2) = \bigoplus_{n, l \geq 2} M(n, l)$. So, by Lemma 3.1(ii), $\varphi(J^{rl(A)}) \subset \ker \widetilde{f} \subset \varphi(J^2) + M(2, 1) + M(3, 1)$.

One can show

$$J^t \subset \varphi^{-1}\varphi(J^t) \subset J^t + \phi\left(\bigoplus_{n} \bigoplus_{l \leq t-1} M^F(n, l)\right) \cap \phi(\ker \pi)$$

for $t \geq 1$. In fact, trivially, $J^t \subset \varphi^{-1}\varphi(J^t)$. On the other hand, $\varphi = \pi\phi^{-1}$ and $\varphi^{-1} = \phi\pi^{-1}$. By Proposition 2.9, $\varphi(J^t) = \bigoplus_{n, l \geq t} M(n, l)$. From the definition of π

in Lemma 2.4, it can be seen that

$$\pi^{-1}\left(\bigoplus_{n,\,l\geq t} M(n,l)\right) \subset \bigoplus_{n,\,l\geq t} M^F(n,l) + \left(\bigoplus_{n}\bigoplus_{l\leq t-1} M^F(n,l)\right) \cap \ker\pi.$$

Thus, by Lemma 2.4, we have

$$\begin{aligned}\varphi^{-1}\varphi(J^t) &= \phi\pi^{-1}\left(\bigoplus_{n,\,l\geq t} M(n,l)\right)\\ &\subset \phi\left(\bigoplus_{n,\,l\geq t} M^F(n,l)\right) + \phi\left(\bigoplus_{n}\bigoplus_{l\leq t-1} M^F(n,l)\right)\cap\phi(\ker\pi)\\ &= J^t + \phi\left(\bigoplus_{n}\bigoplus_{l\leq t-1} M^F(n,l)\right)\cap\phi(\ker\pi).\end{aligned}$$

Hence,

$$\begin{aligned}J^{rl(A)} &\subset \varphi^{-1}\varphi\left(J^{rl(A)}\right) \subset \varphi^{-1}\left(\ker\widetilde{f}\right) = I \subset \varphi^{-1}\varphi\left(J^2\right) + \varphi^{-1}\left(M(2,1)+M(3,1)\right)\\ &\subset J^2 + \phi\left(M^F(3,1)+M^F(2,1)+M^F(1,1)\right)\cap\phi(\ker\pi)\\ &\quad + \varphi^{-1}(M(2,1)+M(3,1))\\ &= J^2 + A\cdot PSE(\Delta_1,\mathscr{A})\cdot A\end{aligned}$$

since $\phi(M^F(1,1))\cap\phi(\ker\pi) = 0$, and then

$$\begin{aligned}&\phi\left(M^F(3,1)+M^F(2,1)+M^F(1,1)\right)\cap\phi(\ker\pi) + \varphi^{-1}(M(2,1)+M(3,1))\\ &\quad = A\cdot PSE(\Delta_1,\mathscr{A})\cdot A.\end{aligned}$$

But it is clear that $J^2 + A\cdot PSE(\Delta_1,\mathscr{A})\cdot A = J$. Therefore, we get:

$$J^{rl(A)} \subset \varphi^{-1}\left(\ker\widetilde{f}\right) = I \subset J.$$

Lastly, by Proposition 2.12, there is a set ρ of relations such that I can be generated by ρ, that is, $I = \langle\rho\rangle$. Hence, $PSE_k(\Delta,\mathscr{A},\rho) = PSE_k(\Delta,\mathscr{A})/\langle\rho\rangle \cong A$ with $\langle\rho\rangle = \ker(\widetilde{f}\varphi)$ and $J^{rl(A)} \subset \langle\rho\rangle \subset J$. □

Usually, for a left Artinian algebra A, the set ρ of relations in Theorem 3.2 is infinite. But when A is finite dimensional, we can show that ρ is finite.

In fact, suppose that A is finite dimensional, so that $\overline{A}_i$ is finite dimensional for all i. Thus the k-space consisting of all $\mathscr{A}$-pseudo paths of a certain length is finite dimensional. It follows that $J^{rl(A)}$ is the ideal finitely generated in $PSE_k(\Delta,\mathscr{A})$ by all $\mathscr{A}$-pseudo paths of length $rl(A)$. Similarly, $PSE_k(\Delta,\mathscr{A})/J^{rl(A)}$ is generated finitely as a k-space by all $\mathscr{A}$-paths of length less than $rl(A)$, and so also is $I/J^{rl(A)}$ as a

k-subspace. Then it is easy to see that I is a finitely generated ideal in $PSE_k(\Delta, \mathscr{A})$. Assume $\{\sigma_1, \ldots, \sigma_p\}$ is a set of finite generators for the ideal I. For the identity $\overline{1}$ of A/r, we have the decomposition into orthogonal idempotents as $\overline{1} = \overline{e}_1 + \cdots + \overline{e}_s$, where $\overline{e}_i$ is the identity of $\overline{A}_i$. Then $\sigma_l = \overline{1} \cdot \sigma_l \cdot \overline{1} = \sum_{1 \le i, j \le s} \overline{e}_i \cdot \sigma_l \cdot \overline{e}_j$, where $\overline{e}_i \sigma_l \overline{e}_j$ can be expanded as a k-linear combination of some such $\mathscr{A}$-pseudo paths which have the same start vertex i and the same end vertex j. So $\sigma^{ilj} = \overline{e}_i \cdot \sigma_l \cdot \overline{e}_j$ is a relation on the $\mathscr{A}$-pseudo path algebra $PSE_k(\Delta, \mathscr{A})$. Moreover, I is generated by all σ^{ilj} since $\sigma_l = \sum_{i,j} \sigma^{ilj}$. Therefore we have a finite set $\rho = \{\sigma^{ilj} : 1 \le i, j \le s, 1 \le l \le p\}$ with $I = \langle \rho \rangle$ such that $PSE_k(\Delta, \mathscr{A}, \rho) = PSE_k(\Delta, \mathscr{A})/\langle \rho \rangle \cong A$. Therefore the following holds.

