14

Deep-inelastic scattering

In this chapter we will show how the operator-product expansion can be used to compute the cross-section for deep-inelastic scattering. Since the calculation is fairly straightforward, this process is one of the classic tests of quantum chromodynamics (QCD).

The process is the scattering of a lepton of momentum l^μ on a hadron of momentum p^μ, with the only observed particle in the final state being the lepton:

$$l + N \to l' + \text{anything}. \tag{14.0.1}$$

In practice one uses a beam of electrons, muons, or neutrinos, and the hadrons in the target are nucleons. There are a number of cases for which there are experimental data, the cases being distinguished by the types of lepton involved:

$$\left.\begin{aligned} &e + N \to e + \text{anything},\\ &\mu + N \to \mu + \text{anything},\\ &\nu + N \to \nu + \text{anything},\\ &\nu + N \to (e \text{ or } \mu) + \text{anything}.\end{aligned}\right\} \tag{14.0.2}$$

The basic reason for measuring the cross-section for these processes is to study the fundamental constituents (quarks and gluons) of the hadron. Suppose we have a scattering of a lepton on a small-sized constituent, and that the momentum transfer is large, so that the scattering happens over a small time-scale. The weak and electromagnetic interactions have a small coupling, so to a good approximation the lepton does not interact again. Moreover the interactions that turn the final state, involving the struck quark, into hadrons happen on a much longer time-scale. So these interactions do not interfere with the basic Born graph. Hence, we should be able to calculate the cross-section for the process (14.0.1) rather simply in terms of the cross-section for lepton–quark scattering (Fig. 14.0.1). The approximation in which final-state interactions of the hadrons are ignored is called the impulse approximation. We can use it because we choose to sum the cross-section over all hadronic final states.

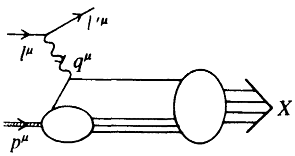

Fig. 14.0.1. Parton model for deep-inelastic scattering.

A more mathematical formulation of the same idea is the parton model, explained in Section 14.2. There are, however, many weak points in the intuitive discussion just given, and we must remedy them. In the remainder of this chapter we will compute the cross-section from the theory of strong interactions (QCD). The parton model will in fact give a qualitatively correct approximation to the real cross-section. Our intuitive argument shows why deep-inelastic scattering is simple enough to permit calculations.

Our treatment in this chapter is based on material to be found, among other places, in Gross (1976) and Treiman, Jackiw & Gross (1972).

14.1 Kinematics, etc.

We will compute the cross-section to lowest order in weak and electromagnetic interactions. Then the amplitudes contributing to the process (14.0.1) have the form of Fig. 14.1.1 where a boson is exchanged between the lepton and the hadron. The boson can be a photon, a W or a Z. At high enough energy, it is also necessary to include Higgs boson exchange. Higher-order weak and electromagnetic corrections do not need to be included, except for soft photon effects. We will ignore the soft photon corrections here, and will concentrate on understanding the strong-interaction corrections.

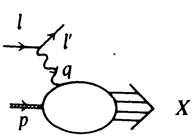

Fig. 14.1.1. Amplitude for process contributing to deep-inelastic scattering.

We first review the kinematics of the process. The two independent Lorentz invariants of the hadron system are chosen to be

$$\begin{aligned} Q^2 &= -q^\mu q_\mu, \\ \nu &= p\cdot q, \end{aligned} \tag{14.1.1}$$

where q^μ is the momentum transfer from the leptons. The mass of the final hadron state X is then

$$m_X^2 = (p+q)^2 = m_N^2 + 2\nu - Q^2. \tag{14.1.2}$$

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In the laboratory frame, where the initial hadron is at rest, we can express Q^2 and ν in terms of the initial and final lepton energies E and E', and of the lepton scattering angle θ:

$$Q^2 = 2EE'(1 - \cos\theta).$$
$$\nu = m_N(E - E'). \tag{14.1.3}$$

We have neglected lepton masses with respect to E and E'. The following inequalities hold

$$\nu \geq Q^2/2 \geq 0. \tag{14.1.4}$$

The region we will investigate is where both Q^2 and m_X^2 get large in a fixed ratio. We define the Bjorken scaling variable $x = Q^2/(2\nu)$. Then we let Q^2 get large with x fixed. This is called the Bjorken limit or the deep-inelastic region. In this region the missing mass m_X is large:

$$m_X^2 = Q^2(1/x - 1), \tag{14.1.5}$$

(where we neglect m_N^2 compared with Q^2). In order that m_X^2 be positive we must have $0 \leq x \leq 1$. The Bjorken limit applies only if x is not at its endpoints. An equivalent variable to x that is sometimes used is $\omega = 1/x$.

The cross-section is given by

$$E'\frac{\mathrm{d}\sigma}{\mathrm{d}^3p'} = \frac{1}{2\pi E'}\frac{\mathrm{d}\sigma}{dE'\mathrm{d}\cos\theta}$$
$$= \frac{\pi}{4ME}g_W^4\sum_X |\langle l'|j_\lambda^{\mathrm{lept}}|l\rangle\langle X|j_\nu^{\mathrm{had}}|N\rangle D^{\lambda\nu}(q)|^2\delta^{(4)}(p_X^\mu - p^\mu - q^\mu)$$
$$= \left(\frac{g_W^2}{4\pi}\right)^2\frac{1}{ME}D^{\kappa\mu}(q)^*D^{\lambda\nu}(q)L_{\kappa\lambda}(l,l')W_{\mu\nu}(p,q). \tag{14.1.6}$$

Here $j_\lambda^{\mathrm{lept}}$ and j_ν^{had} are the currents to which the exchanged vector boson couples, g_W is its coupling, and $D^{\lambda\nu}(q)$ is its propagator. The lepton tensor $L_{\kappa\lambda}$ is easily computed in the tree approximation for

$$L_{\kappa\lambda} = \langle l|j_\kappa^{\mathrm{lept}}(0)|l'\rangle\langle l'|j_\lambda^{\mathrm{lept}}(0)|l\rangle. \tag{14.1.7}$$

The hadron tensor is equal to

$$W_{\mu\nu}(p,q) = \frac{1}{4\pi}\sum_X\langle p|j_\mu(0)^\dagger|X\rangle\langle X|j_\nu(0)|p\rangle(2\pi)^4\delta^{(4)}(p_X^\mu - p^\mu - q^\mu)$$
$$= \frac{1}{4\pi}\int \mathrm{d}^4y\,e^{iq\cdot y}\langle p|j_\mu(y)^\dagger j_\nu(0)|p\rangle, \tag{14.1.8}$$

where the normalization is the standard convention.

