Compositio Mathematica **123:** 243–272, 2000.

The Exterior Cube L-Function for GL(6)*

DAVID GINZBURG[1] and STEPHEN RALLIS[2]
[1] School of Mathematical Sciences, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel
[2] Department of Mathematics, The Ohio State University, Columbus, OH 43210, U.S.A.

(Received: 20 November 1997; in final form: 13 September 1999)

Abstract. We construct a Rankin Selberg integral to represent the exterior cube L function $L(\pi, \Lambda^3, s)$ of an automorphic cuspidal module π of $\mathrm{GL}_6(\mathbb{A}_F)$ (where F is a number field). We determine the poles of this L function and find period conditions for the special value $L(\pi, \Lambda^3, 1/2)$. We use the Siegal Weil formula. We also state an analogue of the Gross–Prasad conjecture concerning a criterion for the nonvanishing of $L(\pi, \Lambda^3, 1/2)$.

Mathematics Subject Classification (2000). 11F67.

Key words. Rankin-Selberg integral, poles of L functions, special value of L functions.

In this paper we study certain properties of the exterior cube L-function of GL_6. If F is a number field we let π be a cuspidal representation of $\mathrm{GL}_6(\mathbb{A}_F)$ with central character ω_π. We let χ be a character of $F^*\backslash\mathbb{A}_F^*$. Let Λ^3 denote the third fundamental representation of GL_6. The space of Λ^3 is a 20-dimensional vector space and let 1_{20} be the map of $\mathrm{GL}_1 \cong \mathbb{C}^*$ into $\mathrm{GL}_{20}(\mathbb{C})$ given by $\lambda \to \lambda \cdot I_{20}$. In this paper we give a Rankin Selberg integral which represents $L^S(\pi \otimes \chi, \Lambda^3 \otimes 1_{20}, s)$ – the twisted partial L function associated to the automorphic representation $\pi \otimes \chi$ of $\mathrm{GL}_6 \times \mathrm{GL}_1$ and the representation $\Lambda^3 \otimes 1_{20}$ of the L-group of $\mathrm{GL}_6 \times \mathrm{GL}_1$ (which is again $\mathrm{GL}_6 \times \mathrm{GL}_1$). Here S is a finite number set of places. To simplify notations we write $L^S(\pi, \Lambda^3 \otimes \chi, s)$ for $L^S(\pi \otimes \chi, \Lambda^3 \otimes 1_{20}, s)$.

Our global integral involves the Siegel Eisenstein series of GSp_6. Since the analytic properties of this Eisenstein series are basically the same as the Siegel Eisenstein series of Sp_6, we can apply the Siegel Weil formula as stated in [K-R]. Using [K-R] we are able to study the analytic properties of $L^S(\pi, \Lambda^3 \otimes \chi, s)$ and the behavior of this L-function at $s = 1/2$ (the center of symmetry of the functional equation). In Section 3 we first prove that $L^S(\pi, \Lambda^3 \otimes \chi, s)$ can have at most a simple pole at $s = 1$ (Theorem 3.2). Using [K-R] we can relate the existence of this pole to a certain period (Theorem 3.3). In Section 4 we study the value of $L^S(\pi, \Lambda^3 \otimes \chi, 1/2)$. When $\omega_\pi\chi^2 = 1$ we relate the nonvanishing of the partial L

*Both authors are supported by a grant from the U.S.A.-Israel Binational Science Foundation.

function with certain periods (Theorem 4.2). In fact, in Section 4 we make a precise conjecture concerning the conditions similar to the Gross Prasad conjecture when $L^S(\pi, \Lambda^3 \otimes \chi, 1/2)$ is nonvanishing.

To predict the criterion for the existence of a pole of $L^S(\pi, \Lambda^3 \otimes \chi, s)$ we consider more closely the representation of GL_6 in $\Lambda^3(\mathbb{C}^6)$. This is an irreducible GL_6 module. In fact there is a nondegenerate symplectic form ω on $\Lambda^3(\mathbb{C}^6)$ (obtained by the obvious pairing $\Lambda^3 \otimes \Lambda^3 \to \Lambda^6 \cong \mathbb{C}$) so that GL_6 embeds into $\mathrm{GSp}(\omega)$ =the similitude group of ω.

We heuristically consider a cuspidal automorphic representation of $\mathrm{GL}_6 \times \mathrm{GL}_1$ as being classified by a homomorphism ρ of the (conjectured) Langlands groups $\mathcal{L}_F$ into $\mathrm{GL}_6 \times \mathrm{GL}_1$. Then the L-function associated to ρ and the representation $\Lambda_3 \otimes 1_{20}$ should admit a pole at $s = 1$ if the image $= (\Lambda^3 \otimes 1_{20} \circ \rho)(\mathcal{L}_F)$ admits a fixed vector in Λ^3. In particular, this means that $(\Lambda^3 \otimes 1_{20}) \circ \rho(\mathcal{L}_F)$ is contained in the group

$$\Big[\Big((\mathrm{GL}_3 \times \mathrm{GL}_3) \ltimes Z_2\Big) \times \mathrm{GL}_1\Big]^0$$

which consists of all tuples $\Big((g_1, g_2, \epsilon), \lambda\Big)$ so that

$$\lambda \det g_1 = \lambda \det g_2 = 1 \quad \text{and} \quad \epsilon = \pm 1$$

and

$$\epsilon(g_1, g_2, 1)\epsilon = \begin{cases} (g_1, g_2, 1) & \text{if} \quad \epsilon\text{=}1, \\ (g_2, g_1, 1) & \text{if} \quad \epsilon\text{=-}1. \end{cases}$$

We note here that the above subgroup embeds into $\mathrm{GL}_6 \times \mathrm{GL}_1$ via the map

$$\Big((g_1, g_2, \epsilon), \lambda\Big) \rightsquigarrow \left(\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \cdot \epsilon' \ , \ \lambda\right),$$

with $\epsilon' = I_6$ or $\epsilon' = \left(\begin{smallmatrix} 0 & I_3 \\ I_3 & 0 \end{smallmatrix}\right)$ according to $\epsilon = +1$ or $\epsilon = -1$. Also the above subgroup in $\mathrm{GL}_6 \times \mathrm{GL}_1$ is the fixator of a generic vector in $\Lambda^3(\mathbb{C}^6)$. We note that the L group of $\mathrm{GL}_3(K)$ (relative to restriction of scalars where K/F is a degree 2 extension) is given by the semi-direct product $(\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})) \ltimes W_F$ with W_F, the Weil group of F (where W_F acts on $\mathrm{GL}_3 \times \mathrm{GL}_3$ by the map $W_F \to W_F/W_K \cong Z_2$ with Z_2 flipping the two coordinates in $\mathrm{GL}_3 \times \mathrm{GL}_3$). In fact let $i_K : {}^L\mathrm{GL}_3(K) \to \mathrm{GL}_6$ given by sending $(g_1, g_2) \to \left(\begin{smallmatrix} g_1 & \\ & g_2 \end{smallmatrix}\right)$ and $W_F \to Z_2$ as above. Thus we emphasize here that $L(\Lambda_3 \otimes 1_{20} \circ \rho, s)$ has a pole at $s = 1$ is equivalent to the fact that ρ factors through the subgroup $[\quad]^0$ above contained in $i_K({}^L\mathrm{GL}_3(K)) \times \mathrm{GL}_1$.

In this paper we give a period condition for the existence of a pole for $L^S(\pi, \Lambda^3 \otimes \chi, s)$ at $s = 1$. We require that $\omega_\pi \chi^2$ has order 2; thus we can determine a quadratic field K/F. The period condition in Theorem 3.3 is determined in terms of the quadratic field K as stated above.

