ON SOME METHODS FOR CONSTRUCTION OF B.I.B. DESIGNS

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1. Summary. In this paper some methods for the construction of balanced incomplete block (b.i.b. for conciseness) designs are given. In the last section it is established that the existence of an affine resolvable b.i.b. design implies the existence of two other b.i.b. designs; §§ 6 and 7 are independent of §§ 3, 4, and 5.

2. Preliminaries and terminology. If M is a module with v elements and if there exist n blocks (or subsets) of elements of M, viz.:

$$[\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ik}], \qquad i = 1, 2, \ldots, n, \quad \alpha_{ij} \in M, \tag{2.1}$$

where no block is a rearrangement of the elements of another block and such that the set of $nk(k-1)$ differences of the form

$$\alpha_{ij} - \alpha_{ij'}, \qquad j \neq j', \quad 1 \leqq j, j' \leqq k, \quad i = 1, 2, \ldots, n, \tag{2.2}$$

contain all the non-zero elements of M, each repeated λ times, we will say that (2.1) is an n-block difference system of strength λ and block size k for the module M. In case $n = 1$, the difference system is called a difference set for M. Obviously, we have

$$\lambda(v-1) = nk(k-1). \tag{2.3}$$

For each prime p and each positive integer δ, $GF(p^\delta)$ denotes a Galois field with p^δ elements. Let $p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_m^{\delta_m}$ be powers of distinct primes. Consider the set S of all m-vectors of the form

$$\xi = (a_1, a_2, \ldots, a_m) \quad \text{with } a_i \in GF(p_i^{\delta_i}), \qquad i = 1, 2, \ldots, m.$$

Define an addition of these vectors in the usual manner. Thus, if

$$\eta = (b_1, b_2, \ldots, b_m),$$

then

$$\xi + \eta = \eta + \xi = (a_1 + b_1, a_2 + b_2 + \ldots + a_m + b_m).$$

Obviously under this addition, S is an abelian additive group with

$$p_1^{\delta_1} p_2^{\delta_2} \ldots p_m^{\delta_m}$$

elements. We may also define a multiplication for these vectors by the rule

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$\xi\eta = \eta\xi = (a_1b_1, a_2b_2, \ldots, a_mb_m)$. Under addition and multiplication as defined above, S is an abelian ring.

If $p_1, p_2, \ldots, p_m$ are all odd primes, we can break up S disjointly as $S = S' \cup S'' \cup (0, 0, \ldots, 0)$ such that if a vector $(x_1, x_2, \ldots, x_m)$ of S belongs to S', then the vector $(-x_1, -x_2, \ldots, -x_m)$ belongs to S''. Each of S' and S'' then contains $(v-1)/2$ vectors where $v = p_1^{\delta_1} p_2^{\delta_2} \ldots p_m^{\delta_m}$.

Of course this decomposition is not unique. We will henceforth assume that the vector $(1, 1, 1, \ldots, 1)$ of S is in S'.

The definition of a b.i.b. design with parameters v, b, r, k, λ is well known. A b.i.b. design has an incidence matrix constructed in the following way. List the varieties in a column and the blocks in a row. Construct a $v \times b$ matrix by inserting 1 in the (i, j) position of the matrix if the ith variety occurs in the jth block and 0 if it does not, $i = 1, 2, \ldots, v$; $j = 1, 2, \ldots, b$. All the row sums of this matrix are r, the column sums k, and the scalar product of any two rows of the matrix is λ. Conversely, existence of such a $(0-1)$-matrix implies the existence of a b.i.b. design with the above parameters.

A b.i.b. design with parameters $v = nk$, $b = nr$, r, k, λ (n a positive integer), is called resolvable if the blocks can be divided into r disjoint classes, each class containing n blocks such that the blocks of each class contain a complete replication of the v varieties (i.e., the n blocks of a class contain in between them each variety once and only once). A resolvable b.i.b. design with parameters $v = nk$, $b = nr = v + r - 1$, $r = k + \lambda$, k, λ is called an affine resolvable b.i.b. design.

An interesting intersection property of blocks of an affine resolvable b.i.b. design, due to Bose (**2**), states that any two blocks belonging to two different replications have k/n varieties in common. (Of course, two blocks belonging to the same replication have no common variety.)