COROLLARY 3.3. *Assume that A is a finite dimensional k-algebra and A/r can be lifted. Then $A \cong PSE_k(\Delta, \mathscr{A}, \rho)$ with $J^s \subset \langle \rho \rangle \subset J$ for some s, where Δ is the quiver of A and ρ is a finite set of relations on $PSE_k(\Delta, \mathscr{A})$.*

When A is elementary, $A_i = A_j = k$ and ${}_iM_j = r/r^2$ is free as a k-linear space. Thus π is an isomorphism, so $\ker \pi = 0$ and $\ker \varphi = 0$. According to the classical Gabriel Theorem, we have $J^{rl(A)} \subset \langle \rho \rangle \subset J^2$, which is a special case of the results of Theorem 3.2 and Corollary 3.3.

By the famous Wedderburn-Malcev Theorem (see [4]), for a left Artinian k-algebra A and its radical r, if $\operatorname{Dim} A/r \le 1$ then A/r can be lifted. Here, $\operatorname{Dim} A$ is the dimension of a k-algebra A and

$$\operatorname{Dim} A = \sup\{n : H_k^n(A, M) \ne 0 \text{ for some } A\text{-bimodule } M\}$$

where $H_k^n(A, M)$ means the nth Hochschild cohomology module of A with coefficients in M. In particular, $\operatorname{Dim} A/r = 0$ if and only if A/r is a separable k-algebra. By [4, Corollary 10.7b], when k is a perfect field (for example, $\operatorname{char} k = 0$ or k is a finite field), A is separable. So, we have the following.

PROPOSITION 3.4. *Assume that A is a left Artinian k-algebra. Then $A \cong PSE_k(\Delta, \mathscr{A}, \rho)$ with $J^s \subset \langle \rho \rangle \subset J$ for some s, where Δ is the quiver of A and ρ is a set of relations of $PSE_k(\Delta, \mathscr{A})$, if one of the following conditions holds:*

(i) $\operatorname{Dim} A/r \le 1$, *where r is the radical of A;*
(ii) *A/r is separable;*
(iii) *k is a perfect field (for example, when* $\operatorname{char} k = 0$ *or k is a finite field).*

Note that in Proposition 3.4, the condition (ii) is a special case of (i), and (iii) is a special case of (ii).

In Theorem 3.2, $A \cong PSE_k(\Delta, \mathscr{A}, \rho)$ holds where Δ is the quiver of A from the corresponding $\mathscr{A}$-pseudo path algebra of the $\mathscr{A}$-path-type pseudo tensor algebra $\mathscr{P}\mathscr{T}(A/r, r/r^2)$ by the definitions in Section 2. Moreover, in the case where

$\langle\rho\rangle \subset J_\Delta^2$, we will discuss the uniqueness of the corresponding pseudo path algebra and quiver of a left Artinian algebra under isomorphism, that is, whether there exists another quiver and its related pseudo path algebra so that the same isomorphism relation is satisfied. In fact, we have the following statement on the uniqueness.

THEOREM 3.5. *Assume that A is a left Artinian k-algebra. Let $A/r(A) = \bigoplus_{i=1}^{p} \overline{A}_i$ with simple algebras $\overline{A}_i$. If there is a quiver Δ and a pseudo path algebra $PSE_k(\Delta, \mathscr{B})$ with a set of simple algebras $\mathscr{B} = \{B_1, \cdots, B_q\}$ and ρ a set of relations such that $A \cong PSE_k(\Delta, \mathscr{B}, \rho)$ with $J_\Delta^t \subset \langle\rho\rangle \subset J_\Delta^2$ for some t and J_Δ the ideal in $PSE_k(\Delta, \mathscr{B})$ generated by all pure paths in $PSE_k(\Delta_1, \mathscr{B})$, then Δ is just the quiver of A and $p = q$ and $\overline{A}_i \cong B_i$ for $i = 1, \ldots, p$ after reindexing.*

PROOF. $PSE_k(\Delta, \mathscr{B})/J_\Delta = B_1 + \cdots + B_q$ by the definition of J_Δ. Since $J_\Delta^t \subset \langle\rho\rangle$, it follows that $(J_\Delta/\langle\rho\rangle)^t = J_\Delta^t/\langle\rho\rangle = 0$. Also,

$$PSE_k\big(\Delta, \mathscr{B}, \rho\big)/\big(J_\Delta/\langle\rho\rangle\big) = \big(PSE_k(\Delta, \mathscr{B})/\langle\rho\rangle\big)/\big(J_\Delta/\langle\rho\rangle\big) = PSE_k(\Delta, \mathscr{B})/J_\Delta$$
$$= B_1 + \cdots + B_q$$

is semisimple. Hence $J_\Delta/\langle\rho\rangle$ is the radical of $PSE_k(\Delta, \mathscr{B}, \rho)$. Thus, from $A \cong PSEk(\Delta, \mathscr{B}, \rho)$, it follows that $A/r(A) \cong PSE_k(\Delta, \mathscr{B})/J_\Delta$. However, $A/r(A) = \bigoplus_{i=1}^{p} \overline{A}_i$ and $PSE_k(\Delta, \mathscr{B})/J_\Delta = B_1 + \cdots + B_q$ where $\overline{A}_i$ and B_j are simple algebras. Therefore $p = q$ and $\overline{A}_i \cong B_i$ for $i = 1, \ldots, p$ after reindexing, according to the Wedderburn-Artin Theorem.