Deep-inelastic scattering is the region where Q gets large with x fixed. This is not the short-distance limit we treated in Chapter 10. There we took all components of q^μ to infinity in a fixed ratio, so that $Q \to \infty$ with $Q/p\cdot q$ fixed,

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i.e., x is proportional to Q. This is not in the physical region $0 \le x \le 1$ for deep-inelastic scattering. Luckily we can use a dispersion relation (Christ, Hasslacher & Mueller (1972)) to relate the deep-inelastic limit of $W_{\mu\nu}$ to the short-distance limit of the time-ordered matrix element

$$T_{\mu\nu} = \int \mathrm{d}^4 y e^{iq\cdot y} \langle p | T j_\mu(y)^\dagger j_\nu(0) | 0 \rangle. \tag{14.1.9}$$

We will perform this analysis in Section 14.3.

Let us now decompose $W_{\mu\nu}$ into tensors with scalar coefficients. We will assume that the hadron is unpolarized, i.e., that we average over its spin states. Then the most general form of $W_{\mu\nu}$ is

$$\begin{aligned} W_{\mu\nu} = {} & W_1(-g_{\mu\nu} + q_\mu q_\nu/q^2) + W_2(p_\mu - q_\mu \nu/q^2)(p_\nu - q_\nu \nu/q^2) \\ & - \mathrm{i} W_3 \varepsilon_{\mu\nu\alpha\beta} p^\alpha q^\beta/(2M^2) + W_4 q_\mu q_\nu/M^2 \\ & + W_5(p_\mu q_\nu + q_\mu p_\nu)/(2M^2) - \mathrm{i} W_6(p_\mu q_\nu - q_\mu p_\nu)/(2M^2). \end{aligned} \tag{14.1.10}$$

The scalar coefficients $W_i(Q^2, \nu)$ are called the structure functions. The normalizations are such that they are dimensionless. A number of properties follow from symmetries and basic quantum mechanics (Treiman, Jackiw, & Gross (1972)):

(1) Each W_i is real.
(2) Time reversal invariance of strong interactions implies that $W_6 = 0$.

In the case of a purely electromagnetic process, the current j_μ^{had} is conserved, so that $q^\mu W_{\mu\nu} = 0$. Hence $W_4 = W_5 = 0$. Moreover the electromagnetic current is a pure vector, and strong interactions are parity invariant. So $W_3 = 0$.

In the case of neutrino scattering, the currents are only conserved if quark masses are zero. In that case the only non-zero structure functions are W_1, W_2 and W_3. It turns out that when the masses are non-zero the contributions of W_4 and W_5 to the cross-section are suppressed by a factor of order $m_l m_q/Q^2$, where m_l and m_q are lepton and quark masses. We will discuss this further in Section 14.8.

When we compute the structure functions in the Bjorken limit, $Q^2 \to \infty$, x fixed, we will find that they behave as a certain power of Q^2 times logarithms. The power is $(Q^2)^0$ for W_1, $1/Q^2$ for W_2 and W_3, and $1/Q^4$ for W_4 and W_5. So it is convenient to anticipate these results and define scaling structure functions $F_i(x, Q^2)$ which depend only logarithmically on Q^2. The standard definitions are:

$$\left.\begin{aligned} F_1(x, Q^2) &= W_1(Q^2, \nu = Q^2/(2x)), \\ F_i(x, Q^2) &= \nu W_i/M^2, \quad \text{for } i = 2 \text{ and } 3, \\ F_i(x, Q^2) &= \nu^2 W_i/M^4, \quad \text{for } i = 4 \text{ and } 5. \end{aligned}\right\} \tag{14.1.11}$$

14.2 Parton model

Before we discuss the true theoretical predictions for deep-inelastic scattering, let us discuss the parton model (Bjorken & Paschos (1969)). This is the simplest model for the process, and contains the essence of the correct physics. One considers the initial state hadron in the overall center-of-mass for the whole scattering. Now time dilation slows down processes in the hadron so that they typically occur on a rather long time-scale T of order Q/m_N^2. Then the scattering happens on a much shorter time-scale $T \sim 1/Q$. This implies that the hadron may be regarded as an assembly of non-interacting point-like constituents. These are what were originally (Feynman (1972)) called partons. (We now identify them as quarks and gluons.) Since the hadron in this frame is ultra-relativistic, we must regard the partons as massless and as each moving parallel to the hadron with a certain fraction z of its momentum.

The structure functions of the hadron are obtained by computing the partons' structure functions in tree approximation and by then summing over all partons weighted by their number density.

Let $f_{a/N}(z)\mathrm{d}z$ be the number of partons of type a in hadron N with fraction z to $z + \mathrm{d}z$ of its momentum. Then, for example, the electromagnetic structure functions are:

$$\begin{aligned} 2F_1^{\mathrm{em}}(x) &= x^{-1}F_2^{\mathrm{em}}(x) \\ &= \tfrac{4}{9}[f_{u/N}(x) + f_{\bar{u}/N}(x)] + \tfrac{1}{9}\sum_{q=d,s}[f_{q/N}(x) + f_{\bar{q}/N}(x)] \\ &\quad + \text{heavy quark terms.} \end{aligned} \tag{14.2.1}$$

The relation $F_2 = 2xF_1$ is characteristic of the spin-$\frac{1}{2}$ of the quarks (Callan & Gross (1969)). Notice that F_1 and F_2 are independent of Q^2. This is the property known as scaling. Experimentally, F_1 and F_2 obey this property approximately.

It is common in the literature to use the terms 'structure function' and 'quark distribution' interchangeably, because of the parton model relation between them (14.2.1), which is also approximately true in QCD – see Section 14.7. However, it is important to distinguish the two terms. Structure functions are coefficients of certain tensors in a current correlation function. They can be defined for other processes, e.g., the Drell–Yan process (Lam & Tung (1978)). On the other hand the quark distribution functions are exactly what their name implies: probability distributions of quarks in a hadron.