We note here that if τ is an automorphic cuspidal representation of $\mathrm{GL}_3(\mathbb{A}_K)$ then if we apply the induction functor i_K to τ (defined above) we get by [A-C] that $i_K(\tau)$ is an automorphic representation of $\mathrm{GL}_6(\mathbb{A}_F)$. Thus we compute

$$L^S(i_K(\tau), \Lambda^3 \otimes \chi, s) = \xi_{K,S}(\omega_\tau \circ \chi_{K/F}, s) L^S(\Lambda^2(\tau) \otimes \tau^\sigma \otimes \chi_{K/F}, s),$$

where $\omega_\tau =$central character of τ, $\chi_{K/F} = \chi\circ \mathrm{Norm}_{K/F}, \tau^\sigma$ is the Galois twist of τ associated to σ in $\mathrm{Gal}(K/F)$, $L^S(\Lambda^2(\tau) \otimes \tau^\sigma \otimes \chi_{K/F}, s)$ is the partial Rankin product of $\Lambda^2(\tau) \otimes \tau^\sigma \cong \omega_\tau^{-1}\tau^\vee \otimes \tau^\sigma$ twisted by $\chi_{K/F}$ and ζ_K^S is the usual partial zeta function associated to the field K. Then we see that $L^S(i_K(\tau), \Lambda^3 \otimes \chi, s)$ admits a pole at $s = 1$ if and only if $\omega_\tau \circ \chi_{K/F} = 1$ or $\omega_\tau\chi_{K/F}\tau^\sigma \cong \tau$ (or $\tau^\sigma \cong (\omega_\tau\chi_{K/F})^{-1}\tau$). The last statement implies that $\omega_{\tau^\sigma} = \omega_\tau(\omega_\tau\chi_{K/F})^3$ or $(\omega_\tau\chi_{K/F})^3\big|_{\mathbb{A}_F^*} = 1$. Thus it follows that if δ is some automorphic character on $\mathbb{A}_K^*$ so that $\delta^3 = 1$ and $\delta = \omega_\tau\chi_{K/F}$ on $\mathbb{A}_F^*$ then for β an automorphic character on $\mathbb{A}_K^*$ satisfying $\frac{\beta/}{/\beta^\sigma} = \delta^{-1}\omega_\tau\chi_{K/F}$ we have that $(\tau \otimes \beta)^\sigma \cong \tau \otimes \beta$. Thus $i_K(\tau \otimes \beta)$ (and hence $i_K(\tau)$ itself) is an noncuspidal representation for $\mathrm{GL}_6(F)$ (i.e. equivalent to an Eisenstein series). Thus $L^S(i_K(\tau), \Lambda^3 \otimes \chi, s)$ admits a pole at $s = 1$ if and only $\omega_\tau \circ \chi_{K/F} = 1$

At this point we also note that the meromorphic properties of $L(\pi, \Lambda^3 \otimes \chi, s)$ can be determined by the Langlands Shahidi method. In contrast to the latter approach, the advantage of the method presented here is that we give this precise location of the possible poles and a period condition for the existence of the pole!

1. The Global Integral

Let F be a global field and $\mathbb{A}$ its ring of adèles. Let π be a cusp form on $\mathrm{GL}_6(\mathbb{A})$ with a central character ω_π. We know that π is generic. Namely, let N be the maximal standard unipotent subgroup of GL_6. Thus N consists of all upper unipotent matrices. Given a nontrivial additive character ψ of $F\backslash\mathbb{A}$ we define a character ψ_N of N by

$$\psi_N(n) = \psi(n_{12} + n_{23} + n_{34} - n_{45} + n_{56}),$$

where $n = (n_{ij}) \in N$. Thus to say that π is generic means that the space of functions generated by

$$W_\varphi(g) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\psi_N(n)\mathrm{d}n$$

is not identically zero. Here $\varphi \in \pi$ and $g \in \mathrm{GL}_6(\mathbb{A})$. We call the space of functions of the above form the Whittaker model of π and denote it by $\mathcal{W}(\pi, \psi)$.

Our construction uses the Siegel Eisenstein series on GSp_6 as constructed in [G]. To describe it let

$$\mathrm{GSp}_6 = \{g \in \mathrm{GL}_6 : {}^t gJg = \mu(g)J,\ \mu(g) \text{ a scalar}\},$$

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

where

$$J = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & -1 & & & \\ & -1 & & & & \\ -1 & & & & & \end{pmatrix}.$$

Let $Q = (\mathrm{GL}_1 \times \mathrm{GL}_3)R$ denote the Siegel parabolic on GSp_6. In terms of matrices we identify $\mathrm{GL}_1 \times \mathrm{GL}_3$ with

$$(\alpha, g) \longrightarrow \begin{pmatrix} \alpha g & \\ & g^* \end{pmatrix} \quad \alpha \in \mathrm{GL}_1 \ , \ g \in \mathrm{GL}_3,$$

where g^* is such that the above matrix is in GSp_6. R can be identified with

$$\left\{ \begin{pmatrix} I & Y \\ & I \end{pmatrix} : \quad Y \in M_3 \quad \text{and} \quad {}^tY \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} Y \right\}.$$

Define a character χ_π of $Q(\mathbb{A})$ as follows. Let χ be a unitary character of $F^* \backslash \mathbb{A}^*$. Define χ_π on $\mathrm{GL}_1(\mathbb{A}) \times \mathrm{GL}_3(\mathbb{A})$ as

$$\chi_\pi\big((\alpha, g)\big) = (\omega_\pi \chi^3)(\alpha)(\omega_\pi \chi^2)(\det g),$$

where $\alpha \in \mathrm{GL}_1(\mathbb{A})$ and $g \in \mathrm{GL}_3(\mathbb{A})$. We extend χ_π to $Q(\mathbb{A})$ by letting it act trivially on $R(\mathbb{A})$. Given $s \in \mathbb{C}$ set

$$I(s, \chi) = \mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{GSp}_6(\mathbb{A})} \delta_Q^s \chi_\pi,$$

where δ_Q is the modular function Q. Given $f_s \in I(s, \chi_\pi)$ we define the Siegel Eisenstein series as (at least for $\mathrm{Re}(s)$ large)

$$E(g, f_s, \chi, s) = \sum\nolimits_{\gamma \in Q(F) \backslash \mathrm{GSp}_6(F)} f_s(\gamma g) \quad g \in \mathrm{GSp}_6(\mathbb{A}) \ .$$

Denote by

$$w = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \in \mathrm{GSp}_6$$

and set $j(g) = wgw^{-1}$ for $g \in \mathrm{GSp}_6$. Our global integral is

$$I(\varphi, f_s, \chi, s) = \int_{Z(\mathbb{A})\mathrm{GSp}_6(F) \backslash \mathrm{GSp}_6(\mathbb{A})} \varphi\big(j(g)\big) E\big(g, f_s, \chi, s\big) \mathrm{d}g \ .$$

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

Here Z is the center of GSp_6 which is also the center of GL_6. Finally define

$$X(r) = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ && r & 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}.$$

We have:

PROPOSITION 1.1. *The integral* $I(\varphi, f_s, \chi, s)$ *converges absolutely for all* $s \in \mathbb{C}$ *except for those* s *for which the Eisenstein series has a pole. For* $\mathrm{Re}(s)$ *large we have*

$$I(\varphi, f_s, \chi, s) = \int_{Z(\mathbb{A})V(\mathbb{A})\backslash \mathrm{GSp}_6(\mathbb{A})} \int_{\mathbb{A}} W_\varphi\big(X(r)j(g)\big) f_s(g)\,\mathrm{d}r\,\mathrm{d}g.$$

where V *is the maximal unipotent of* GSp_6 *such that* $V \subset N$.

Proof. The standard argument using the cuspidality of φ allows us to assert that the integral

$$\int_{Z(\mathbb{A})\mathrm{GSp}_6(F)\backslash \mathrm{GSp}_6(\mathbb{A})} |\varphi(j(g))| E(g, s)|\mathrm{d}g$$

is finite provided $\mathrm{Re}(s)$ is sufficiently large. Then we can proceed with the standard unfolding of the Eisenstein series in the integral to obtain

$$I(\varphi, f_s, \chi, s) = \int_{Z(\mathbb{A})Q(F)\backslash \mathrm{GSp}_6(\mathbb{A})} \varphi\big(j(g)\big) f_s(g)\,\mathrm{d}g$$

$$= \int_{Z(\mathbb{A})\mathrm{GL}_1(F)R(\mathbb{A})\backslash \mathrm{GSp}_6(\mathbb{A})} \int_{R(F)\backslash R(\mathbb{A})} \varphi\big(j(rg)\big) f_s(g)\,\mathrm{d}r\,\mathrm{d}g.$$

Consider the Fourier expansion

$I(\varphi, f_s, \chi, s) =$

$$\int \sum_{\alpha,\beta,\gamma \in F} \int_{(F\backslash \mathbb{A})^3} \varphi\left[j\left(\left(\begin{pmatrix} 1 &&& 0 & 0 & 0 \\ & 1 && 0 & 0 & x_1 \\ && 1 & 0 & x_2 & x_3 \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} rg \right) \right) \right] \psi(\alpha x_1 + \beta x_2 + \gamma x_3) f_s(g)\,\mathrm{d}x_i\,\mathrm{d}r\,\mathrm{d}g,$$

where r and g are integrated as before.