Without loss of generality, we will suppose that the first set of n blocks of an affine resolvable b.i.b. design makes up the first replication, the second set of n blocks makes up the second replication and so on. Moreover, we will assume that the 1st, $(n+1)$th, $(2n+1)$th, $\ldots$, $(r-1)(n+1)$th blocks all contain the first variety. Also, we will suppose that the first block contains the first k varieties. If A is the incidence matrix of the above affine resolvable b.i.b. design, then the 1st, $(n+1)$th, $(2n+1)$th, $\ldots$, $(r-1)(n+1)$th positions of its first row are occupied by 1 and the rest of the positions by 0. Denote the column vectors of A by $\xi_1, \xi_2, \ldots, \xi_{v+r-1}$. We will use this matrix in proving Theorems 15 and 16.

3. Application of a difference system of a module.

A difference system for a module can be utilized for the construction of a b.i.b. design. This method is sometimes called the method of difference. We will use the following theorem due to Bose (**3**).

THEOREM 1. *If there exists an n-block difference system of strength* λ *and*

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block size k for a module M of order v, then there exists a b.i.b. design (with no two blocks identical) having parameters v, $b = vn$, $r = nk$, k, λ.

Proof. The proof is very simple. Identify in any manner each of the v varieties with an element of M. Let (2.1) be the n-block difference system for M. These n blocks are the initial n blocks of the b.i.b. design. Next we "develop" each initial block by adding successively the non-zero elements of M. Consequently, all the vn blocks of the design can be compactly given as

$$[\alpha_{i1} + y, \alpha_{i2} + y, \ldots, \alpha_{ik} + y], \qquad i = 1, 2, \ldots, n,$$

where y runs through all the elements of M.

Consider a pair of varieties μ, ν say, $\mu, \nu \in M$. Let $\mu - \nu = \pi \neq 0$. Now π occurs λ times amongst the differences (2.2). Suppose that

$$\pi = a_{i_t j_t} - a_{i_t j_t'}, \qquad t = 1, 2, \ldots, \lambda.$$

Here $i_1, i_2, \ldots, i_\lambda$ may not be all distinct but if $i_e = i_t$, then $j_e \neq j_t$ and $j_e' \neq j_t'$. Determine $\zeta_1, \zeta_2, \ldots, \zeta_\lambda$ in M such that $\mu = a_{i_t j_t} + \zeta_t$, $t = 1, 2, \ldots, \lambda$. Note that $\zeta_1, \zeta_2, \ldots, \zeta_\lambda$ are all distinct elements of M. It is easy to see that μ and ν occur together in each of the following blocks (and in these alone)

$$[\alpha_{i_t 1} + \zeta_t, \alpha_{i_t 2} + \zeta_t, \ldots, \alpha_{i_t k} + \zeta_t], \qquad t = 1, 2, \ldots, \lambda.$$

Each variety appears k times in the blocks developed from an initial block, so it appears nk times in the whole design. Thus $r = nk$. Also, note that no two blocks in the design can be identical (i.e., the varieties in one block are not a rearrangement of the varieties in another block; otherwise, the two initial blocks from which they were developed would be identical. Hence we get a b.i.b. design from the difference system.

4. Existence of certain difference systems. In this section we establish the existence of difference systems for the ring S of § 2 and Galois field $GF(p^\delta)$.

THEOREM 2. *If* $v = p_1^{\delta_1} p_2^{\delta_2} \ldots p_m^{\delta_m}$ *(canonical factorization of v) and if* $k \leqq \min(p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_m^{\delta_m})$, *then there exists an n-block difference system of strength* λ *and block size k (where* $n = v - 1$, $\lambda = k(k-1)$*) for the ring of* § 2.

Proof. We will construct the blocks of the difference system. Choose k vectors $\alpha_1, \alpha_2, \ldots, \alpha_k$ from S as follows. Let

$$\alpha_i = (a_{i1}, a_{i2}, \ldots, a_{im}), \qquad i = 1, 2, \ldots, k, \tag{4.1}$$

where $a_{ij} \neq a_{i'j}$ if $i \neq i'$. Such a choice is possible since

$$k \leqq \min(p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_m^{\delta_m}).$$

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Denote the non-null vectors of S by $\beta_1 = (1, 1, 1, \ldots, 1)$, $\beta_2, \beta_3, \ldots, \beta_n$, where $n = v - 1$. Then

$$(4.2) \qquad [\beta_i\alpha_1, \beta_i\alpha_2, \ldots, \beta_i\alpha_k], \qquad i = 1, 2, \ldots, n,$$

is the desired difference system for the ring S containing v vectors.