The parton model is correct in a super-renormalizable theory (Drell, Levy and Yan (1969, 1970a, b, c)). However, in a renormalizable theory like QCD, there are processes inside a hadron that happen significantly on all time scales down to zero. Then the basic assumption of the parton model does not hold. However, the fact that it is short times that cause the problems allows the operator-product expansion to come into play to solve the problem. There are two approaches, essentially equivalent for deep-inelastic scattering:

(1) Use a dispersion relation to show that moments of the structure functions (i.e.,

$$F_{i,N}(Q^2) = \int_0^1 \mathrm{d}x\, x^{N-1} F_i\,(x, Q^2)) \qquad (14.2.2)$$

can be directly computed by the Wilson expansion. This approach was initiated by Christ, Hasslacher & Mueller (1972), and it is the one we will use.

(2) One can generalize the derivation of the operator-product expansion (Amati, Petronzio & Veneziano (1978), Ellis *et al.* (1979), Libby & Sterman (1978), Stirling (1978)). For deep-inelastic scattering, the result is equivalent to the first method without the taking of moments.

14.3 Dispersion relations and moments

Consider the time-ordered Green's function $T_{\mu\nu}$ defined by (14.1.9). It can be expanded in scalar structure functions $T_1, T_2, T_3, T_4, T_5, T_6$, just like $W_{\mu\nu}$. We will only need T_1, T_2, and T_3. The operator-product expansion derived in Chapter 10 can be applied to $T_{\mu\nu}$ when Q^2 and ν get large with Q^2/ν^2 fixed. As we have seen, this is not the scaling region, for we have $x \to \infty$ instead of x fixed. However, we will relate $T_{\mu\nu}$ to $W_{\mu\nu}$ by a dispersion relation. Then we will see that information on $W_{\mu\nu}$ in the physical region can be obtained from the operator-product expansion for $T_{\mu\nu}$.

If Q^2 is fixed and positive then each T_i is analytic in the ν-plane. There are cuts going out to infinity from the thresholds $\nu = \pm Q^2/2$. See Fig. 14.3.1. (This is a standard property. It can be proved by expressing

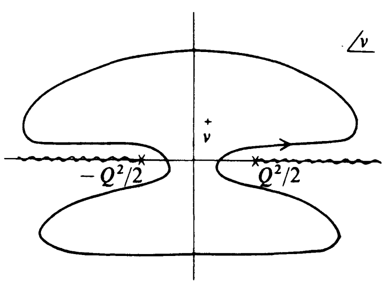

Fig. 14.3.1. Analyticity of $W_{\mu\nu}$ and contour to derive (14.3.1).

$\langle N|Tj_\mu(y)j_\nu(0)|N\rangle$ in terms of $W_{\mu\nu}$ and noting that $W_{\mu\nu}$ is zero if $|\nu| \langle Q^2/2$.) The discontinuities across the cut are:

$$\left.\begin{aligned} T_{\mu\nu}(p,q)\big|_{\nu-\mathrm{i}\varepsilon}^{\nu+\mathrm{i}\varepsilon} &= 4\pi W_{\mu\nu}(p,q), \quad (\text{if } \nu>0). \\ T_{\mu\nu}(p,q)\big|_{\nu+\mathrm{i}\varepsilon}^{\nu-\mathrm{i}\varepsilon} &= 4\pi \bar{W}_{\nu\mu}(p,q), \quad (\text{if } \nu<0), \end{aligned}\right\} \tag{14.3.1}$$

where $\bar{W}_{\mu\nu}$ is $W_{\mu\nu}$ with j replaced by its hermitian conjugate $j^\dagger$:

$$\bar{W}_{\mu\nu} = \frac{1}{4\pi}\int \mathrm{d}^4 y \mathrm{e}^{\mathrm{i}q\cdot y}\langle p|j_\mu(y)j_\nu(0)^\dagger|p\rangle. \tag{14.3.2}$$

For the electromagnetic or neutral-current processes the current is hermitian, so that $\bar{W}_{\mu\nu} = W_{\mu\nu}$. But if $W_{\mu\nu}$ is for charged-current neutrino scattering then $\bar{W}_{\mu\nu}$ gives the structure functions for antineutrino scattering.

By Cauchy's theorem we have

$$T_i(Q^2,\nu) = \frac{1}{2\pi\mathrm{i}}\int_C \frac{\mathrm{d}\nu'}{\nu'-\nu} T_i(Q^2,\nu'), \tag{14.3.3}$$

where C is any contour enclosing ν, as shown in Fig. 14.3.1. We will be able to compute the T_i's in the short-distance limit $\nu/Q^2 \to 0$. So suppose we expand $T_i(Q^2,\nu)$ in a power series in $1/x = 2\nu/Q^2$:

$$T_1 = \sum_{n=0}^{\infty} T_{1,n}(Q^2)x^{-n},$$

$$\nu T_i/M^2 = \sum_{n=0}^{\infty} T_{i,n}(Q^2)x^{-n} \quad (i=2 \text{ or } 3). \tag{14.3.4}$$

(We expand $\nu T_i/M^2$ (if i is 2 or 3) in analogy with (14.1.11).) Then from (14.3.3)

$$T_{i,n}(Q^2) = \frac{1}{2\pi\mathrm{i}}\int_C \frac{\mathrm{d}\nu'}{\nu'}\left(\frac{Q^2}{2\nu'}\right)^n T_i(Q^2,\nu')(\nu'/M^2)^{a_i}, \tag{14.3.5}$$

where $a_i = 0$ if $i=1$ and $a_i = 1$ if $i = 2$ or 3. If n is large enough to give convergence at $|\nu'| = \infty$, then we can deform the contour and pick up only the contribution from the discontinuity of T_i across the cuts:

$$\begin{aligned} T_{i,n}(Q^2) = {} & \frac{2}{\mathrm{i}}\int_{Q^2/2}^{\infty} \frac{\mathrm{d}\nu'}{\nu'}\left(\frac{Q^2}{2\nu'}\right)^n W_i(Q^2,\nu')\left(\frac{\nu'}{M^2}\right)^{a_i} \\ & + \frac{2}{\mathrm{i}}\int_{-\infty}^{-Q^2/2} \frac{\mathrm{d}\nu'}{\nu'}\left(\frac{Q^2}{2\nu'}\right)^n \bar{W}_i(Q^2,-\nu')\left(\frac{\nu'}{M^2}\right)^{a_i}. \end{aligned} \tag{14.3.6}$$