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

We note here that by standard estimates given in [M-W] and [J-R] the series

$$(*) = \sum_{W \in M_{33}(k)} \left| \int_{M_{3,3}(k)\backslash M_{3,3}(\mathbb{A})} \varphi\left(j\left(\begin{bmatrix} I_3 & Z \\ 0 & I_3 \end{bmatrix} g\right)\right)\psi(\mathrm{tr}(ZW))\mathrm{d}Z \right|$$

is dominated by a finite sum of terms of the form

$$\sum_k c_k \left(\int_{M_{3,3}(k)\backslash M_{3,3}(\mathbb{A})} \left| (D_\infty^k * \varphi)\left[j\begin{pmatrix} I_3 & X \\ 0 & I_3 \end{pmatrix} g \right] \right|^2 \mathrm{d}x \right)^{1/2},$$

where D_∞ is some element in the enveloping algebra of G_R here (with each $c_k > 0$) depending only on the K_∞ types in φ. Thus using the fact that $D_\infty^k * \varphi$ remains cuspidal when φ is cuspidal we have that for any m positive integer

$$\left| (D_\infty^k * \varphi)(g) \right| \leqslant \sup\{1, \delta_Q(g)\}^{-m},$$

where δ_Q is the modular function associated to the parabolic Q.

Thus we deduce that the series $(*)$ is dominated by (for m sufficiently large) $\sup\{1, \delta_Q(g)\}^{-m}$ and in turn the integral

$$\int_{Z(\mathbb{A})Q(F)R(\mathbb{A})\backslash \mathrm{GSp}_6(\mathbb{A})} \sup\{1, \delta_Q(g)\}^{-m} |f_s(g)| \mathrm{d}g$$

is finite for $\mathrm{Re}(s)$ sufficiently large!

Thus we can replace φ by its Fourier expansion in the integral above.

The group $\mathrm{GL}_1 \times \mathrm{GL}_3$ acts on the root spaces x_1, x_2 and x_3 modulo elements in R, with two orbits. The trivial one contributes zero by cuspidality. Indeed, we obtain

$$\int_{M_3(F)\backslash M_3(\mathbb{A})} \varphi\left(j\left(\begin{pmatrix} I & X \\ & I \end{pmatrix}\right)\right)\mathrm{d}X$$

as an inner integral. The other orbit contributes

$$I(\varphi, f_s, \chi, s)$$

$$= \int_{Z(\mathbb{A})\mathrm{GL}_2(F)L(F)R(\mathbb{A})\backslash \mathrm{GSp}_6(\mathbb{A})} \int_{M_3(F)\backslash M_3(\mathbb{A})} \varphi\left(j\left(\begin{pmatrix} I & X \\ & I \end{pmatrix} g\right)\right)\psi_1(X) f_s(g)\mathrm{d}x\mathrm{d}g,$$

where ψ_1 is defined as follows. If

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix},$$

then $\psi_1(X) = \psi(x_4 - x_8)$. The stabilizer in $\mathrm{GL}_1 \times \mathrm{GL}_3$ of ψ_1 is $\mathrm{GL}_2 L$ which is

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

embedded in GSp_6 as

$$\begin{pmatrix} |g| & & & \\ & g & & \\ & & g^* & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & r_1 & r_2 & & & \\ & 1 & 0 & & & \\ & & 1 & & & \\ & & & 1 & 0 & -r_2 \\ & & & & 1 & -r_1 \\ & & & & & 1 \end{pmatrix}, \tag{1.1}$$

where $g \in \mathrm{GL}_2$ and $r_1, r_2 \in F$. Let L_1 be the unipotent subgroup of GL_6 consisting of the matrices $I_6 + m_1 e_{56} + m_2 e_{46}$ where $m_1, m_2 \in F$ and e_{ij}, is the 6×6 matrix with one at the (i, j) position and zero otherwise. In $I(\varphi, f_s, \chi, s)$ consider the Fourier expansion along $L_1(F)\backslash L_1(\mathbb{A})$ (we can substitute again the Fourier expansion by similar arguments as used above) The above $\mathrm{GL}_2(F)$ acts on the character group of $L_1(F)\backslash L_1(\mathbb{A})$ with two orbits. It is easy to see that the trivial one contributes zero to $I(\varphi, f_s, \chi, s)$. Let $N_1 = L \cdot L_1 \begin{pmatrix} I & X \\ & I \end{pmatrix}$ where $X \in M_3$. Define a character of N_1 as

$$\psi_{N_1}(n) = \psi\big(n_{12} + n_{24} - n_{35} + n_{56}\big),$$

where $n = (n_{ij})$. Notice that $\psi_{N_1}\begin{pmatrix} I & X \\ & I \end{pmatrix} = \psi_1(X)$. Thus

$$I(\varphi, f_s, \chi, s) = \int_{Z(\mathbb{A})\mathrm{GL}_1(F)L_2(F)V_1(\mathbb{A})\backslash \mathrm{GSp}_6(\mathbb{A})} \int_{N_1(F)\backslash N_1(\mathbb{A})} \varphi\big(j(ng)\big)\psi_{N_1}(n) f_s(g)\,\mathrm{d}n\mathrm{d}g\,.$$

Here $\mathrm{GL}_1 \cdot L_2$ is the stabilizer of ψ_{N_1} in $\mathrm{GL}_2 L$. Thus $\mathrm{GL}_1 \cdot L_2$ is embedded in GSp_6 as

$$\begin{pmatrix} \alpha & & & & & \\ & \alpha & & & & \\ & & 1 & & & \\ & & & \alpha & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & & & \\ & 1 & \beta & & & \\ & & 1 & & & \\ & & & 1 & -\beta & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \quad \alpha \in \mathrm{GL}_1 \quad \beta \in F\,. \tag{1.2}$$

Also V_1 is the unipotent subgroup of V defined by

$$V_1 = \begin{pmatrix} 1 & & & & & \\ & 1 & 0 & & * & \\ & & 1 & & & \\ & & & 1 & 0 & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix},$$

where $*$ indicates that the above matrix is an arbitrary GSp_6 matrix.

Let $N_2 = L_2 N_1$. Thus

$$I(\varphi, f_s, \chi, s) = \int_{Z(\mathbb{A})\mathrm{GL}_1(F)V(\mathbb{A})\backslash \mathrm{GSp}_6(\mathbb{A})} \int_{N_2(F)\backslash N_2(\mathbb{A})} \varphi\big(j(ng)\big)\psi_{N_2}(n) f_s(g)\,\mathrm{d}r\mathrm{d}g\,.$$

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

Here ψ_{N_2} is a character on N_2 trivially extended from ψ_{N_1}. In the above inner integral consider the Fourier expansion with respect to the unipotent group $j(I + me_{23})$. Thus

$$\int_{N_2(F)\backslash N_2(\mathbb{A})} \varphi\big(j(ng)\big)\psi_{N_2}(n)\mathrm{d}n$$

$$= \int_{N_2(F)\backslash N_2(\mathbb{A})} \sum_{\alpha\in F} \int_{F\backslash\mathbb{A}} \varphi\big(j(X_1(m)ng)\big)\psi_{N_2}(n)\psi(\alpha m)\mathrm{d}m\mathrm{d}n,$$

where $X_1(m) = I + me_{23}$. Let N_3 be the unipotent subgroup of N_2 for which $n_{34} = 0$. Thus

$$N_2 = N_3\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 & r && \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}.$$

Notice that

$$X(r) = j\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 & r && \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}.$$

Thus, using the left invariance property of φ under rational points, we obtain

$$\int_{N_3(F)\backslash N_3(\mathbb{A})} \sum_{\alpha\in F} \int_{(F\backslash\mathbb{A})^2} \varphi[X(\alpha)j(X_1(m)n)X(r)j(g)]\psi_{N_3}(n)\psi(\alpha m)\mathrm{d}r\mathrm{d}m\mathrm{d}n$$