On the other hand, $A/r(A)^2 \cong PSE_k(\Delta, \mathscr{B})/J_\Delta^2$. Thus the quivers of $A/r(A)^2$ and $PSE_k(\Delta, \mathscr{B})/J_\Delta^2$ are the same.

But

$$PSE_k(\Delta, \mathscr{B})/J_\Delta^2 = \big(PSE_k(\Delta, \mathscr{B})/\langle\rho\rangle\big)/\big(J_\Delta^2/\langle\rho\rangle\big) = PSE_k(\Delta, \mathscr{B}, \rho)/\big(J_\Delta^2/\langle\rho\rangle\big)$$

and the radical of $PSE_k(\Delta, \mathscr{B}, \rho)$ is $J_\Delta/\langle\rho\rangle$. Then the radical of $PSE_k(\Delta, \mathscr{B})/J_\Delta^2$ is $(J_\Delta/\langle\rho\rangle)/(J_\Delta^2/\langle\rho\rangle) \cong J_\Delta/J_\Delta^2$. So, the quivers of $PSE_k(\Delta, \mathscr{B})/J_\Delta^2$ are that of the $\mathscr{A}$-path-type pseudo tensor algebra

$$\mathscr{P}\mathscr{T}\big((PSE_k(\Delta, \mathscr{B})/J_\Delta^2)/(J_\Delta/J_\Delta^2),\ (J_\Delta/J_\Delta^2)/(J_\Delta^2/J_\Delta^2)\big)$$
$$\cong \mathscr{P}\mathscr{T}\big(PSE_k(\Delta, \mathscr{B})/J_\Delta,\ J_\Delta/J_\Delta^2\big).$$

Now, we consider the quiver Γ of $\mathscr{P}\mathscr{T}\big(PSE_k(\Delta, \mathscr{B})/J_\Delta, J_\Delta/J_\Delta^2\big)$. From the definition of the quiver associated to an $\mathscr{A}$-path-type pseudo tensor algebra in Section 2, we know that $\Gamma_0 = \{1, \cdots, q\} = \Delta_0$. For any $i, j \in \Gamma_0$, the number of arrows from i to j in Γ is the rank r_{ij} of ${}_iM_j = B_i \cdot J_\Delta/J_\Delta^2 \cdot B_j$ as a finitely generated B_i-B_j-bimodule. However, by the definition of J_Δ, $B_i \cdot J_\Delta/J_\Delta^2 \cdot B_j$ can be constructed as an B_i-B_j-linear expansion of all $\mathscr{A}$-pseudo-paths of length 1 from i to j in $PSE_k(\Delta_1, \mathscr{B})$. It means that r_{ij} is equal to the number of arrows from i to j in Δ. Thus the number of arrows from i to j in Γ is equal to the number of arrows from i to j in Δ. Then $\Gamma_1 = \Delta_1$. Therefore $\Gamma = \Delta$.

The above discussion implies that the quiver of $A/r(A)^2$ is just Δ. Moreover,

$$A/r(A) = \left(A/r(A)^2\right)/\left(r(A)/r(A)^2\right) \quad \text{and}$$
$$r(A)/r(A)^2 = \left(r(A)/r(A)^2\right)/\left(r(A)/r(A)^2\right)^2,$$

where $r(A)/r(A)^2$ is the radical of $A/r(A)^2$. So the quiver Δ of $A/r(A)^2$ is also that of

$$\mathscr{P}\mathscr{T}\left(\left(A/r(A)^2\right) / \left(r(A)/r(A)^2\right),\ \left(r(A)/r(A)^2\right) / \left(r(A)/r(A)^2\right)^2\right).$$

But

$$\begin{aligned}&\mathscr{P}\mathscr{T}\left(A/r(A), r(A)/r(A)^2\right)\\ &\quad\cong \mathscr{P}\mathscr{T}\left(\left(A/r(A)^2\right) / \left(r(A)/r(A)^2\right), \left(r(A)/r(A)^2\right) / \left(r(A)/r(A)^2\right)^2\right).\end{aligned}$$

It follows that Δ is the quiver of A. □

According to this theorem, we see that for a left Artinian algebra A, the existence of the pseudo path algebra such that A is isomorphic to its quotient algebra (see Theorem 3.2) can imply its uniqueness. That is, such pseudo path algebra, whose quotient is isomorphic to A, is uniquely determined by the quiver and the semisimple decomposition of A.