To establish this, let ξ be any non-null vector of S, $\xi = (d_1, d_2, \ldots, d_m)$. Take a pair of distinct integers j, j' from $1, 2, \ldots, k$. In S consider the equation

$$(4.3) \qquad x(\alpha_j - \alpha_{j'}) = \xi.$$

If $\alpha_j - \alpha_j' = (c_1, c_2, \ldots, c_m)$, then $c_1 \neq 0$, $c_2 \neq 0, \ldots, c_m \neq 0$. Setting $x = (x_1, x_2, \ldots, x_m)$, we see that (4.3) is equivalent to the equations

$$(4.4) \qquad x_t c_t = d_t; \quad x_t, c_t, d_t \in \mathrm{GF}(p_t^{\delta_t}), \qquad t = 1, 2, \ldots, m.$$

Whatever d_t might be, each of these equations has a unique solution x_t. Thus $x = (x_1, x_2, \ldots, x_m)$ is the unique solution of (4.3) in S. Consequently, for every choice of distinct integers, j, j', from $1, 2, \ldots, k$, there is a unique β_i in S such that $\beta_i\alpha_j - \beta_i\alpha_{j'} = \xi$.

Thus, the $nk(k - 1)$ differences $\beta_i\alpha_j - \beta_i\alpha_{j'}$, $j \neq j'$, $1 \leqq j, j' \leqq k$, $i = 1, 2, \ldots, n$, give all the non-zero vectors of S, each repeated $\lambda = k(k - 1)$ times. So (4.2) is a difference system for S.

THEOREM 3. *If v is odd and $v = p_1^{\delta_1} p_2^{\delta_2} \ldots p_m^{\delta_m}$ (canonical factorization of v) and if $k \leqq \min(p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_m^{\delta_m})$, then there exists a $[(v - 1)/2]$-block difference system of strength λ and block size k (where $\lambda = k(k - 1)/2$) for the ring of* § 2.

Proof. Choose the vectors $\alpha_1, \alpha_2, \ldots, \alpha_k$ from S as in the above proof. Denote the vectors of S' (defined in § 2) by $\beta_1 = (1, 1, 1, \ldots, 1)$, $\beta_2, \beta_3, \ldots, \beta_{(v-1)/2}$. Then a difference system for S is

$$(4.5) \qquad [\beta_i\alpha_1, \beta_i\alpha_2, \ldots, \beta_i\alpha_k], \qquad i = 1, 2, \ldots, (v - 1)/2.$$

To see this, let ξ be any non-null vector of S. Choose any pair of distinct integers j, j', $1 \leqq j, j' \leqq k$. In S, consider the equations

$$(4.6) \qquad x(\alpha_j - \alpha_{j'}) = \xi,$$

$$(4.7) \qquad y(\alpha_{j'} - \alpha_j) = \xi.$$

If $x = (x_1, x_2, \ldots, x_m)$ is the solution of (4.6), then $y = (-x_1, -x_2, \ldots, -x_m)$ is the solution of (4.7). Thus one and only one of the equations (4.6) and (4.7) has a solution in S'. Therefore, the $k(k - 1)(v - 1)/2$ differences $\beta_i\alpha_j - \beta_i\alpha_{j'}$, $j \neq j'$, $1 \leqq j, j' \leqq k$, $i = 1, 2, \ldots, (v - 1)/2$, give all the non-zero vectors of S, each repeated $\lambda = k(k - 1)/2$ times. Hence (4.5) is a difference system for S.

A weak version of this theorem (viz. with $k \leqq \min(p_1, p_2, \ldots, p_m)$ was established by Betty Jane Gassner (**4**) by a different method. Theorems 3 and 9 have been obtained independently by Ramanujacharyulu (**7**).

We now discuss a difference system for a Galois field $\mathrm{GF}(p^\delta)$. Let g be a primitive root of $\mathrm{GF}(p^\delta)$ (i.e., a generator of the multiplicative group G of the non-zero elements of $\mathrm{GF}(p^\delta)$). Then all the non-zero elements of $\mathrm{GF}(p^\delta)$ can be expressed as powers of g.