One can check, using matrix multiplication that this equals (also a change of variables is needed)

$$\int_{N_3(F)\backslash N_3(\mathbb{A})} \sum_{\alpha\in F} \int_{(F\backslash\mathbb{A})^2} \varphi[j(X_1(m)n)X(\alpha + r)j(g)]\psi_{N_3}(n)\mathrm{d}r\mathrm{d}m\mathrm{d}n\ .$$

Collapsing the summation and integration over α an r we obtain

$$\int_{\mathbb{A}} \int_{N_4(F)\backslash N_4(\mathbb{A})} \varphi(j(n)X(r)j(g))\psi_{N_4}(n)\mathrm{d}n\mathrm{d}r,$$

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

where N_4 is the unipotent subgroup of N given by all matrices of the form

$$\begin{pmatrix} 1 & & & & & \\ & 1 & & & * & \\ & & 1 & 0 & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}.$$

Also ψ_{N_4} is extended trivially from ψ_{N_3}. Finally, we consider the Fourier expansion in $I(\varphi, f_s, \chi, s)$ with respect to $j(I + me_{43})$ with $m \in F\backslash\mathbb{A}$. Thus

$$I(\varphi, f_s, \chi, s) = \iint_{\mathbb{A}} \sum_{\alpha \in F} \int_{F\backslash\mathbb{A}} \int_{N_4(F)\backslash N_4(\mathbb{A})} \varphi\Big(j\Big(\big(I + me_{43}\big)n\Big)X(r)j(g)\Big) \\ \times \psi_{N_4}(n)\psi(\alpha m)f_s(g)\mathrm{d}n\mathrm{d}m\mathrm{d}r\mathrm{d}g,$$

where g is integrated as before. The GL_1 as defined in (1.2) acts on the group character of $j(I + me_{43})$ with two orbits. The trivial one contributes zero by cuspidality whereas the open one yields the identity we need to prove. □

Let $\pi = \otimes_v \pi_v$, $\chi = \otimes_v \chi_v$ and $I(s, \chi_\pi) = \otimes_v I_v(s, \chi_v)$ where the product is over all places of F. Also $\mathcal{W}(\pi, \psi) = \otimes_v \mathcal{W}(\pi_v, \psi_v)$. Let φ and f_s be factorizable vectors. Thus $\varphi = \otimes_v \varphi_v$ and $f_s = \otimes_v f_s^{(v)}$. It follows from Proposition 1.1. that

$$I(\varphi, f_s, \chi, s) = \prod_v I_v\big(W_v, f_s^{(v)}, \chi_v, s\big),$$

where

$$I_v\big(W_v, f_s^{(v)}, \chi_v, s\big) = \int_{Z(F_v)V(F_v)\backslash \mathrm{GSp}_6(F_v)} \int_{F_v} W_v(X(r)j(g))f_s^{(v)}(g)\mathrm{d}r\mathrm{d}g$$

and $W_\varphi = \otimes_v W_v$.

In the next section, we will study this local integral.

2. Some Local Theory

In this section, we will study the local integral obtained from the factorization of the global integral. We shall carry out the unramified computation and prove some nonvanishing result.

Let F be a local field. Let π be an admissible generic representation of $\mathrm{GL}_6(F)$ with central character ω_π. We shall write GL_6 for $\mathrm{GL}_6(F)$ etc. Let χ be a unitary character of F^*. As in the global case, we let $I(s, \chi_\pi) = \mathrm{Ind}_Q^{\mathrm{GSp}_6} \delta_Q^s \chi_\pi$. Thus $f_s \in I(s, \chi_\pi)$ is a smooth function which satisfies

$$f_s\Big((\alpha, g)rh\Big) = (\omega_\pi \chi^3)(\alpha)(\omega_\pi \chi^2)(\det g)\delta_Q^s((\alpha, g))f_s(h)$$

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

for all $(\alpha, g) \in \mathrm{GL}_1 \times \mathrm{GL}_3$, $r \in R$ and $h \in \mathrm{GSp}_6$. Given a reductive group G we let $K(G)$ denote its standard maximal compact subgroup. If F is a nonarchimedean field, $\mathcal{O}$ will denote the ring of integers in F and p a generator of the maximal ideal in $\mathcal{O}$. We let $q^{-1} = |p|$. Also, if μ is an unramified character of F^*, let $L(\mu, s) = (1 - \mu(p)q^{-s})^{-1}$.

Thus our aim in this section is to study the local integral

$$I(W, f_s, \chi, s) = \int_{ZV\backslash \mathrm{GSp}_6} \int_F W(X(r)j(g))f_s(g)\mathrm{d}r\mathrm{d}g$$

where $W \in \mathcal{W}(\pi, \psi)$ and $f_s \in I(s, \chi)$.

We start with:

2.1. THE UNRAMIFIED COMPUTATION

Let F be a nonarchimedean field. In this section, we assume all data to be unramified. Thus there exists a unique $W \in \mathcal{W}(\pi, \psi)$ such that $W(k) = W(e) = 1$ for all $k \in K(\mathrm{GL}_6)$ and similarly $f_s \in I(s, \chi)$ with $f_s(k) = f_s(e) = 1$ for all $k \in K(\mathrm{GSp}_6)$. Thus ω_π and χ are unramified characters.

From general theory, we may assume that $\pi = \mathrm{Ind}_B^{\mathrm{GL}_6} \delta_B^{1/2}\mu$ where B is the standard Borel subgroup of GL_6, i.e. the group of upper diagonal matrices. Also μ is defined as follows. There exists unramified characters μ_i of F^* such that

$$\mu(\mathrm{diag}(t_1, \ldots, t_6)n) = \prod_{i=1}^{6} \mu_i(t_i), \quad t_i \in F^*, \quad n \in N \ .$$

Thus we may attach to π a semi-simple conjugacy class t_π in $\mathrm{GL}_6(\mathbb{C})$ whose representative is chosen to be $\mathrm{diag}\big(\mu_1(p), \mu_2(p), \ldots, \mu_6(p)\big)$. Next we define the local L-function we shall study. Let Λ^3 denote the exterior cube representation of $\mathrm{GL}_6(\mathbb{C})$. This representation has dimension 20. Define the local twisted exterior cube L-function by

$$L\big(\pi \otimes \chi, \Lambda^3, s\big) = \det\big[I - \Lambda^3(t_\pi)\chi(p)q^{-s}\big]^{-1},$$

where I is the 20×20 identity matrix. We have

$$L\big(\pi \otimes \chi, \wedge^3, s\big) = \prod_{i<j<k} \big(1 - (\mu_i\mu_j\mu_k)(p)\chi(p)q^{-s}\big)^{-1} \ .$$

We have

PROPOSITION 2.1. *For all unramified data and for* $\mathrm{Re}(s)$ *large,*

$$I(W, f_s, \chi, s) = \frac{L(\pi \otimes \chi, \Lambda^3, 2s - 1/2)}{L(\omega_\pi\chi^2, 4s)L(\omega_\pi^2\chi^4, 8s - 2)} \ .$$

Proof. We start by writing the Iwasawa decomposition of GSp_6. Let

$$t' = \mathrm{diag}(abc, ac, 1, a, c^{-1}, b^{-1}c^{-1}) \quad a, b, c \in F^*$$

be a parameterization of the maximal torus in GSp_6. We have

$$\delta_{B'}(t') = |a^4 b^6 c^{10}|, \qquad \delta_P(t') = |a^2 b^4 c^8|,$$

where B' is the standard Borel subgroup of GSp_6. We have

$$I(W, f_s, \chi, s) = \int_{(F^*)^3} \int_F W\left[\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ && x & 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} j(t')\right] |a^2 b^4 c^8|^s$$

$$\chi(ab^2c^4)\omega_\pi(bc^2)|a^4b^6c^{10}|^{-1} \mathrm{d}x \mathrm{d}^*a \mathrm{d}^*b \mathrm{d}^*c$$

Here the measure on $K(\mathrm{GSp}_6)$ is chosen so that $\int_{K(\mathrm{GSp}_6)} \mathrm{d}k = 1$. Conjugating the torus to the left we obtain

$$I(W, f_s, \chi, s) = \int_{(F^*)^3} \int_F W\left[j(t')\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ && x & 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}\right] |a^2 b^4 c^8|^s$$

$$|a^5b^6c^{10}|^{-1}\chi(ab^2c^4)\omega_\pi(bc^2) \mathrm{d}x \mathrm{d}^*a \mathrm{d}^*b \mathrm{d}^*c$$