Our main result, Theorem 3.2, means that when the quotient algebra of a left Artinian algebra is lifted, the algebra can be covered by a pseudo path algebra under an algebra homomorphism. We know that a generalized path algebra must be a homomorphic image of a pseudo path algebra and its definition seems to be more concise than that of pseudo path algebra. So it is natural to ask why we do not look for a generalized path algebra to cover a left Artinian algebra. In fact, this is our original idea. However, unfortunately, in general, as shown by the following counter-example, a left Artinian algebra with lifted quotient may not be a homomorphic image of its corresponding $\mathscr{A}$-path-type tensor algebra. Thus one cannot use the above method (that is, through Proposition 2.10) to gain a generalized path algebra in order to cover the left Artinian algebra. The following counter-example was given by W. Crawley-Boevey.

EXAMPLE 1. There is an example of a finite dimensional algebra A over a field k such that

(a) A is split over its radical r, that is, A/r can be lifted;

https://doi.org/10.1017/S144678870003799X Published online by Cambridge University Press

(b) there is **no** surjective algebra homomorphism from $T(A/r, r/r^2)$ to A, that is, A cannot be equivalent to some quotient of $T(A/r, r/r^2)$.

Concretely, we describe A in the following eight steps.

(1) Let F/k be a finite field extension, and let $\delta : F \to F$ be a nonzero k-derivation. For example, one can take $k = \mathbf{Z}_2(t)$, $F = \mathbf{Z}_2(\sqrt{t})$ and $\delta(p + q\sqrt{t}) = q$ for $p, q \in \mathbf{Z}_2(t)$ where $\mathbf{Z}_2$ denotes the prime field of characteristic 2. It is easy to check that δ is a k-derivation since $\operatorname{char} k = 2$.

(2) Define $E = F \oplus F$ and consider it as an F-F-bimodule with the actions:

$$f(x, y) = (fx, fy), \quad (x, y)f = (xf + y\delta(f), yf)$$

for $x, y, f \in F$. Let θ and ϕ be F-F-bimodule homomorphisms respectively from F to E and from E to F satisfying

$$\theta(x) = (x, 0), \quad \phi(x, y) = y$$

for $x, y \in F$. Then we have a nonsplitting extension of F-F-bimodules:

$$0 \to F \xrightarrow{\theta} E \xrightarrow{\phi} F \to 0$$

In fact, if there were an F-F-bimodule homomorphism $\psi : E \to F$ with $\psi \cdot \theta = 1_F$ then for all $f \in F$ we would have

$$\begin{aligned}\delta(f) &= \psi\theta(\delta(f)) = \psi(\delta(f), 0) = \psi(\delta, f) - \psi(0, f) = \psi((0, 1)f) - \psi(f(0, 1)) \\ &= \psi(0, 1)f - f\psi(0, 1) = 0,\end{aligned}$$

and hence $\delta = 0$, which contradicts the assumption on δ.

(3) Define $A = F \oplus F \oplus E$ with multiplication given by

$$(x, y, e)\left(x', y', e'\right) = \left(xx', xy' + yx', xe' + \theta(yy') + ex'\right).$$

Let $S = \{(x, 0, 0) : x \in F\}$. Then S is a subalgebra of A isomorphic to F.
Let $r = \{(0, y, e) : y \in F, e \in E\}$. Then r is an ideal in A with

$$r^2 = \{(0, 0, e) : e \in \operatorname{im}(\theta)\} \quad \text{and} \quad r^3 = 0.$$

Thus r is the radical of A, and $A = S \oplus r$, so A is split over r.

(4) As an F-F-module, r/r^2 is isomorphic to $F \oplus F$ due to the surjective F-F-module homomorphism $\pi : r \to F \oplus F$ satisfying $\pi(0, y, e) = (y, \phi(e))$ with $\ker \pi = r^2$.

(5) By (3) and (4), the $\mathscr{A}$-path-type tensor algebra $T(A/r, r/r^2) \cong T(F, F \oplus F)$. Let $s = (1, 0)$ and $t = (0, 1)$, so that $F \oplus F \cong Fs \oplus Ft$. Thus $T(F, F \oplus F)$ (equivalently, say $T(A/r, r/r^2)$) can be considered as the free associative algebra $F\langle s, t\rangle$ generated by two variables s, t over F. It follows that the centre $Z\big(T(A/r, r/r^2)\big)$ of $T(A/r, r/r^2)$ is equal to F.

(6) If $(x, y, e) \in Z(A)$ the centre of A, then for all $e' \in E$, (x, y, e) commutes with $(0, 0, e')$, thus $(0, 0, xe') = (0, 0, e'x)$, so $xe' = e'x$. Taking $e' = (0, 1)$, we get $x(0, 1) = (0, 1)x$. But by (2), $x(0, 1) = (0, x)$ and $(0, 1)x = (\delta(x), x)$. It follows that $\delta(x) = 0$. Therefore, we have

$$Z(A) \subset \{(x, y, e) : x, y \in F, e \in E, \delta(x) = 0\}.$$

(7) If L is a subalgebra of $Z(A)$ and is a field, then $\dim_k L \overset{<}{\neq} \dim_k F$.

In fact, the composition

$$L \hookrightarrow Z(A) \hookrightarrow \{(x, y, e) : x, y \in F, e \in E, \delta(x) = 0\} \to \{x : \delta(x) = 0\}$$

is an algebra homomorphism. Assume that $l = (x, y, e) \in L$ is in the kernel of this composition. Then $x = 0$ and $l = (0, y, e)$, so $l \in r$ the radical of A. By (3), $l^3 = 0$. But L is a field, so $l = 0$ which means that this composition is a one-one map. Therefore,

$$\dim_k L \leq \dim_k\{x : \delta(x) = 0\} \overset{<}{\neq} \dim_k F$$

where "$\neq$" is from $\delta \neq 0$.