If $p^\delta = kf + 1$, then $1, g^f, g^{2f}, \ldots, g^{(k-1)f}$ form a subgroup C_0 of G. With the aid of C_0 we can break up G into f disjoint cosets C_i, $i = 0, 1, 2, \ldots, f-1$. C_i consists of the elements $g^i, g^{i+f}, g^{i+2f}, \ldots, g^{i+(k-1)f}$ and

$$G = C_0 \cup C_1 \cup \ldots \cup C_{f-1}.$$

If p is an odd prime and k is odd, the elements of $C_{f/2}$ are negatives of the elements of C_0. For, remembering that $g^{kf/2} = -1$, we have $g^{f/2} = -g^{(k+1)f/2}$, $g^{f/2+f} = -g^{(k+3)f/2}, \ldots, g^{f/2+(k-3)f/2} = -g^{(k-1)f}$ and $g^{f/2+(k-1)f/2} = -1$, $g^{f/2+(k+1)f/2} = -g^f, \ldots, g^{f/2+(k-1)f} = -g^{(k-1)f/2}$. Similarly, the elements of $C_{f/2+1}, C_{f/2+2}, \ldots, C_{f-1}$ are, respectively, the negatives of the elements of $C_1, C_2, \ldots, C_{f/2-1}$.

In the following theorems we will use the cosets as blocks for a difference system for $\mathrm{GF}(p^\delta)$.

THEOREM 4. *If $p^\delta = kf + 1$, p odd prime, k odd, g a primitive root of the Galois field* $\mathrm{GF}(p^\delta)$, *then the $f/2$ blocks*

$$\text{(4.8)} \qquad [g^i, g^{i+f}, g^{i+2f}, \ldots, g^{i+(k-1)f}], \qquad i = 0, 1, 2, \ldots, f/2 - 1,$$

form an $(f/2)$-block difference system of strength $\lambda = (k-1)/2$ and block size k, for $\mathrm{GF}(p^\delta)$.

Proof. Note that the above blocks are the cosets $C_0, C_1, C_2, \ldots, C_{f/2-1}$ of G. Take the element g^i of C_i, where i is any one of $0, 1, 2, \ldots, f-1$. Consider the equation (in j, j')

$$\text{(4.9)} \qquad g^{jf} - g^{j'f} = g^i, \qquad 0 \leqq j, j' \leqq k - 1.$$

Let the number of solutions (i.e., number of pairs j, j' satisfying (4.9)) be N. We claim that if a is any other element of C_i, then the equation (in j, j')

$$\text{(4.10)} \qquad g^{jf} - g^{j'f} = a, \qquad 0 \leqq j, j' \leqq k - 1,$$

has N solutions also. This is easy to see. For, if $a \in C_i$, then $a = g^{i+sf}$ for some s, $1 \leqq s \leqq k - 1$. Therefore, if

$$\text{(4.11)} \qquad g^{j_t f} - g^{j_t' f} = g^i, \qquad t = 1, 2, \ldots, N,$$

then $g^{(j_t+s)f} - g^{(j_t'+s)f} = g^{i+sf} = a$, $t = 1, 2, \ldots, N$, where the exponents $(j_t + s)f$ and $(j_t' + s)f$ are to be reduced mod k as $g^{kf} = 1$. Hence, if an element of C_i can be represented in N_i ways as $g^{jf} - g^{j'f}$, $0 \leqq j, j' \leqq k - 1$, then every element of C_i can be represented in the above manner in N_i ways, $i = 0, 1, 2, \ldots, f - 1$.

Next, if $g^{jf} - g^{j'f} = a$, then $g^{j'f} - g^{jf} = -a$ and conversely. Hence, if a can be represented in N ways, $-a$ also can be represented in N ways as a difference of the elements of the first block. Recalling the fact that the elements $C_{f/2}$ are negatives of the elements of C_0 we infer that $N_0 = N_{f/2}$. In a similar fashion we get

$$(4.12) \qquad N_0 = N_{f/2}, N_1 = N_{f/2+1}, N_2 = N_{f/2+2}, \ldots, N_{f/2-1} = N_{f-1}.$$

Take a fixed t, $1 \leqq t \leqq f/2 - 1$ and a fixed i, $1 \leqq i \leqq f - 1$. If $i \geqq t$, consider the representation of the element g^i of C_i as a difference of two elements of the $(t + 1)$th block, viz., as

$$(4.13) \qquad g^{t+jf} - g^{t+j'f} = g^i, \qquad 0 \leqq j, j' \leqq k - 1.$$

Clearly, g^i has N_{i-t} representations. If, on the other hand, $i < t$, consider the representation of the element g^{i+f} of C_i as

$$(4.14) \qquad g^{t+jf} - g^{t+j'f} = g^{i+f}, \qquad 0 \leqq j, j' \leqq k - 1.$$

This element has N_{i+f-t} representations. So we can now state compactly that any element of C_i has N_e representations as a difference of the elements of the $(t + 1)$th block, where $0 \leqq e \leqq f - 1$ and $e \equiv (i - t) \pmod f$.