We have, for $|x| > 1$

$$\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ && x & 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && x^{-1} &&& \\ &&& x && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 & x && \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} k$$

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

with $k \in K(\mathrm{GSp}_6)$. Thus

$$I(W, f_s, \chi, s) = \int_{(F^*)^3} W\Big(j(t')\Big)\left|a^2b^4c^8\right|^s\left|a^5b^6c^{10}\right|^{-1}\chi(ab^2c^4)\omega_\pi(bc^2)d^*ad^*bd^*c+$$

$$+\int_{(F^*)^3}\int_{|x|>1} W\left(j(t')\begin{pmatrix}1&&&&&\\&1&&&&\\&&x^{-1}&&&\\&&&x&&\\&&&&1&\\&&&&&1\end{pmatrix}\begin{pmatrix}1&&&&&\\&1&&&&\\&&1&x&&\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}\right)$$

$$\times\left|a^2b^4c^8\right|^s\left|a^5a^6c^{10}\right|^{-1}.$$

$$\chi(ab^2c^4)\omega_\pi(bc^2)\mathrm{d}x\mathrm{d}^*a\mathrm{d}^*b\mathrm{d}^*c\ .$$

Here the measure on F is chosen so that $\int_{|x|\leqslant 1} dx = 1$. Denote

$$t = j(t') = \mathrm{diag}\left(abc, ac, a, 1, c^{-1}, b^{-1}c^{-1}\right).$$

Changing variables $a \to ax^2$ $c \to cx^{-1}$ we obtain

$$I(W, f_s, \chi, s) = \int_{(F^*)^3} W(t)\omega_\pi(bc^2)\chi(ab^2c^4)\left|a^2b^4c^8\right|^s\left|a^5b^6c^{10}\right|^{-1}H(a)\mathrm{d}^*a\mathrm{d}^*b\mathrm{d}^*c,$$

where

$$H(a) = 1 + \int_{|x|>1}\omega_\pi^{-1}\chi^{-2}|x|^{-4s}\psi(ax)\mathrm{d}x\ .$$

It follows as in [G] Proposition 3.1 that

$$H(a) = \frac{1-\omega_\pi\chi^2(p)q^{-4s}}{1-\omega_\pi\chi^2(p)q^{-4s+1}}\left(1-\omega_\pi\chi^2(a)|a|^{4s-1}\omega_\pi\chi^2(p)q^{-4s+1}\right).$$

Let $K(t) = \delta_B^{-1/2}W(t)$ where B is the standard Borel subgroup of GL_6. Thus $\delta_B(t) = \left|a^9b^{10}c^{16}\right|$. Denote

$$d(n_1, n_2, n_3) = \mathrm{diag}\left(p^{n_1+n_2+n_3}, p^{n_1+n_3}, p^{n_1}, 1, p^{-n_3}, p^{-n_2-n_3}\right).$$

Since $W(t) = 0$ if $|a| > 1$ or $|b| > 1$ or $|c| > 1$ we obtain

$$I(W, f_s, \chi, s)$$

$$= \frac{L\big(\omega_\pi\chi^2, 4s-1\big)}{L(\omega_\pi\chi^2, 4s)} \sum_{n_1,n_2,n_3=0}^{\infty} K(d(n_1, n_2, n_3))\chi(p)^{n_1+2n_2+4n_3}\omega_\pi(p)^{n_2+2n_3} \times$$

$$\times\, q^{(-2s+1/2)n_1+(-4s+1)n_2+(-8s+2)n_3}\big(1-(\omega_\pi\chi^2)(p)^{n_1+1}q^{(-4s+1)(n_1+1)}\big)\ .$$

Here we choose the measure on a, b and c so that $\int_{|\epsilon|=1} d\epsilon = 1$. Let $x = \chi(p)q^{-2s+1/2}$. Thus

$$I(W, f_s, \chi, s) = \frac{L(\omega_\pi\chi^2, 4s-1)}{L(\omega_\pi\chi^2, 4s)} \sum\nolimits_{n_1,n_2,n_3=0}^{\infty} K\big(d(n_1, n_2, n_3)\big) \times$$

$$\times\, \omega_\pi(p)^{n_2+2n_3}x^{n_1+2n_2+4n_3}\big(1-\omega_\pi(p)^{n_1+1}x^{2(n_1+1)}\big).$$

On the other hand, we have

$$L(\pi\otimes\chi, \Lambda^3, 2s-1/2) = \sum_{n=0}^{\infty} \mathrm{tr}S^n(t_\pi)\chi(p)^n q^{(-2s+1/2)n},$$

where S^n denotes the symmetric nth power operation. Thus we need to prove the identity

$$(1-\omega_\pi(p)x^2)(1-\omega_\pi^2(p)x^4)\sum_{n=0}^{\infty}\mathrm{tr}S^n(t_\pi)x^n$$

$$= \sum_{n_1,n_2,n_3=0}^{\infty} K\big(d(n_1, n_2, n_3)\big)\omega_\pi(p)^{n_2+2n_3}x^{n_1+2n_2+4n_3}\left(1-\omega_\pi(p)^{n_1+1}x^{2(n_1+1)}\right)\ .$$

Let $\widetilde{\omega}_i$ $1 \leqslant i \leqslant 5$ denote the ith fundamental representation of $\mathrm{GL}_6(\mathbb{C})$. Let $(0, \ldots, 1, \ldots, 0)$, one in the ith position and zero elsewhere, denote the character of the representation $\widetilde{\omega}_i$ evaluated at t_π. We use the Casselman-Shalika formula [C-S] to deduce that

$$K\big(d(n_1, n_2, n_3)\big) = (n_2, n_3, n_1, n_3, n_2)$$

Thus we need to prove

$$(1-\omega_\pi(p)x^2)(1-\omega_\pi^2(p)x^4)\sum_{n=0}^{\infty}\mathrm{tr}S^n(t_\pi)x^n$$

$$=\sum_{n_1,n_2,n_3=0}^{\infty}(n_2,n_3,n_1,n_3,n_2)\omega_\pi(p)^{n_2+2n_3}x^{n_1+2n_2+4n_3}\left(1-\omega_\pi(p)^{n_1+1}x^{2(n_1+1)}\right). \tag{$*$}$$

It follows from the result of Brion [B] page 13 that

$$\mathrm{tr}S^r(t_\pi)=\sum(n_2,n_3,n_1+n_4,n_3,n_2)\omega_\pi(p)^{n_2+2n_3+n_4+2n_5}$$

where the sum is over all $n_i\in N$, $1\leqslant i\leqslant 5$ satisfying $n_1+2n_2+3n_4+4n_3+4n_4=r$. Thus

$$\sum_{r=0}^{\infty}\mathrm{tr}S^r(t_\pi)x^r=\sum_{\substack{n_i=0\\1\leqslant i\leqslant 5}}^{\infty}(n_2,n_3,n_1+n_4,n_3,n_2)\omega_\pi(p)^{n_2+2n_3+n_4+2n_5}x^{n_1+2n_2+3n_4+4n_3+4n_4}.$$

At this point, we refer the reader to [G] formulas (3.4) and (3.5) and the discussion there. One can check that the same argument there applied to our case will prove (*). □

2.2. A NONVANISHING RESULT

In this section, we shall prove that data can be chosen so that $I(W,f_s,\chi,s)$ is nonzero at $s=s_0$. We prove:

PROPOSITION 2.2. *Let f_s be a $K(\mathrm{GSp}_6)$ standard section i.e. its restriction to $K(\mathrm{GSp}_6)$ is independent of s. Let W be a smooth vector in the Whittaker space of π. Then $I(W,f_s,\chi,s)$ converges absolutely for* Re(s) *large. If f_s is $K(\mathrm{GSp}_6)$ finite then $I(W,f_s,\chi,s)$ has a meromorphic continuation to the whole complex plane. Finally, given $s_0\in\mathbb{C}$, there is a choice of W and a $K(\mathrm{GSp}_6)$ finite section f_s so that $I(W,f_s,\chi,s)$ is nonzero at $s=s_0$.*

Proof. We note two facts concerning the smooth Whittaker vector W in π.