(8) If there were a surjective algebra homomorphism $\lambda : T(A/r, r/r^2) \to A$, it would induce a homomorphism of the centre $Z(T(A/r, r/r^2))$ of $T(A/r, r/r^2)$ into the centre $Z(A)$ of A. By (5), $Z(T(A/r, r/r^2)) = F$. Thus, $L = \lambda(F)$ would be a field and a subalgebra of $Z(A)$. By (7), we have $\dim_k L \overset{<}{\neq} \dim_k F$. On the other hand, if there is an x satisfying $0 \neq x \in \ker \lambda|_F$, that is, $\lambda(x) = 0$, then, since F is a field, we get $\lambda(1) = \lambda(1/x)\lambda(x) = 0$, which implies $\lambda = 0$ since λ is an algebra homomorphism. This is impossible since λ is surjective. Hence $\ker \lambda|_F = 0$, that is, $\lambda|_F$ is injective, so $F \overset{\lambda|_F}{\cong} L$ which contradicts $\dim_k L \overset{<}{\neq} \dim_k F$.

This completes the description of Example 1. Due to this example, we know that a general left Artinian algebra with lifted quotient cannot be covered by its corresponding $\mathscr{A}$-path-type tensor algebra. This is the reason that we introduce pseudo path algebras and $\mathscr{A}$-path-type pseudo tensor algebras to replace generalized path algebras and $\mathscr{A}$-path-type tensor algebras in order to cover left Artinian algebras with lifted quotients.

However, there still exist some special classes of left Artinian algebras which can be covered by the corresponding $\mathscr{A}$-path-type tensor algebras and moreover by a generalized path algebra. This point can be seen in the next section, but we will have to restrict a left Artinian algebra to be finite dimensional.

4. When the radical is 2-nilpotent

In this section, we will always assume that the radical r of a finite dimensional algebra A is 2-nilpotent, that is, $r \neq 0$ but $r^2 = 0$. Also, suppose that A is split over its radical r such that $A = B \oplus r$ with $\widehat{B} = \{B_1, \ldots, B_s\}$ the set of primitive orthogonal simple subalgebras of A as constructed in Section 3. For $\bar{x} = x + r \in A/r$, let $\bar{x} \cdot r \overset{\text{def}}{=} xr$ and $r \cdot \bar{x} = rx$. Then r is a finitely generated A/r-bimodule. If $A/r = \bigoplus_{i=1}^{s} \overline{A}_i$ where $\overline{A}_i$ is a simple subalgebra for each i, then, for each pair (i, j), r is a finitely generated $\overline{A}_i$-$\overline{A}_j$-bimodule whose rank is written as l_{ij}. Now

$$r = A/r \cdot r \cdot A/r = \sum_{i,j=1}^{s} \overline{A}_i \cdot r \cdot \overline{A}_j = \sum_{i,j=1}^{s} {}_iM_j$$

where ${}_iM_j \overset{\text{def}}{=} \overline{A}_i \cdot r \cdot \overline{A}_j$. Then, for $k \neq i$,

$$\overline{A}_k \cdot {}_iM_j = \overline{A}_k \cdot \left(\overline{A}_i \cdot r \cdot \overline{A}_j\right) = B_k B_i r B_j = 0,$$

so $\overline{A}_k \cdot {}_iM_j = 0$; similarly, for $k \neq j$, ${}_iM_j \cdot \overline{A}_k = 0$. Hence, we get the $\mathscr{A}$-path-type tensor algebra $T(A/r, r)$ and the corresponding $\mathscr{A}$-path algebra $k(\Delta, \mathscr{A})$ with $\mathscr{A} = \{\overline{A}_i : i \in \Delta_0\}$. Δ is called the *quiver of* A. In a manner similar to the proof of Lemma 3.1, we obtain the following results.

LEMMA 4.1. *Assume that A is a finite dimensional k-algebra with 2-nilpotent radical $r = r(A)$ and A is split over the radical r. Let $\widehat{B} = \{B_1, \ldots, B_s\}$ be the set of primitive radical-orthogonal simple subalgebras of A as constructed above. Write $A/r = \bigoplus_{i=1}^{s} \overline{A}_i$, where $\overline{A}_i$ are simple algebras for all i. Then the following statements hold.*

(i) *If $\{r_1, \ldots, r_t\}$ is a set of generators of the A/r-bimodule r then $B_1 \cup \cdots \cup B_s \cup \{r_1, \cdots, r_t\}$ generates A as a k-algebra;*

(ii) *There is a surjective k-algebra homomorphism $\widetilde{f} : T(A/r, r) \to A$ with $\ker \widetilde{f} = \bigoplus_{j \geq 2} (r)^j$, where $(r)^j$ denotes $r \otimes_{A/r} r \otimes_{A/r} \cdots \otimes_{A/r} r$ with j copies of r.*

PROOF. It is easy to see that r is an (A/r)-bimodule with the action given by $\overline{A}_i \cdot r = B_i r$. Note that $\overline{A}_i \overline{A}_j \cdot r = 0 \cdot r = 0$ and, on the other hand,

$$\overline{A}_i \overline{A}_j \cdot r = (B_i B_j + r) \cdot r = B_i B_j r \subset rr = 0,$$

so that this action is well-defined. The proof of (i) can be given in a manner similar to the proof of Lemma 3.1(i) in the case $\mathrm{rl}(A) = 2$.