Finally we write the right-hand side in terms of the scaling functions $F_i(x,Q^2)$:

$$\begin{aligned} T_{i,n}(Q^2) &= -2\mathrm{i}\int_0^1 \mathrm{d}x' x'^{n-1}[F_i(x',Q^2) + (-1)^{n+a_i}\bar{F}_i(x',Q^2)] \\ &= -2\mathrm{i}[F_{i,n}(Q^2) + (-1)^{n+a_i}\bar{F}_i(Q^2)]. \end{aligned} \tag{14.3.7}$$

This is the dispersion relation referred to earlier. It relates the power series expansion of $T_{\mu\nu}$ about $\nu = 0$ to the moments of the F's, which are defined by

$$F_{i,N}(Q^2) = \int_0^1 \mathrm{d}x x^{N-1} F_i(x, Q^2). \tag{14.3.8}$$

The relation (14.3.7) only applies if the integral is convergent. For small enough values of n it diverges. If $\nu^{a_i} W_i$ behaves like ν^p as $\nu \to \infty$, then we have convergence only if n is greater than p. Now the limit $\nu \to \infty$ at fixed Q^2 is a Regge limit (elastic scattering of a virtual boson off a hadron at energy m_X^2). So there is a standard expectation (Treiman, Jackiw & Gross (1972)) that $p = 1$ for $i = 1$ or 3 and $p = 0$ for $i = 2$. This is equivalent to $F_1, F_2/x, F_3$ all behaving like $1/x$ as $x \to 0$. In the parton model this would correspond to a $1/x$ behavior for quark distributions, and is roughly what is measured experimentally.

Our theoretical predictions will give all the terms in the series expansions of the T_i's, e.g. (14.3.4). Those coefficients for which (14.3.7) does not apply will not have any direct implications for deep-inelastic cross-sections.

14.4 Expansion for scalar current

To explain without a slew of indices the method for computing moments of structure functions, let us first work out the case where j_μ is replaced by a hermitian *scalar* operator j. For example, j might be $ZZ_m \bar{q}_i q_i$, appropriate to the coupling of a scalar boson to a particular flavor i of quark. We have a single scalar structure function:

$$F(x, Q^2) = W(\nu, Q^2) = (1/4\pi) \int \mathrm{d}^4 y \mathrm{e}^{\mathrm{i}q \cdot y} \langle p | j(y) j(0) | p \rangle, \tag{14.4.1}$$

while the time-ordered function is:

$$T(\nu, Q^2) = \int \mathrm{d}^4 y \mathrm{e}^{\mathrm{i}q \cdot y} \langle p | T j(y) j(0) | p \rangle. \tag{14.4.2}$$

(Notice that we choose j to be a renormalized operator. As is the case for the ϕ^2 operator for a scalar field, the renormalization factor for $\bar{q}q$ is the same as the mass renormalization factor, so $[\bar{q}q] = ZZ_m \bar{q}q = Z_m \bar{q}_0 q_0$, where q_0 is the bare quark field.)

The dispersion-relation argument applied to the series

$$T(Q^2/(2x), Q^2) = \sum_{n=0}^{\infty} T_n(Q) x^{-n} \tag{14.4.3}$$

gives

$$T_n(Q) = -2\mathrm{i}[F_n(Q) + (-1)^n \bar{F}_n(Q)]. \tag{14.4.4}$$

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Since the current is hermitian, we have $F(x, Q) = \bar{F}(x, Q)$, so that

$$T_n(Q) = \begin{cases} -4iF_n(Q), & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{14.4.5}$$

Regge theory suggests $T \sim 1/x$ as $x \to 0$, so this equation is valid only if n is bigger than 1.

We now apply the operator-product expansion. The results of Chapter 10 apply in the limit $Q^2 \to \infty$ with ν^2/Q^2 fixed, i.e., with x = constant times Q. Now

$$T_n x^{-n} = Q^{-n}[T_n(2\nu/Q)^n], \tag{14.4.6}$$

so that it would appear that all but the $n = 0$ term are non-leading by a power of Q^2 and that we only have a reliable prediction for $n = 0$. But the relation (14.4.5) is not expected to hold unless $n > 1$.

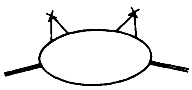

Fig. 14.4.1. $T(\nu, Q)$.

To remedy this problem, we must find an object for which $x^{-n}T_n$ contains the leading-power behavior as $Q \to \infty$. This is done by making a partial wave decomposition in the t-channel. That is, we treat $T(\nu, Q)$ (Fig. 14.4.1) as we would treat a scattering amplitude, and decompose it in terms of angular momenta:

$$\langle p|Tj(y)j(0)|0\rangle = \sum_{J=0}^{\infty} \langle p|V_J|p\rangle M_J(y\cdot q/q^2). \tag{14.4.7}$$

Here $\langle p|V_J|p\rangle$ is the reduced matrix element of some operator $V_{J,m}$ of spin J, and the M_J are appropriate polynomials in $y\cdot q/q^2$. (The operator V_J is not necessarily local.)

Now we perform an operator-product expansion of $Tj(y)j(0)$, keeping the leading-power behavior for each spin:

$$\langle p|Tj(y)j(0)|p\rangle \sim -2i\sum_{J,a}(-i)^J\langle p|\mathcal{O}^{(J,a)}_{\mu_1\ldots\mu_J}|p\rangle C^{Ja}(y^2)y^{\mu_1}\ldots y^{\mu_J}. \tag{14.4.8}$$

Here the operator $\mathcal{O}^{Ja}_{\mu_1\ldots\mu_J}$ is a local operator of spin J. The label 'a' denotes different operators of the same spin. Only the symmetric part of the operator is relevant, and in order that it be of definite spin, it must be traceless. Since

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the hadron is unpolarized the matrix element has the form

$$\langle p|\mathcal{O}^{Ja}_{\mu_1\ldots\mu_J}|p\rangle = \langle p|\mathcal{O}^{Ja}|p\rangle(p_{\mu_1}\ldots p_{\mu_J} - \text{traces}), \tag{14.4.9}$$

where $\langle p|\mathcal{O}^{Ja}|p\rangle$ is the reduced matrix element – a scalar quantity. The normalizations in (14.4.8) have been adjusted for later convenience.