First each such W can be expressed as a convolution of the following form. Given W there exists a $K(\mathrm{GL}_6)$ finite function W_K and a function $f\in S(\mathrm{GL}_6)$ (the Schwartz space of GL_6) so that $W=\pi(f)(W_K)$. In concrete terms this means

$$W(g)=\int_{\mathrm{GL}_6}W_K(gx)f(x)\mathrm{d}x\,.$$

Thus if we convolve any $\varphi\in S(V)$ (the Schwartz space of a unipotent subgroup $V\subset\mathrm{GL}_6$) into f then $\varphi*f\in S(\mathrm{GL}_6)$. Thus $\pi(\varphi)(W)=\pi(\varphi*f)(W_K)$.

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

On the other hand, there is yet a more explicit way to present W (assuming π is unitary). If $g = utk$ is the Iwasawa decomposition of g in GL_6 then we get an asymptotic formula:

$$W(tk) = \sum \Phi_\chi(t, k)\chi(t),$$

where $\Phi_\chi \in S(\mathbb{R}^5 \times K(\mathrm{GL}_6))$ and χ a toral finite function on D (diagonal matrices) in GL_6. This formula is used to get estimates etc. in local Ranking Selberg integrals.

We start with the convergence. Using the Iwasawa decomposition, it is enough to prove that

$$\int_{(F^*)^3}\int_F \left| W\left[t \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ && x & 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} \right] \right| |a|^{2s-5}|b|^{4s-6}|c|^{8s-10}\mathrm{d}x\mathrm{d}^*a\mathrm{d}^*b\mathrm{d}^*c \tag{2.1}$$

converges for $\mathrm{Re}(s)$ large. Here

$$t = \mathrm{diag}(abc, ac, a, 1, c^{-1}, b^{-1}c^{-1}).$$

Set $\mu_s(a, b, c) = |a|^{2s-5}|b|^{4s-6}|c|^{8s-10}$. If F is nonarchimedean we break the x integration to $|x| \leqslant r$ and $|x| > r$ for large constant r. Since $|x| \leqslant r$ is a compact set the absolute convergence of

$$\int_{(F^*)^3}\int_{|x|\leqslant r} \left| W\left[t \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ && x & 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} \right] \right| \mu_s(a, b, c)\mathrm{d}x\mathrm{d}^*a\mathrm{d}^*b\mathrm{d}^*c$$

for $\mathrm{Re}(s)$ large follows from the asymptotic expansion of the Whittaker function given in [J-S] Section 4. When $|x| > r$ we get, after using the GL_2 Iwasawa decomposition

$$\begin{pmatrix} 1 & \\ y & 1 \end{pmatrix} = \begin{pmatrix} -y^{-1} & 1 \\ & y \end{pmatrix} k_y \quad |y| > 1\ ,\ k_y \in K(\mathrm{GL}_2),$$

the contribution

$$\int_{(F^*)^3}\int_{|x|>r} |W(t)|\mu_s(a, b, c)|x|^{-4s}\mathrm{d}x\mathrm{d}^*a\mathrm{d}^*b\mathrm{d}^*c,$$

which ones again converge for $\mathrm{Re}(s)$ large.

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

If F is archimedean we write

$$\begin{pmatrix} 1 & \\ y & 1 \end{pmatrix} = \begin{pmatrix} (1+y^2)^{-1/2} & * \\ & (1+y^2)^{1/2} \end{pmatrix} k'_y \quad y \neq 0 \quad k'_y \in K(\mathrm{GL}_2)\ .$$

Plugging this to (2.1) we get

$$\int_{(F^*)^3} \int_F |W(tk''_x)| \mu_s(a,b,c)(1+x^2)^{-2s} \mathrm{d}x \mathrm{d}^*a \mathrm{d}^*b \mathrm{d}^*c,$$

where $k''_x \in K(\mathrm{GL}_6)$. Once again due to the asymptotic formula of W given above, the integral converges for $\mathrm{Re}(s)$ large. We use here that the Φ_χ are bounded functions.

To study the meromorphic continuation we write

$$I(W, f_s, \chi, s) = \int_{ZV\backslash \mathrm{GSp}_6} \int_F (\rho(w)W) \left(w \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 & x && \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} g \right) f_s(g) \mathrm{d}x \mathrm{d}g,$$

where $\rho(w)W$ denotes the right translation of W by w. Let $\widetilde{W} = \rho(w)W$. Thus

$$I(W, f_s, \chi, s) = \int_{ZV\backslash \mathrm{GSp}_6} \widetilde{W}(g) \int_F f_s \left(w \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 & x && \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} g \right) \psi(x) \mathrm{d}x \mathrm{d}g. \tag{2.2}$$

We shall prove the meromorphic continuation of the right hand side of (2.2). Since $f_s(g)$ is $K(\mathrm{GSp}_6)$ finite it is enough to study the continuation of

$$\int_{(F^*)^3} \widetilde{W}(t)\mu(t) \int_F f_s \left(w \begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && 1 & x & \\ &&& 1 & \\ &&&& 1 \end{pmatrix} t \right) \psi(x) \mathrm{d}x \mathrm{d}^*t, \tag{2.3}$$

where $t = \mathrm{diag}(abc, ac, a, 1, c^{-1}, b^{-1}c^{-1})$ and $\mu(t) = |a|^{n_1}|b|^{n_2}|c|^{n_3}$, where $n_i \in \mathbb{Z}$.

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

Denote

$$W_s(g) = \int_F f_s\left(w\begin{pmatrix}1&&&&\\&1&&&\\&&1&x&\\&&&1&\\&&&&1\end{pmatrix}\begin{pmatrix}|g|&&&&\\&|g|&&&\\&&g&&\\&&&1&\\&&&&1\end{pmatrix}\right)\psi(x)\mathrm{d}x,$$

where $g \in \mathrm{GL}_2$ and $|g| = \det g$. Thus (2.3) equals

$$\int_{(F^*)^3} \widetilde{W}(t)\mu_s(a, b, c)W_s\begin{pmatrix}a&\\&1\end{pmatrix}\mathrm{d}t, \tag{2.4}$$

where $\mu_s(a, b, c)$ depends on the absolute value of a b, c to some power of s and also on χ and w_π. From the definition of W_s we may view it as the Whittaker model of a GL_2 induced representation.

Next we use the standard integral representation of W_s on GL_2 given by

$$W_s\begin{pmatrix}a&\\&1\end{pmatrix} = \left(\int \Phi(at, t^{-1})|t|^s d^*(t)\right)|a|^{s+1/2},$$

where Φ is a Schwartz function (in $S(k^2)$). Then we substitute this expression into (2.4) and then with the suitable change of coordinates and use of asymptotic expansion we express (2.4) in terms of Tate integrals (to obtain the continuation).

Finally, to finish the proof of Proposition 2.2. we need to show that given $s_0 \in \mathbb{C}$, data can be chosen so that $I(W, f_s, \chi, s)$ is nonzero at $s = s_0$. Define, for Re(s) large,

$$I_1(W, \chi, s, k) = \int_{Z(V\cap\mathrm{GL}_3)\backslash\mathrm{GL}_1\times\mathrm{GL}_3}\int_F W\Big(X(r)j\big((\alpha, g)\big)k\Big)\times$$
$$\times\, \omega_\pi\chi^3(\alpha)\omega_\pi\chi^2(\det g)|\alpha|^{6s-6}|\det g|^{4s-4}\mathrm{d}r\mathrm{d}^*\alpha\mathrm{d}g\ .$$

Here $k \in K(\mathrm{GSp}_6)$ and $(\alpha, g) \in \mathrm{GL}_1 \times \mathrm{GL}_3$. Thus for Re($s$) large,

$$I(W, f_s, \chi, s) = \int_{\mathrm{GL}_3\cap K(\mathrm{GSp}_6)\backslash K(\mathrm{GSp}_6)} I_1(W, \chi, s, k)f_s(k)\mathrm{d}k\ .$$

We note that $I_1(W, \chi, s, k)$ admits a continuation in s and such continuation in s as a function in k variable is locally constant (smooth in the archimedean case). In the nonarchimedean case this follows directly from the relation between $I_1(W, \chi, s, k)$ and $I(W, f_s, \chi, s)$. In the archimedean case this point is more subtle and we sketch a brief proof. Indeed given a smooth Whittaker function (not necessarily K finite) W (belonging to π) we can write by the Dixmier Malliavan criterion applied to the action of $K(\mathrm{GSp}_6)$ on π $W = \sum W_i * \varphi_i$, where W_i lies in the smooth