From the above we see that any element of C_0 has $N_0, N_{f-1}, N_{f-2}, \ldots, N_{f/2+1}$ representations as a difference of the elements of the 1st, 2nd, 3rd, ..., $(f/2)$th block in (4.8). Therefore, the element appears

$$N_0 + N_{f-1} + N_{f-2} + \ldots + N_{f/2+1}$$

times amongst the differences of the elements of the blocks in (4.8). (Of course, we do not take the difference of two elements belonging to two blocks.) Using (4.12), we deduce that this number is equal to $N_0 + N_1 + N_2 + \ldots + N_{f/2-1}$. Similarly, any element of C_1 appears $N_1 + N_0 + N_{f-1} + N_{f-2} + \ldots + N_{f/2+2}$ times amongst all the differences of elements of all blocks in (4.8). Again, this number, by (4.12), is equal to $N_0 + N_1 + N_2 + \ldots + N_{f/2-1}$. The same holds for every element of every coset C_i, $i = 0, 1, 2, \ldots, f - 1$.

Thus, each non-zero element of $GF(p^\delta)$ appears $\lambda = N_0 + N_1 + \ldots + N_{f/2-1}$ times amongst the differences of the elements of the blocks in (4.8). Clearly, $\lambda = fk(k - 1)/(2kf) = (k - 1)/2$. Therefore (4.8) is a difference system of strength $\lambda = k - 1$, block size k for the Galois field $GF(p^\delta)$.

The following theorem is useful when $p = 2$ or k is even.

THEOREM 5. *If $p^\delta = kf + 1$, g a primitive root of the Galois field $GF(p^\delta)$, then the f blocks*

$$(4.15) \qquad [g^i, g^{i+f}, g^{i+2f}, \ldots, g^{i+(k-1)f}], \qquad i = 0, 1, 2, \ldots, f - 1,$$

form an f-block difference system of strength $\lambda = k - 1$ and block size k, for $GF(p^\delta)$.

Proof. Using the notation of the previous proof we see, as before, that any element of C_i has N_e, $0 \leqq e \leqq f-1$, $e \equiv (i-t) \pmod f$, representations as a difference of two elements of the $(t+1)$th block, $0 \leqq t \leqq f-1$.

Any element of C_i (as before) appears $\lambda = N_i + N_{i-1} + N_{i-2} + \ldots + N_1 + N_0 + N_{f-1} + N_{f-2} + \ldots + N_{i+1}$ times amongst the differences of the elements of all blocks in (4.15). Thus, every non-zero element of $\mathrm{GF}(p^\delta)$ has $\lambda = N_0 + N_1 + \ldots + N_{f-1} = fk(\kappa - 1)/(kf) = \kappa - 1$ representations as a difference of the elements of all blocks in (4.15). Consequently, (4.15) is a difference system of strength $\lambda = \kappa - 1$ and block size k for $\mathrm{GF}(p^\delta)$.

THEOREM 6. *If $p^\delta = \kappa f + 1$, p an odd prime, κ odd, g a primitive root* $\mathrm{GF}(p^\delta)$, *then the $f/2$ blocks*

$$(4.16) \qquad [0, g^i, g^{i+f}, g^{i+2f}, \ldots, g^{i+(k-1)f}], \qquad i = 0, 1, 2, \ldots, f/2 - 1,$$

form an $(f/2)$-block difference system of strength $\lambda = (\kappa+1)/2$ and block size $k = \kappa + 1$ for $\mathrm{GF}(p^\delta)$.

Proof. Using the notation of Theorem 4, any element of C_0 has $N_0' = N_0 + 1$ representations as differences of pairs of elements of the first block while any element of C_i has $N_i' = N_i$ such representations, $i = 1, 2, \ldots, f-1$.

As before, we see that amongst all the differences arising from all the $f/2$ blocks, any element of C_i occurs $N_0' + N_i' + \ldots + N_{f/2-1}' = N_0 + N_1 + \ldots + N_{f/2-1} + 1$ times, $i = 0, 1, 2, , 3, \ldots, f-1$.