The leading power of y as $y \to 0$ is obtained by using operators of the minimum possible dimension. In QCD these are

$$\left.\begin{aligned}\mathcal{O}^{Jq}_{\mu_1\ldots\mu_J} &= 2^{1-J}\bar{q}\gamma_{\mu_1}\mathrm{i}D_{\mu_2}\ldots\mathrm{i}D_{\mu_J}q, \text{ symmetrized minus traces,}\\ \mathcal{O}^{Jg}_{\mu_1\ldots\mu_J} &= -2^{3-J}G_{\mu_1\nu}\mathrm{i}D_{\mu_2}\ldots\mathrm{i}D_{\mu_{J-1}}G^{\nu}_{\mu_J}, \text{ symmetrized minus traces.}\end{aligned}\right\} \tag{14.4.10}$$

In accordance with the results of Section 12.8 we have kept only gauge-invariant operators. (We use D_μ to denote the covariant derivative, and q to denote the field of a quark of flavor q. Sums over color indices are implicit in (14.4.10). Only hermitian operators are needed.)

Our usual power-counting arguments imply that the behavior of the scalar coefficient $C^{Ja}(y^2)$ in (14.4.8) is

$$(y^2)^{-\dim(C)} \text{ times logarithms,}$$

with

$$\begin{aligned}\dim(C) &= 2\dim(j) - \dim(\mathcal{O}^{Ja}_{\mu_1\ldots\mu_J}) + J\\ &= 2\dim(j) - \dim[\langle p|\mathcal{O}^{Ja}|p\rangle/\langle p|p\rangle].\end{aligned} \tag{14.4.11}$$

The dimension minus the spin of an operator is evidently the important quantity here; it is called the twist of the operator. The leading twist is two, for the operators of (14.4.10), and for them $C(y) \sim y^{-4}$ modulo logarithms.

Fourier transformation of (14.4.8) now gives

$$T(\nu, Q^2) = -2\mathrm{i}\sum_{J,a}\langle p|\mathcal{O}^{Ja}|p\rangle\tilde{C}^{Ja}(Q)x^{-J} + \text{non-leading powers of } Q, \tag{14.4.12}$$

where we define

$$\tilde{C}^{Ja}(Q) = Q^{2J}(\partial/\partial Q^2)^J\int \mathrm{d}^4y\,\mathrm{e}^{\mathrm{i}q\cdot y}C^{Ja}(y^2). \tag{14.4.13}$$

Perturbative calculations of $T(\nu, Q^2)$ will give us $\tilde{C}^{Ja}$ in (14.4.13). We can then obtain moments of $F(x, Q)$ from the dispersion relation:

$$2F_J(Q) = \sum_a\langle p|\mathcal{O}^{Ja}|p\rangle\tilde{C}^{Ja}(Q)[1 + O(1/Q)], \tag{14.4.14}$$

if J is greater than one.

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14.5 Calculation of Wilson coefficients

There are two parts to the calculations. The first part is to do low-order calculations of the Wilson coefficients

$$\tilde{C}^{Ja}(g, Q/\mu),$$

where we have now explicitly indicated the dependence on all parameters. Higher-order corrections have logarithms of Q/μ, so the second part of the calculation is to compute the anomalous dimensions and then to do a renormalization-group transformation to set μ to be of order Q. Thus we write

$$2F_n(Q) = \sum_{a,a'} \langle p|\mathcal{O}^{Ja}|p\rangle_{(\mu)} \tilde{C}^{Ja'}(g(Q), 1) M_{a,a'}(g(Q), Q/\mu). \tag{14.5.1}$$

The subscript (μ) on the matrix element denotes renormalization with unit of mass μ. The matrix M is obtained by solving the renormalization group equation for C, and the $\tilde{C}^{Ja'}(g(Q), 1)$ is well approximated by its lowest-order term in perturbation theory. Measurements of deep-inelastic scattering at one value of Q are enough to give $\langle p|\mathcal{O}^{Ja}|p\rangle$, and then (14.5.1) predicts the moments of the structure functions at other values of Q.

14.5.1 *Lowest-order Wilson coefficients*

The Wilson coefficients are independent of the target, so we may calculate them with the hadron state replaced by a quark state. In tree approximation we have the expansion sketched in Fig. 14.5.1. The scalar 'current' j is the renormalized operator $ZZ_m\bar{q}_iq_i = [\bar{q}_iq_i]$, for a particular quark flavor i. Since the Wilson coefficients are independent of mass, we set quark masses to zero. Then from the graphs of Fig. 14.5.1 we find

$$\begin{aligned} T &= \frac{\mathrm{i}}{2}\frac{\mathrm{tr}\,[\not{p}(\not{p}+\not{q})]}{(p+q)^2} + \frac{\mathrm{i}}{2}\frac{\mathrm{tr}\,[\not{p}(\not{p}-\not{q})]}{(p-q)^2} \\ &= -2\mathrm{i}\frac{1/x^2}{1-1/x^2}. \end{aligned} \tag{14.5.2}$$

The factor $\frac{1}{2}$ comes from averaging over the spin of the quark. For a quark of any other flavor than i, or for a gluon, we have $T = 0$ to this order.

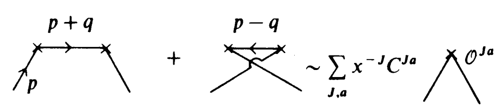

Fig. 14.5.1. Wilson expansion of $\langle 0|Tjj|0\rangle$ to lowest order.

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To compute the Wilson coefficients we also need the matrix elements of the operators defined by (14.4.10). In a quark state with flavor i we have

$$\begin{aligned}\langle pi|\mathcal{O}^{Ji'}_{\mu_1\ldots\mu_J}|pi\rangle &= \tfrac{1}{2}\delta_{ii'}\,\mathrm{tr}\,(\not p\gamma_{\mu_1}p_{\mu_2}\cdots p_{\mu_J}) \\ &= 2\delta_{ii'}p_{\mu_1}\cdots p_{\mu_J},\end{aligned} \tag{14.5.3}$$

in tree approximation, with the label i' denoting a quark flavor or a gluon. For a gluon state and gluon operator

$$\langle pg|O^{Jg}_{\mu_1\ldots\mu_J}|pg\rangle = 2p_{\mu_1}\cdots p_{\mu_J}. \tag{14.5.4}$$

All other matrix elements are zero. Hence the non-zero reduced matrix elements are all equal to 2.