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

Whittaker spaces of π and $\varphi_i \in C^\infty(K(\mathrm{GSp}_6))$. Here

$$W_i * \varphi_i(g) = \int_{K(\mathrm{GSp}_6)} W_i(gk)\varphi(k)\mathrm{d}k \ .$$

Then we write

$$I_1(W, \chi, s, k_1) = \sum I_1(W_i * \varphi_i, \chi, s, k_1)$$

and we have that

$$\begin{aligned} &I_1(W_i * \varphi_i, \chi, s, k_1) \\ &\quad = \int_{Z(V\cap \mathrm{GL}_3)\backslash \mathrm{GL}_1\times \mathrm{GL}_3} \int_F \left(\int_{K(\mathrm{GSp}_6)} W(X(r)j((\alpha, g)k_1k)\varphi_i(k)\mathrm{d}k \right) \times \\ &\qquad \times \omega_\pi \chi^3(\alpha)\omega_\pi\chi^2(\det g)|\alpha|^{6s-6}|\det g|^{4s-4}\mathrm{d}r\mathrm{d}^*\alpha \mathrm{d}g \ . \end{aligned}$$

Then by changes of variables $k \to k_1 k$ and by the use of Fubini we deduce that the above integral equals

$$\int_{Z(V\cap \mathrm{GL}_3)\backslash \mathrm{GL}_1\times \mathrm{GL}_3} \int_F \left[\int_{K(\mathrm{GSp}_6)} W(X(r)j((\alpha, g))k) \left(\int_{K(\mathrm{GL}_1\times \mathrm{GL}_3)} \varphi_i(k_1^{-1}uk)du \right) dk \right] \omega_\pi\, \chi^3(\alpha) \times$$

$$\times\ \omega_\pi\chi^2(\det g)|\alpha|^{6s-6}|\det g|^{4s-4}\mathrm{d}r\mathrm{d}^*\alpha\mathrm{d}g$$

$$\begin{aligned} = \int_{K(\mathrm{GSp}_6)} &\left\{ \left[\int_{Z(V\cap \mathrm{GL}_3)\backslash \mathrm{GL}_1\times \mathrm{GL}_3} \int_F W(X(r)j((\alpha, g))k) \times \right.\right. \\ &\left.\left. \times\ \omega_\pi\chi^3(\alpha)\omega_\pi\chi^2(\det g)|\alpha|^{6s-6}|\det g|^{4s-4}\mathrm{d}r\mathrm{d}^*\alpha\mathrm{d}g \right] \left(\int_{K(\mathrm{GL}_1\times \mathrm{GL}_3)} \varphi_i(k_1^{-1}uk)du \right) \right\} \mathrm{d}k. \end{aligned}$$

Thus if we define

$$\mathbb{F}_{\varphi_i}(k_1, k) = \int_{K(\mathrm{GL}_1\times \mathrm{GL}_3)} \varphi_i(k_1^{-1}uk)\mathrm{d}u$$

and if we choose a C^∞ section for the induced GSp_6 module $I(s, \chi)$ given by

$$\mathbb{F}_{\varphi_i}(k_1, g, s) = \mathbb{F}_{\varphi_i}(k_1, k)(\delta_Q^s\chi_\pi)(\mathbb{A}_{\mathrm{GL}_1\times \mathrm{GL}_3}(g)) ,$$

where $A_{\mathrm{GL}_1\times\mathrm{GL}_3}(\quad)$ is the Levi component of g relative to the Iwasawa decomposition $g \in Q(\mathbb{R})K(\mathrm{GSp}_6)$. We note here $\mathbb{F}_{\varphi_i}$ is not necessarily a $K(\mathrm{GSp}_6)$ finite function.

Thus with the above calculations we have shown that

$$I_1(W, \chi, s, k_1) = \sum I(W_i, \mathbb{F}_{\varphi_i}(k_1, s), \chi, s) \ .$$

Thus we have shown here each integral in the sum above admits a continuation in s and as function in the k_1 variable it is C^∞. For this we just adapt the method of proof of continuation given above. We are now assuming W and $\mathbb{F}_{\varphi_i}$ are not $K(\mathrm{GSp}_6)$ finite data in the problem. Then following the same line of arguments as above (use here asymptotic expansion of W stated above) we reduce to an integral of the form

$$\int_{T\times K(\mathrm{GSp}_6)} \Phi_{\widetilde{\chi}}((b, c, a, c, b), k)\widetilde{\chi}(t)\widetilde{\mu}_s(t)\left(\int_F \mathbb{F}_{\varphi_i}\left(k_1, wn(x)\begin{pmatrix} a & \\ & 1\end{pmatrix}k, s\right)\psi(x)\mathrm{d}x\right)\mathrm{d}^*t\mathrm{d}k.$$

Here

$$n(x) = \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}$$

belong to the internal GL_2 as given in (2.3) and (2.4). Moreover $\widetilde{\mu}_s(t) = \widetilde{\mu}_s(a, b, c)$ depends on the absolute values of a, b and c to some power of s and also in χ and w_π. Also $\widetilde{\chi}$ is a T finite function in (a, b, c) variables ($\Phi_{\widetilde{\chi}}$ defined above in the asymptotic expansion of W). We note here by a similar argument as above we can find $\varphi_i \in S(K(\mathrm{GSp}_6) \times M_{6,3}(\mathbb{R}))$ so that

$$\mathbb{F}_{\varphi_i}(k_1, g, s) = |\det g|^{s_1} \int_{M_{3,3}(\mathbb{R})} \varphi_i\Big[k_1, [0 \mid X]g\Big]|\det X|^{s_2}\mathrm{d}X$$

for appropriate s_1 and s_2. Then we deduce that $(*)$ becomes an integral of the form

$$\left\{\int_{T\times K(\mathrm{GSp}_6)} \Phi_{\widetilde{\chi}}\Big[(b, c, a, c, b), k\Big]\widetilde{\chi}(t)\widetilde{\mu}_s(a, b, c)\right.$$
$$\left.\varphi\left[k_1, \left[\begin{array}{ccc|ccc} 0 & 0 & a_{11}a & a_{11}x & a_{12} & a_{13} \\ 0 & 0 & a_{21}a & a_{21}x & a_{22} & a_{23} \\ 0 & 0 & a_{31}a & a_{31}x & a_{32} & a_{33}\end{array}\right] k\right]|\det(A)|^{s_2}\,\psi(x)\mathrm{d}^*t\mathrm{d}x\mathrm{d}^*A\mathrm{d}k\right\}.$$

Then using appropriate differential operators in the a, b, c and A variables one checks that the integral has meromorphic continuation in s and in fact becomes for each s (more precisely its highest order term in s expanded about any point s_0) an C^∞ function in the variable k_1!

Assume that $I(W, f_s, \chi, s)$ is zero at $s = s_0$ for all choice of data. Since $f_s(k)$ is independent of s we obtain that $\int I_1(W, \chi, s, k)\sigma(k)\mathrm{d}k$ is zero at $s = s_0$ for all smooth functions σ on $(\mathrm{GL}_3 \cap K(\mathrm{GSp}_6))\backslash K(\mathrm{GSp}_6)$. Thus $I_1(W, \chi, s, k)$ is zero at $s = s_0$ for all

W. Put $k=e$. Thus the meromorphic continuation of

$$I_1(W,\chi,s)=\int_{ZV\cap \mathrm{GL}_3\backslash \mathrm{GL}_1\times \mathrm{GL}_3}\int_F W\Big(X(r)j\big((\alpha,g)\big)\Big)\mu_1(\alpha,g,s)\mathrm{d}r\mathrm{d}^*\alpha\mathrm{d}g$$

is zero at $s=s_0$ for all W. Here $\mu_1(\alpha,g,s)=\omega_\pi\chi^3(\alpha)\omega_\pi\chi^2(\det g)|\alpha|^{6s-6}|\det g|^{4s-4}$. Replace W in $I_1(W,\chi,s)$ by

$$W_1(h)=\int_{F^3} W\left(hj\left(\left(\begin{pmatrix}1&&&0&0&0\\&1&&0&0&r_1\\&&1&0&r_2&r_3\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}\right)\right)\right)\phi(r_1,r_2,r_3)\mathrm{d}r_i\,,$$

where ϕ is a smooth function of compact support on F^3 and $h\in \mathrm{GL}_6$. Thus