Again, as in Theorem 4, we conclude that (4.16) is the desired difference system with $\lambda = (f/2)\kappa(\kappa+1)/(\kappa f) = (\kappa+1)/2$.

Analogous to Theorem 5 we have the following theorem (proved as above). It is useful when p is even or k is even.

THEOREM 7. *If $p^\delta = \kappa f + 1$, p a prime, g a primitive root of* $\mathrm{GF}(p^\delta)$, *then the f blocks*

$$(4.17) \qquad [0, g^i, g^{i+f}, g^{i+2f}, g^{i+(\kappa-1)f}], \qquad i = 0, 1, 2, \ldots, f-1,$$

form an f-block difference system of strength $\lambda = \kappa + 1$ and block size $k = \kappa + 1$ for $\mathrm{GF}(p^\delta)$.

Theorems 4, 5, and 6 were obtained by Sprott (**8**) in a different way.

5. Applications of difference systems for the construction of b.i.b. designs. Using Theorem 1, we deduce from Theorems 2, 3, 4, 5, 6, and 7, respectively, Theorems 8, 9, 10, 11, 12, and 13.

THEOREM 8. *If $v = p_1^{\delta_1} p_2^{\delta_2} \ldots p_m^{\delta_m}$, the p's distinct primes, and if $k \leqq \min(p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_m^{\delta_m})$, then there exists a b.i.b. design (with no two blocks identical) having parameters $v, b = v(v-1), r = k(v-1), k, \lambda = k(k-1)$.*

THEOREM 9. *If v is odd and $v = p_1^{\delta_1} p_2^{\delta_2} \ldots p_m^{\delta_m}$, the p's distinct primes, and if $k \leqq \min(p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_m^{\delta_m})$, then there exists a b.i.b. design (with no two blocks identical) having parameters*

$$v, \quad b = v(v-1)/2, \quad r = (v-1)k/2, \quad k, \quad \lambda = k(k-1)/2.$$

THEOREM 10. *If $p^\delta = kf + 1$, p an odd prime, k odd, then there exists a b.i.b. design (with no two blocks identical) having parameters*

$$v = p^\delta, \quad b = p^\delta(p^\delta - 1)/2k, \quad r = (p^\delta - 1)/2, \quad k, \quad \lambda = (k-1)/2.$$

THEOREM 11. *If $p^\delta = kf + 1$, p a prime, then there exists a b.i.b. design (with no two blocks identical) having parameters*

$$v = p^\delta, \quad b = p^\delta(p^\delta - 1)/k, \quad r = p^\delta - 1, \quad k, \quad \lambda = k - 1.$$

THEOREM 12. *If $p^\delta = \kappa f + 1$, p an odd prime, κ odd, then there exists a b.i.b. design (with no two blocks identical) having parameters*

$$v = p^\delta, \quad b = p^\delta(p^\delta - 1)/(2\kappa), \quad r = (\kappa + 1)(p^\delta - 1)/(2\kappa),$$
$$k = \kappa + 1, \qquad \lambda = (\kappa + 1)/2.$$

THEOREM 13. *If $p^\delta = \kappa f + 1$, p any prime, f a positive integer, then there exists a b.i.b. design (with no two blocks identical) having parameters*

$$v = p^\delta, \quad b = p^\delta(p^\delta - 1)/\kappa, \quad r = (\kappa + 1)(p^\delta - 1)/\kappa, \quad k = \kappa + 1, \quad \lambda = \kappa + 1.$$

6. Method of partial replacement for certain b.i.b. designs. Suppose we have a b.i.b. design with parameters $v = 2k$, b, r, k, λ. Without loss of generality, we will henceforth assume that each of the first r blocks contains the first variety. The method of partial replacement consists of replacing all the k varieties of the ith block by the other $v - k = 2k - k = k$ varieties, $i = 1, 2, \ldots, r$, while the other $b - r$ blocks are kept unchanged. Notice that the first variety does not occur in the design so obtained, but all the other $v - 1$ varieties occur in it, each of its blocks containing k of these varieties.