Comparison of (14.5.2)–(14.5.4) with the operator-product expansion (14.4.12) shows that

$$\left.\begin{aligned}\tilde{C}^{Ji} &= 1 + O(g^2), \quad \text{if} \quad J \geq 1 \text{ and is even,} \\ \tilde{C}^{Ji'} &= O(g^2), \quad \text{if} \quad i' \neq i, \text{ or if } i' \text{ is a gluon.}\end{aligned}\right\} \tag{14.5.5}$$

14.5.2 *Anomalous dimensions*

The anomalous dimension of the operator $ZZ_m\bar{q}_iq_i$ is γ_m:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}(ZZ_m\bar{q}q) = \gamma_m(g)ZZ_m\bar{q}q, \tag{14.5.6}$$

and we let $\gamma^J_{aa'}(g)$ be the anomalous dimension matrix of the $\mathcal{O}^{Ja}$'s:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\mathcal{O}^{Ja} = \sum_{a'}\gamma_{aa'}(g)\mathcal{O}^{Ja'}. \tag{14.5.7}$$

(Operators of different spin do not mix.) As shown in Section 10.5 in a simpler case, the Wilson expansion then implies a renormalization-group equation for the Wilson coefficients:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\right]C^{Ja}(g, Q/\mu) = -\sum_{a'}\gamma_{a'a}C^{Ja'} + \gamma_mC^{Ja} \tag{14.5.8}$$

This equation would be trivial to solve were it not that it is a matrix equation.

The lowest-order counterterms for the operators are generated by the graphs of Fig. 14.5.2. It is evident that the different operators mix. To solve the RG equation we must diagonalize the anomalous dimension matrix at order g^2. The first step is to recall that the counterterms are independent of quark masses. So the renormalizations respect the $SU(n_{\mathrm{fl}})$ symmetry of the flavor space. Therefore let us now choose a new basis for the twist-2 operators.

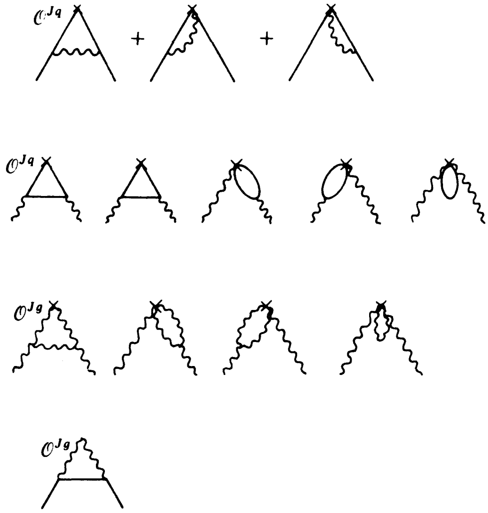

Fig. 14.5.2. Lowest-order divergences of the operators in the Wilson expansion of $\langle 0|Tjj|0\rangle$.

There is a multiplet of non-singlet operators:

$$\mathcal{O}^{J,NS,a} = 2^{1-J}\bar{\psi}\gamma_{\mu_1}\mathrm{i}\overleftrightarrow{D}_{\mu_2}\ldots\mathrm{i}\overleftrightarrow{D}_{\mu_J}\lambda^a\psi, \tag{14.5.9}$$

where the λ^a's are the $n^{\mathrm{fl}} \times n^{\mathrm{fl}}$ matrices that generate the flavor symmetry. There are two singlet operators:

$$\left.\begin{aligned} &\mathcal{O}^{Jg}, \text{ defined by (14.4.10),} \\ &\mathcal{O}^{JS} = \sum_{\mathrm{flav},i} \mathcal{O}^{Ji}. \end{aligned}\right\} \tag{14.5.10}$$

The renormalizations are

$$\begin{pmatrix} [\mathcal{O}^{Jg}] \\ [\mathcal{O}^{JS}] \\ [\mathcal{O}^{JNS}] \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} & 0 \\ Z_{21} & Z_{22} & 0 \\ 0 & 0 & Z_{NS} \end{pmatrix} \begin{pmatrix} \mathcal{O}_0^{Jg} \\ \mathcal{O}_0^{JS} \\ \mathcal{O}_0^{JNS} \end{pmatrix}, \tag{14.5.11}$$

$$[\mathcal{O}_\alpha^J] = \sum_\beta Z_{\alpha\beta}\mathcal{O}_{\beta,0}^J,$$

from which the anomalous dimension matrix

$$\gamma_{\alpha\beta} = \begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 \\ \gamma_{21} & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{NS} \end{pmatrix} \tag{14.5.12}$$

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is obtained by

$$\sum_{\beta} \gamma_{\alpha\beta} Z_{\beta\gamma} = \beta \frac{\partial}{\partial g} Z_{\alpha\gamma}. \tag{14.5.13}$$

A calculation of the divergences of the graphs of Fig. 14.5.2 gives (Gross (1976))

$$\begin{aligned}
\gamma^J_{22} = \gamma^J_{NS} &= \frac{g^2}{6\pi^2}\left[1 - \frac{2}{J(J+1)} + 4\sum_{k=2}^{J}\frac{1}{k}\right], \\
\gamma^J_{11} &= \frac{g^2}{8\pi^2}\left[1 - \frac{12}{J(J-1)} - \frac{12}{(J+1)(J+2)} + 12\sum_{2}^{J}\frac{1}{k} + \frac{2}{3}n_{f1}\right], \\
\gamma^J_{21} &= -\frac{g^2 n_{f1}}{4\pi^2}\frac{(J^2+J+2)}{J(J+1)(J+2)}, \\
\gamma^J_{12} &= -\frac{g^2}{3\pi^2}\frac{(J^2+J+2)}{J(J^2-1)}.
\end{aligned} \tag{14.5.14}$$

14.5.3 *Solution of RG equation – non-singlet*

The Wilson coefficients of the non-singlet operators evolve very simply:

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} C^J_{NS}(g(\mu), Q/\mu) = (-\gamma^J_{NS} + \gamma_m) C^J_{NS}. \tag{14.5.15}$$

An approximate solution can be found by taking the one-loop approximation for the anomalous dimensions. This gives

$$C^J_{NS}(g(\mu), Q/\mu) = C^J_{NS}(g(Q), 1)\left[\frac{\ln(Q/\Lambda)}{\ln(\mu/\Lambda)}\right]^{[\gamma^{(1)}_{NS,J} - \gamma^{(1)}_m]/(2A_1)} \tag{14.5.16}$$

where $\gamma^{(1)}_{NS,J}$ and $\gamma^{(1)}_m$ denote the coefficients of $g^2/4\pi^2$ in $\gamma^J_{NS}(g)$ and $\gamma_m(g)$, and $-A_1$ is the coefficient of $g^3/(4\pi^2)$ in $\beta(g)$. We may replace $C^{J,NS}(g(Q), 1)$ by its value in tree approximation. The accuracy of (14.5.16) may be systematically improved by taking more terms in the perturbation expansions of β, γ_{NS}, γ_m, and C.