Next, we prove (ii). $r = A/r \cdot r \cdot A/r = \sum_{i,j=1}^{s} \overline{A}_i \cdot r \cdot \overline{A}_j = \sum_{i,j=1}^{s} B_i r B_j$ is a direct sum decomposition since $B_i^2 = B_i$ and $B_i B_j = 0$ for $i \neq j$.

$(A/r)\oplus r$ is a k-algebra in which the multiplication is derived from the A/r-module action of r and the multiplication of A/r and r.

For each pair of integers i, j with $1 \le i, j \le s$, choose elements $\{y_u^{ij}\}$ to form a k-basis in $B_i r B_j$. Then $\bigcup_{i,j=1}^{s}\{y_u^{ij}\}$ is a basis for r.

Define $f : (A/r) \oplus r \to A$ by $f|_r = id_r$, that is, $f(y_u^{ij}) = y_u^{ij}$, and $f|_{\overline{A}_i} = \eta^{-1}$. Then, $f|_{A/r} : A/r \to B = f(A/r)$ is a k-algebra isomorphism since $B \overset{\eta|_B}{\cong} A/r$, and $f|_r$: $r \to f(r) = r \subset A$ is an embedded homomorphism of A/r-bimodules. Hence, by Lemma 2.3, there is a unique algebra morphism $\widetilde{f} : T(A/r, r) \to A$ such that $\widetilde{f}|_{(A/r)\oplus r} = f$.

Firstly, $\bigcup_{i,j,u}\{y_u^{i,j}\} \subset \widetilde{f}(r)$ and $B_1 \cup \cdots \cup B_s \subset \widetilde{f}(A/r)$. From (i), it follows that $\bigcup_{i,j,u}\{y_u^{i,j}\} \cup B_1 \cup \cdots \cup B_s$ generates A as a k-algebra and then $\widetilde{f}$ is surjective. On the other hand, $f|_{A/r}$ and $f|_r$ are monomorphic, so $\widetilde{f}|_{(A/r)\oplus r} : (A/r) \oplus r \to A$ is a monomorphism. Then $\ker \widetilde{f} \subset \bigoplus_{j\ge 2}(r)^j$. Moreover, $\widetilde{f}\big((r)^j\big) = 0$ for $j \ge 2$ since $r^j = 0$ in this case. Therefore, $\bigoplus_{j\ge 2}(r)^j \subset \ker \widetilde{f}$. Thus, $\ker \widetilde{f} = \bigoplus_{j\ge 2}(r)^j$. □

In the proof of this lemma, since $f|_r = \mathrm{id}_r$, it is naturally a A/r=homomorphism. So, the condition of Lemma 2.2 is satisfied by $T(A/r, r)$. In general, this is not true for $T(A/r, r)$ in the case that $r^2 \neq 0$.

THEOREM 4.2 (Generalized Gabriel's Theorem With 2-Nilpotent Radical). *Assume that A is a finite dimensional k-algebra with 2-nilpotent radical $r = r(A)$ and A is split over the radical r. Then, $A \cong k(\Delta, \mathscr{A}, \rho)$ with $\widetilde{J}^2 \subset \langle\rho\rangle \subset \widetilde{J}^2 + \widetilde{J} \cap \ker\widetilde{\varphi}$ where Δ is the quiver of A and ρ is a set of relations of $k(\Delta, \mathscr{A})$, $\widetilde{\varphi}$ is defined as in Proposition* 2.10.

PROOF. Let Δ be the associated quiver of A. By Lemma 4.1(ii), we have the surjective k-algebra homomorphism $\widetilde{f} : T(A/r, r) \to A$. By Proposition 2.10, there is a surjective k-algebra homomorphism $\widetilde{\varphi} : k(\Delta, \mathscr{A}) \twoheadrightarrow T(A/r, r)$ such that $\widetilde{\varphi}(\widetilde{J}^t) = \bigoplus_{j\ge t}(r)^j$ for all $t \ge 1$. Then $\widetilde{f}\widetilde{\varphi} : k(\Delta, \mathscr{A}) \to A$ is a surjective k-algebra morphism where $I = \ker\big(\widetilde{f}\widetilde{\varphi}\big) = \widetilde{\varphi}^{-1}\big(\bigoplus_{j\ge 2}(r)^j\big)$ since $\ker \widetilde{f} = \bigoplus_{j\ge 2}(r)^j = \widetilde{\varphi}(\widetilde{J}^2)$.

As a special case of the corresponding part of the proof of Theorem 3.2, we have

$$\widetilde{J}^t \subset \widetilde{\varphi}^{-1}\widetilde{\varphi}\big(\widetilde{J}^t\big) \subset \widetilde{J}^t + \widetilde{\phi}\left(\bigoplus_{j\le t-1}(r)^{Fj}\right) \cap \widetilde{\phi}(\ker \pi)$$

for $t \ge 1$. Hence,

$$\begin{aligned}\widetilde{J}^2 &\subset \widetilde{\varphi}^{-1}\widetilde{\varphi}\big(\widetilde{J}^2\big) = \widetilde{\varphi}^{-1}\big(\ker \widetilde{f}\big) = I \subset \widetilde{J}^2 + \widetilde{\phi}\left(\bigoplus_{j\le 1}(r)^{Fj}\right) \cap \widetilde{\phi}(\ker \pi)\\ &\subseteq \widetilde{J}^2 + \widetilde{J} \cap \widetilde{\phi}(\ker \pi).\end{aligned}$$

But, $\widetilde{\phi}(\ker\pi) = \widetilde{\phi}(\pi^{-1}(0)) = \widetilde{\varphi}^{-1}(0) = \ker\widetilde{\varphi}$. Then we get $\widetilde{J}^2 \subset I \subset \widetilde{J}^2 + \widetilde{J} \cap \ker\widetilde{\varphi}$.