THEOREM 14. *If there is a b.i.b. design with parameters*

$$v = 2k, \quad b, \quad r, \quad k, \quad \lambda, \tag{6.1}$$

then the method of partial replacement gives another b.i.b. design with parameters

$$v' = 2k - 1, \quad b' = b, \quad r' = 2(r-1), \quad k' = k, \quad \lambda' = r - \lambda. \tag{6.2}$$

Proof. Let A be the incidence matrix of the first b.i.b. design. Denote by $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_{v-1}$ the row vectors of the $v \times r$ submatrix of A, situated on its left part. Similarly, denote by $\beta, \beta_1, \beta_2, \ldots, \beta_{v-1}$ the row vector of the $v \times (b - r)$ submatrix of A situated on its right part. Note that $\alpha = (1\,1\,1\,1 \ldots 1)$, $\beta = (0\,0\,0 \ldots 0)$ and the sums of the elements of α_i and β_i are, respectively, $\alpha\alpha_i' = \lambda$ and $r - \lambda$. Each column sum of these submatrices is k.

If the $(v-1) \times b$ matrix B is the incidence matrix of the design obtained by the method of partial replacement then the ith row of B can be exhibited as

$$(\alpha - \alpha_i\beta_i). \tag{6.3}$$

The column sums of B are all k, row sums are $r' = r - \lambda + r - \lambda = 2(r-\lambda)$ and the scalar product of any two distinct row vectors is

$$\lambda' = (\alpha - \alpha_i)(\alpha - \alpha_j)' + \beta_i\beta_j' = r - \lambda.$$

Therefore, the b.i.b. design (6.2) exists, its incidence matrix being given by B.

As an application, we take the b.i.b. design

$$v = 2k, \quad b = 4k - 2, \quad r = 2k - 1, \quad k = k, \quad \lambda = k - 1,$$

which has been constructed by Bose (**3**; or **6**) for every k. From it, by the method of partial replacement, we get the b.i.b. design

$$v' = 2k - 1, \quad b' = 4k - 2, \quad r' = 2k, \quad k' = k, \quad \lambda = k.$$

This result was also discovered by Bhagwandas (**1**).

7. Derivation of b.i.b. designs from an affine resolvable b.i.b. design. Though affine resolvable b.i.b. designs were introduced by Bose more than twenty years ago and have been studied and used extensively since then, the following result managed to escape detection for all these years.

THEOREM 15. *If there exists an affine resolvable b.i.b. design with parameters* $v = nk$, $b = nr = v + r - 1$, $r = k + \lambda$, k, λ, *then there exists a b.i.b. design with parameters* $v' = r$, $b' = v - 1$, $r' = k - 1$, $k' = \lambda$, *and* $\lambda' = (k/n) - 1$.

Proof. We will use the matrix A (given in § 2) of the affine resolvable b.i.b. design. Consider the $v \times r$ submatrix A_1, consisting of the column vectors

$$\xi_1, \xi_{n+1}, \xi_{2n+1}, \xi_{3n+1}, \ldots, \xi_{(r-1)n+1}.$$

The first row of A_1 is $1, 1, 1, \ldots, 1, 1$. The scalar product of this row with any other row of A_1 is clearly λ. Therefore the row sums of A_1 are $r, \lambda, \lambda, \ldots, \lambda, \lambda$, respectively. The column sums of A_1 are all k and the scalar product of any two columns of A_1 is k/n, by Bose's intersection property.

Omitting the first row of A_1 we get a $(v - 1) \times r$ submatrix. Let the transpose of this new submatrix be denoted by B. Then B is a $(0 - 1)$-matrix of size $r \times (v - 1)$, with all column sums equal to λ, all row sums equal to $k - 1$ and with scalar product of any two of its rows equal to $(k/n) - 1$. Therefore, B gives an incidence matrix of a b.i.b. design with parameters

$$v' = r, \quad b' = v - 1, \quad r' = k - 1, \quad k' = \lambda, \quad \lambda' = (k/n) - 1.$$

We include the following known theorem (**5**) for the sake of completeness.

THEOREM 16. *If there exists an affine resolvable b.i.b. design with parameters* $v = nk$, $b = v + r - 1$, $r = k + \lambda$, k, λ, *then there exists a resolvable b.i.b. design with parameters* $v'' = k$, $b'' = v + r - 1 - n$, $r'' = r - 1 = k + \lambda - 1$, $k'' = k/n$, $\lambda'' = \lambda - 1$.

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Proof. The $k \times (b - n)$ submatrix situated at the upper right-hand part of the incidence matrix A of the affine resolvable b.i.b. design is easily shown to be an incidence matrix of the stated resolvable b.i.b. design.

Using these two theorems we can construct two b.i.b. designs from every existent affine resolvable b.i.b. design.

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