The singlet coefficients may be obtained by diagonalizing the 2×2 matrix of anomalous dimensions. Then there are two linear combinations of singlet coefficients that have simple behavior like (14.5.16).

14.6 OPE for vector and axial currents

We will now apply the operator-product expansion to the structure functions of T for a weak or electromagnetic current. The argument is a simple generalization of the treatment in Sections 14.4 and 14.5 of deep-inelastic scattering with a scalar current. We have

$$
\begin{aligned}
T_{\mu\nu}(p,q) \sim & -2\mathrm{i}\int \mathrm{d}^4 y e^{\mathrm{i}q\cdot y} \sum_{J,a} \langle p|\mathcal{O}^{Ja}_{\mu_1 \ldots \mu_J}|p\rangle(-\mathrm{i})^J \times \\
& \times \left\{ y^{\mu_1} \ldots y^{\mu_J}\left(-g_{\mu\nu}\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial y^\mu \partial y^\nu}\right) C_1^{Ja}(y^2) \right. \\
& - y^{\mu_3} \ldots y^{\mu_J}\left(g_\mu^{\mu_1}\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial y^\mu \partial y_{\mu_1}}\right)\left(g_\nu^{\mu_2}\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial y^\nu \partial y_{\mu_2}}\right) C_2^{Ja}(y^2) \\
& \left. - \mathrm{i} y^{\mu_2} \ldots y^{\mu_J} \varepsilon_{\mu\nu}{}^{\mu_1 \beta}\frac{\partial}{\partial y^\beta} C_3^{Ja}(y^2) \right\}.
\end{aligned}
\tag{14.6.1}
$$

The derivatives acting on the C_i^{Ja}'s are arranged to be in combinations with zero divergence, to correspond to the condition $q^\mu T_{\mu\nu} = 0$. Fourier transformation as at (14.4.13), with suitable factors of Q^2, gives

$$
\begin{aligned}
T_{\mu\nu} \sim & -2\mathrm{i}\sum_{J,a} \langle p|\mathcal{O}^{Ja}|p\rangle \times \\
& \times \{(-g_{\mu\nu} + q_\mu q_\nu/q^2) x^{-J} \tilde{C}_1^{Ja}(Q) \\
& + (1/\nu)(p_\mu - q_\mu \nu/q^2)(p_\nu - q_\nu \nu/q^2) x^{1-J} \tilde{C}_2^{Ja}(Q) \\
& - \mathrm{i}\varepsilon_{\mu\nu\alpha\beta} p^\alpha q^\beta (1/2\nu) x^{-J} \tilde{C}_3^{Ja}(Q)\}.
\end{aligned}
\tag{14.6.2}
$$

The set of leading twist operators is the same as in Section 14.4, and we have arranged normalizations so that the $\tilde{C}_i$'s are dimensionless in leading twist.

Hence the moments of the structure functions satisfy

$$
\begin{aligned}
F_{1,J}(Q) + (-1)^J \bar{F}_{1,J}(Q) &= \sum_a \tilde{C}_1^{Ja}(Q)\langle p|\mathcal{O}^{Ja}|p\rangle + \text{correction}, \\
F_{2,J-1}(Q) + (-1)^J \bar{F}_{2,J-1}(Q) &= \sum_a \tilde{C}_2^{Ja}(Q)\langle p|\mathcal{O}^{Ja}|p\rangle + \text{correction}, \\
F_{3,J}(Q) + (-1)^{J+1} \bar{F}_{3,J}(Q) &= \sum_a \tilde{C}_3^{Ja}(Q)\langle p|\mathcal{O}^{Ja}|p\rangle + \text{correction}.
\end{aligned}
\tag{14.6.3}
$$

These equations are valid for $J > 1$, and the corrections are of order $1/Q^2$ times logarithms.

14.6.1 *Wilson coefficients – electromagnetic case*

To compute the lowest-order Wilson coefficient we consider deep-inelastic scattering on a quark target. The graphs are the same as in the case of a scalar current, Fig. 14.5.1, except that the current operator is now the elec-

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tromagnetic current. We find

$$T_{\mu\nu}^{(\text{lowest order})} = \tfrac{1}{2}e_q^2\left\{\mathrm{tr}\left[\not p\gamma_\mu\frac{\mathrm{i}(\not p+\not q)}{(p+q)^2}\gamma_\nu\right] + \mathrm{tr}\left[\not p\gamma_\nu\frac{\mathrm{i}(\not p-\not q)}{(p-q)^2}\gamma_\mu\right]\right\}$$

$$= 2\mathrm{i}e_q^2\left\{(-g_{\mu\nu} + q_\mu q_\nu/q^2)\frac{1/x^2}{1/x^2-1}\right.$$

$$\left. + (p_\mu - q_\mu\nu/q^2)(p_\nu - q_\nu\nu/q^2)\frac{1}{\nu}\frac{2/x}{1/x^2-1}\right\}. \quad (14.6.4)$$

Expanding about $1/x = 0$ gives

$$T_1 = -2\mathrm{i}e_q^2\sum_{n=0}^{\infty}(1/x)^{2n+2} + O(g^2),$$

$$\nu T_2/M^2 = -4\mathrm{i}e_q^2\sum_{n=0}^{\infty}(1/x)^{2n+1} + O(g^2). \quad (14.6.5)$$

The Wilson coefficients are therefore

$$\left.\begin{array}{l}\tilde C_1^{Jq} = e_q^2/2 + O(g^2)\\ \tilde C_2^{Jq} = e_q^2 + O(g^2)\end{array}\right\}\ \text{if } J \text{ is even and} \geq 2,$$

$$C_3 = 0,$$

$$\tilde C_i^{Jg} = 0 + O(g^2). \quad (14.6.6)$$

The relation $\tilde C_2 = 2\tilde C_1$ corresponds to the Callan–Gross relation $F_2 = 2xF_1$ in the parton model. Since the renormalization-group equation is the same for both $\tilde C_1^{Ja}$ and $\tilde C_2^{Ja}$, the Callan–Gross relation is true in QCD with corrections of $O(g(Q)^2)$. These corrections are from the Wilson coefficient, and have been calculated. See Buras (1981) for an up-to-date list of references.