The ideal $\widetilde{J}^2$ is finitely generated in $k(\Delta, \mathscr{A})$ by all $\mathscr{A}$-paths of length 2, while the k-linear space $k(\Delta, \mathscr{A})/\widetilde{J}^2$ is finitely generated by all $\mathscr{A}$-paths of length less than 2, as is $I/\widetilde{J}^2$ as a k-subspace. Then I is a finitely generated ideal in $k(\Delta, \mathscr{A})$. Assume that $\{\sigma_1, \dots, \sigma_p\}$ is its finite set of generators. Moreover, $\sigma_l = \sum_{1\le i,j\le s} \overline{e}_i\sigma_l\overline{e}_j$ where $\overline{e}_i\sigma_l\overline{e}_j$ is a relation on the $\mathscr{A}$-path algebra $k(\Delta, \mathscr{A})$. Therefore, for

$$\rho = \left\{\overline{e}_i\sigma_l\overline{e}_j : 1 \le i, j \le s, 1 \le l \le p\right\},$$

we get $I = \langle\rho\rangle$. Hence $k(\Delta, \mathscr{A}, \rho) = k(\Delta, \mathscr{A})/\langle\rho\rangle \cong A$ with $\langle\rho\rangle = \ker\left(\widetilde{f}\widetilde{\varphi}\right)$ and $\widetilde{J}^2 \subset \langle\rho\rangle \subset \widetilde{J}^2 + \widetilde{J} \cap \ker\widetilde{\varphi}$. □

COROLLARY 4.3. *Assume that A is a finite dimensional k-algebra with 2-nilpotent radical $r = r(A)$. Then, $A \cong k(\Delta, \mathscr{A}, \rho)$, with $\widetilde{J}^2 \subset \langle\rho\rangle \subset \widetilde{J}$ where Δ is the quiver of A and ρ is a set of relations of $k(\Delta, \mathscr{A})$, if one of the following conditions hold:*

(i) $\operatorname{Dim} A/r \le 1$ *for the radical r of A;*
(ii) *A/r is separable;*
(iii) *k is a perfect field (for example, when* $\operatorname{char} k = 0$ *or k is a finite field).*

As in the case of Theorem 3.5, in the case that $\langle\rho\rangle \subseteq J_\Delta^2$, we have the uniqueness of the corresponding $\mathscr{A}$-path algebra and quiver of a finite dimensional algebra. That is, we have the following statement.

THEOREM 4.4. *Assume that A is a finite dimensional k-algebra. Let $A/r(A) = \bigoplus_{i=1}^p \overline{A}_i$, where each $\overline{A}_i$ is a simple algebra. If there is a quiver Δ and a generalized path algebra $k(\Delta, \mathscr{B})$ with a set of simple algebras $\mathscr{B} = \{B_1, \dots, B_q\}$ and a set ρ of relations such that $A \cong k(\Delta, \mathscr{B}, \rho)$ with $J_\Delta^t \subset \langle\rho\rangle \subset J_\Delta^2$ for some t and J_Δ the ideal in $k(\Delta, \mathscr{B})$ generated by all elements in $k(\Delta_1, \mathscr{B})$, then Δ is the quiver of A and $p = q$ such that $\overline{A}_i \cong B_i$ for $i = 1, \dots, p$ after reindexing.*

This theorem can be proved in the same way as Theorem 3.5: we only need to replace $\mathscr{A}$-path-type tensor algebra and $\mathscr{A}$-path with $\mathscr{A}$-path-type pseudo tensor algebra and $\mathscr{A}$-pseudo path respectively.

By Fact 2.6, an $\mathscr{A}$-path-type tensor algebra or an $\mathscr{A}$-path algebra can be covered respectively by $\mathscr{A}$-path-type pseudo tensor algebra or $\mathscr{A}$-pseudo path algebra. Thus we can also state a Generalized Gabriel's Theorem With 2-Nilpotent Radical for $\mathscr{A}$-pseudo path algebras. As a corollary of Theorem 4.2, one has the following.

PROPOSITION 4.5. *Assume that A is a finite dimensional k-algebra with 2-nilpotent radical $r = r(A)$ and A/r can be lifted. Then*

$$A \cong PSE_k(\Delta, \mathscr{A}, \rho) \quad \textit{with} \quad J^2 \subset \langle\rho\rangle \subset J^2 + J \cap \ker\varphi$$

https://doi.org/10.1017/S144678870003799X Published online by Cambridge University Press

where Δ is the quiver of A, ρ is a set of relations on $PSE_k(\Delta, \mathscr{A})$ and φ is defined as in Proposition 2.9.

PROOF. We have the composition of surjective homomorphisms:

$$PSE_k(\Delta, \mathscr{A}) \xrightarrow{\iota} k(\Delta, \mathscr{A}) \xrightarrow{\widetilde{f}\widetilde{\varphi}} A.$$

Then $A \cong PSE_k(\Delta, \mathscr{A})/\ker\left(\widetilde{f}\widetilde{\varphi}\iota\right)$, where $\ker\left(\widetilde{f}\widetilde{\varphi}\iota\right) = \iota^{-1}\left(\ker(\widetilde{f}\widetilde{\varphi})\right)$.