The renormalization-group equations are the same as in the scalar case, except that γ_m is replaced by zero since the anomalous dimension of a conserved current is zero.

14.7 Parton interpretation of Wilson expansion

The use of moments in comparing theory and experiment is not very convenient, since cross-sections are needed outside the range in which they are measured. A more convenient form can be derived in which an expansion is obtained for the structure functions themselves. We will just summarize the results. More details can again be found in Buras (1981).

It is sufficient to examine the case of a scalar current for which the expansion is

$$F(x, Q) = \sum_a \int_x^1 \frac{\mathrm{d}z}{z} f_{a/N}(z, \mu) C_a(z/x; Q/\mu, g(\mu)). \quad (14.7.1)$$

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The sum is over all species of parton, i.e., flavors of quark and antiquark, and gluon. The generalized Wilson coefficient C_a is effectively a structure function for deep-inelastic scattering of a parton state, while $f_{a/N}(z, \mu)$ is a parton distribution. Lowest-order calculations reproduce the parton-model result, so that

$$C_q = C_{\bar{q}} = e_q{}^2 \delta(z/x - 1)/2,$$
$$C_g = 0. \tag{14.7.2}$$

The renormalization group equation for f has the form

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} f_{a/N}(x, \mu) = \sum_b \int_x^1 \frac{\mathrm{d}z}{z} \gamma_{ba}(z/x, g) f_{b/N}(z, \mu). \tag{14.7.3}$$

This is called the Altarelli–Parisi (1977) equation – it was first derived by these authors in leading logarithmic approximation from an heuristic argument. Later derivations (Collins & Soper (1982a), and Curci, Furmanski & Petronzio (1980)) are more complete.

Integro-differential equations like (14.7.3) are not particularly easy to work with. One mathematical simplification that can be made is to take moments, with the result that the operator-product expansion of Sections 14.5 and 14.6 is recovered. This will enable us to see that the two methods are essentially equivalent. However, the mathematically more complicated method using convolutions, as in (14.7.3), gives more physical insight, can be extended to other processes (Buras (1981)), and can be used without knowing structure functions at small x.

To see the equivalence of the two methods, let us define the following moments:

$$f^{(n)}_{a/N}(\mu) = \int_0^1 \mathrm{d}z\, z^{n-1} f_{a/N}(z, \mu),$$
$$C^{(n)}_a = \int_0^1 \mathrm{d}(x/z)(x/z)^{n-1} C_a(z, x; Q/\mu, g(\mu)),$$
$$\gamma^{(n)}_{ba} = \int_0^1 \mathrm{d}(x/z)(x/z)^{n-1} \gamma_{ba}(z, x; g(\mu)). \tag{14.7.4}$$

Then (14.7.1) implies

$$F_n(Q) = \sum_a f^{(n)}_{a/N}(\mu) C^{(n)}(g, Q/\mu). \tag{14.7.5}$$

This expansion has the same form as (14.4.14). In fact, moments of the f's are the same as the matrix elements of the twist-2 operators:

$$\langle N | \mathcal{O}^{Ja} | N \rangle = \int_0^1 \mathrm{d}x\, x^{J-1} [f_{a/N}(x) + f_{\bar{a}/N}(x)], \tag{14.7.6}$$

where there is a contribution from each type of parton and antiparton. This equation can be proved (Collins & Soper (1982a), and Curci, Furmanski & Petronzio (1980)) provided only that the same renormalization prescription (e.g., minimal subtraction) is used for both the operators and for the parton distributions.

14.8 W_4 and W_5

So far we have ignored F_4 and F_5. They are zero if the current j^μ in $W_{\mu\nu}$ is conserved. But the weak-interaction currents are not conserved if quark masses are non-zero:

$$\begin{aligned}\partial_\mu(\bar{\psi}\gamma^\mu(1-\gamma_5)\lambda^a\psi) &= \frac{\mathrm{i}}{2}\bar{\psi}([m,\lambda^a]-\gamma_5\{m,\lambda^a\})\psi \\ &\equiv \mathrm{i}D^a/2. \end{aligned}\tag{14.8.1}$$

Here m is the quark mass matrix. It follows that W_4 and W_5 are non-zero in neutrino scattering. However, we should regard the ratio m/Q as setting the scale for their effect on the cross-section (Jaffe & Llewellyn-Smith (1973) and Llewellyn-Smith (1972)). They therefore give a small contribution to the deep-inelastic cross-section. Since W_4 and W_5 are therefore non-leading in the Bjorken limit, they are difficult to compute directly.

A convenient technique to compute W_4 and W_5 is to consider

$$q^\mu W_{\mu\nu} = (q^2 W_4 + p\cdot q W_5/2)q_\nu/M^2 + q^2 p_\nu W_5/2M^2. \tag{14.8.2}$$

We have the operator formula

$$q^\mu W_{\mu\nu} = \int \mathrm{d}^4 y \mathrm{e}^{\mathrm{i}q\cdot y}\langle p|j_\nu(y)D(0)|p\rangle, \tag{14.8.3}$$

so we can compute W_4 and W_5 by making an operator-product expansion for $j_\mu(y)D(0)$. Since there is an explicit factor of quark mass m in the expression for D, this operator behaves as a dimension 3 rather than a dimension 4 operator. This suppresses the Wilson coefficients for W_4 and W_5 by a power of Q^2. Since, in (14.1.10), the tensors multiplying W_4 and W_5 have a q_μ or q_ν in them, a similar suppression occurs because the lepton masses are much less than Q, as is seen by contracting q_μ or q_ν with the lepton tensor (14.1.7). The result is that W_4 and W_5 make a negligible contribution to the cross-section at large Q.

A detailed treatment of W_4 and W_5 within QCD can be made but has not yet been published. It generalizes the results of Jaffe & Llewellyn-Smith (1973), who worked within the parton model.