By Theorem 4.2, $\widetilde{J}^2 \subset \ker(\widetilde{f}\widetilde{\varphi}) \subset \widetilde{J}^2 + \widetilde{J} \cap \ker\widetilde{\varphi}$. Thus,

$$\iota^{-1}\left(\widetilde{J}^2\right) \subset \iota^{-1}\left(\ker\left(\widetilde{f}\widetilde{\varphi}\right)\right) \subset \iota^{-1}\left(\widetilde{J}^2\right) + \iota^{-1}\left(\widetilde{J} \cap \ker\widetilde{\varphi}\right).$$

But, since $\iota^{-1}(\widetilde{J}) = J$, it follows that $\iota^{-1}(\widetilde{J}^2) = J^2$ and $\iota^{-1}\left(\widetilde{J} \cap \ker\widetilde{\varphi}\right) = J \cap \ker\varphi$. Thus we get

$$J^2 \subset \ker\left(\widetilde{f}\widetilde{\varphi}\iota\right) \subset J^2 + J \cap \ker\varphi.$$

By Proposition 2.12(ii), there is a set ρ of relations on $PSE_k(\Delta, \mathscr{A})$ such that $\ker\left(\widetilde{f}\widetilde{\varphi}\iota\right) = \langle\rho\rangle$. Then

$$A \cong PSE_k(\Delta, \mathscr{A})/\ker\left(\widetilde{f}\widetilde{\varphi}\iota\right) = PSE_k(\Delta, \mathscr{A})/\langle\rho\rangle = PSE_k(\Delta, \mathscr{A}, \rho)$$

and $J^2 \subset \langle\rho\rangle \subset J^2 + J \cap \ker\varphi$. □

So far, in Section 3 and this section, we have established isomorphisms between an algebra and its $\mathscr{A}$-pseudo path algebra or $\mathscr{A}$-path algebra with relations (see Theorem 3.2 and Proposition 4.5) in the cases where this algebra is left Artinian with splitting over its radical or moreover, is finite-dimensional with 2-nilpotent radical. However, it seems to be difficult to discuss the same question for an arbitrary algebra. Our question is whether it would be possible to characterize an arbitrary finite-dimensional algebra which is split over its radical through the combination of the two methods for a left Artinian algebra with splitting over its radical and a finite-dimensional algebra with 2-nilpotent radical.

In fact, for such a finite-dimensional algebra A, we can start from $B = A/r^2$ where $r = r(A)$ is the radical of A. Consider $r(A/r^2) = r/r^2$, denoted by $\widehat{r}$. Then $\widehat{r}^2 = r^2/r^2 = 0$. By Lemma 4.1(ii), there is a surjective homomorphism of algebras $\widetilde{f} : T\left((A/r^2)/(r/r^2), r/r^2\right) \to A/r^2$.

But we have $(A/r^2)/(r/r^2) \cong A/r$, so

$$\widetilde{f} : T\left(A/r, r/r^2\right) \to A/r^2$$

is a surjective homomorphism of algebras.

https://doi.org/10.1017/S144678870003799X Published online by Cambridge University Press

On the other hand, according to the method in Section 3, in order to obtain the corresponding Gabriel Theorem for this A, the key is to find an algebra homomorphism α corresponding to $\widetilde{f}$ in Lemma 3.1. Therefore, this problem may be regarded as the problem of finding a surjective homomorphism of algebras α such that the following diagram commutes

$$\begin{tikzcd} & T(A/r, r/r^2) \arrow[dl, "\alpha"'] \arrow[d, "\widetilde{f}"] & \\ A \arrow[r, "\pi"] & A/r^2 \arrow[r] & 0, \end{tikzcd}$$

where π denotes the natural homomorphism. If such an α exists, the generalized Gabriel Theorem should hold for this finite-dimensional algebra A.

Acknowledgements

This project was supported by the Program for New Century Excellent Talents in University (No. 04-0522) and the National Natural Science Foundation of China (No. 10571153).

The author takes this opportunity to express thanks to Prof. Shao-xue Liu for his kind advice. His assistance has been important for this work. Also, he would like to thank Prof. W. Crawley-Boevey, Prof. Z. Z. Lin, Prof. B. Zhu, Dr G. X. Liu and Dr L. L. Chen for their helpful conversations and suggestions.

Finally, my thanks goes especially to Jason Whyte for the excellent typing of the manuscript.

References

[1] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras* (Cambridge University Press, 1995).

[2] F. U. Coelho and S. X. Liu, *Generalized path algebras*, volume 210 of *Lecture Notes in Pure and Appl. Math.* (Dekker, New York, 2000).

[3] Y. A. Drozd and V. V. Kirichenko, *Finite Dimensional Algebras* (Springer-Verlag, Berlin, 1994).

[4] R. S. Pierce, *Associative Algebras* (Springer-Verlag, New York, 1982).

[5] C. M. Ringel, *Tame algebras and integral quadratic forms*, volume 1099 of *Lecture Notes in Math.* (Springer-Verlag, 1984).

Department of Mathematics
Zhejiang University
Hangzhou, Zhejiang 310027
China
e-mail: fangli@zju.edu.cn