$I_1(W_1,\chi,s)=$

$$\int\int_{F^3} W\left(X(r)j\left((\alpha,g)\begin{pmatrix}1&&&0&0&0\\&1&&0&0&r_1\\&&1&0&r_2&r_3\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}\right)\right)\phi(r_1,r_2,r_3)\mu_1(\alpha,g,s)\mathrm{d}r_i\mathrm{d}^*\alpha\mathrm{d}g\,,$$

where α, g and r are integrated as before. Conjugating the upper unipotent matrix to the left, we obtain for $\mathrm{Re}(s)$ large

$$I_1(W_1,\chi,s)=\int W\Big(X(r)j(\alpha,g)\Big)\widehat{\phi}\Big(\alpha(1,0,0)g\Big)\mu_1(\alpha,g,s)\mathrm{d}r\mathrm{d}^*\alpha\mathrm{d}g\,,$$

where $\alpha(1,0,0)g$ indicates the usual matrix multiplication and

$$\widehat{\phi}(t_1,t_2,t_3)=\int_{F^3}\phi(r_1,r_2,r_3)\psi(r_1t_1+r_2t_2+r_st_3)\,\mathrm{d}r_i\ .$$

The function $\widehat{\phi}(\alpha(1,0,0)g)$ is an arbitrary smooth function on $\mathrm{GL}_2L\backslash \mathrm{GL}_1\times \mathrm{GL}_3$ where GL_2L is embedded in GL_6 as in (1.2). Thus arguing as before we get that the meromorphic continuation of

$$I_2(W,\chi,s)=\int_{(\mathrm{GL}_2\cap V)\backslash \mathrm{GL}_2}\int_F W\Big(X(r)j(g)\Big)\mu_2(g,s)\mathrm{d}r\mathrm{d}g$$

is zero at $s=s_0$ for all W. Here μ_2 is the restriction of μ_1 to GL_2. Next, replacing W

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

by

$$W_1(h) = \int_{F^2} W\left(hj\left(\left(\begin{pmatrix} 1 & r_1 & r_2 & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & -r_2 \\ & & & & 1 & -r_1 \\ & & & & & 1 \end{pmatrix}\right)\right)\right)\phi(r_1, r_2)\mathrm{d}r_i\,,$$

we obtain that the meromorphic continuation of

$$I_3(W, \chi, s) = \int_{F^*}\int_F W\left[X(r)\begin{pmatrix} a & & & & & \\ & a & & & & \\ & & a & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}\right]\mu_3(a, s)\mathrm{d}r\mathrm{d}^*a$$

is zero for all W (μ_3 the restriction of μ_1 to center (GL_3)). Finally, using the unipotent subgroup $I + me_{23}$ for $X(r)$ and $I + me_{34}$ for a we obtain, by arguing as above, that $W(e) = 0$ for all W. This is a contradiction. □

3. The Analytic Properties of the Partial L-Function

In this section we study the analytic properties of the partial exterior cube L-function on GL_6. Let $\pi = \otimes_v \pi_v$ and $I(s, \chi_\pi) = \otimes_v I_v(s, \chi_\pi)$. Let S be a finite set including the archimedean places such that outside of S all data is unramified. Given a character $\mu = \otimes \mu_v$ of $F^*\backslash A^*$ we denote $L^S(\mu, s) = \prod_{v\notin S} L_v(\mu_v, s)$ where $L_v(\mu_v, s)$ is the local degree one L-function of μ_v. As in [G] we set

$$E^*(g, f_s, \chi, s) = L_S(\omega_\pi\chi^2, 4s)L_S(\omega_\pi^2\chi^4, 8s-2)E(g, f_s, \chi, s)$$

and

$$I^*(\varphi, f_s, \chi, s) = L_S(\omega_\pi\chi^2, 4s)L_S(\omega_\pi^2\chi^4, 8s-2)I(\varphi, f_s, \chi, s)\ .$$

We have:

PROPOSITION 3.1. *Let f_s be a standard $K(\mathrm{GSp}_6)$ finite section which is an unramified outside of S. Then:*

(a) *If $\omega_\pi\chi^2 = 1$ or $\omega_\pi^2\chi^4 \neq 1$ then $I^*(\varphi, f_s, \chi, s)$ is entire.*
(b) *If $\omega_\pi^2\chi^4 = 1$ but $\omega_\pi\chi^2 \neq 1$ then $I^*(\varphi, f_s, \chi, s)$ can have at most a simple pole at $s = 1/4$ or $s = 3/4$.*

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

Proof. To prove this we use Lemmas 5.4 and 5.5 in [G]. We use the notations there. If $s = 1$ is a pole then the residue of $I^*(\varphi, f_s, \chi, s)$ at $s = 1$ is zero, since it follows from [J-R] that a cusp form on GL_6 integrated over GSp_6 is zero. If $\omega_\pi \chi^2 = 1$ then arguing as in [G] we have

$$\mathrm{Res}_{s=3/4}\, I^*(\varphi, f_s, \chi, s) = \int_{Z(\mathbb{A})\mathrm{GSp}_6(F)\backslash \mathrm{GSp}_6(\mathbb{A})} \varphi(j(g)) E(g, \widetilde{f}, s_1) \mathrm{d}g\,.$$

As in [G] Lemma 5.1 formula (5.2), we can show that the above integral is zero. □

Thus as in [G] Theorem 5.6 we have:

THEOREM 3.2. *Let π be a cusp form on $\mathrm{GL}_6(\mathbb{A})$. Let S be as above. Then*

$$L^S(\pi, \Lambda^3 \otimes \chi, s) = \prod_{v \notin S} L_v(\pi_v, \Lambda^3 \otimes \chi_v, s)$$

is entire unless $\omega_\pi^2 \chi^4 = 1$ and $\omega_\pi \chi^2 \neq 1$. In this case the L-function can have at most a simple pole at $s = 0$ or $s = 1$.

To study the residue at $s = 1$ of the partial L-function let $\mu = \omega_\pi \chi^2$. According to Theorem 3.2, $L^S(\pi, \Lambda^3 \otimes \chi, s)$ has a pole at $s = 1$ if $\mu \neq 1$ but $\mu^2 = 1$. Assume this is the case and suppose the partial L-function has a pole at $s = 1$. Then according to our global construction we deduce that there is $\varphi \in \pi$ and $f_s \in I(s, \chi)$ such that the residue at $s = 1$ of

$$\int_{Z(\mathbb{A})\mathrm{GSp}_6(F)\backslash \mathrm{GSp}_6(\mathbb{A})} \varphi(g) E(g, f_s, \chi, s) \mathrm{d}g$$

is nonzero. This implies that the residue at $s = 1$ of

$$\int_{\mathrm{Sp}_6(F)\backslash \mathrm{Sp}_6(\mathbb{A})} \varphi(g) E(g, f_s, \chi, s) \mathrm{d}g$$

is nonzero.

To study the residue of the Eisenstein series at $s = 1$ we apply Corollary 6.3 in [K-R]. Let $\theta_\phi(h)$ denote the theta function of $\widetilde{\mathrm{Sp}}_{12}(\mathbb{A})$. Here $\phi \in S(\mathbb{A}^6)$ the Schwartz space on $\mathbb{A}^6$.

We note here that we can by class field theory associate to μ a unique quadratic field F_μ/F. We let $O_2(\mu)$ be the orthogonal group associated to the norm form associated to F_μ. That is $O_2(\mu)(F)$ is an $\mathbb{Z}_2$ extension of the norm one elements of F_μ^*. In fact, the quotient $O_2(\mu)(F)\backslash O_2(\mu)(\mathbb{A})$ is a compact quotient (since the norm form of F_μ is a global anisotropic form). Thus the Siegel-Weil formula states ([K-R]) that

$$\mathrm{Res}_{s=1}\, E(g, f_s, \chi, s) = \int_{O_2(\mu)(F)\backslash O_2(\mu)(\mathbb{A})} \theta_\phi(g,h)\mathrm{d}h\ .$$

Thus we may conclude

THEOREM 3.3. *Suppose that $\omega_\pi\chi^2 \neq 1$ but $\omega_\pi^2\chi^4 = 1$. If $L^S(\pi, \Lambda^3 \otimes \chi, s)$ has a pole at $s = 1$ then there is a choice of data such that the integral*

$$\int_{\mathrm{Sp}_6(F)\backslash \mathrm{Sp}_6(\mathbb{A})} \int_{O_2(\mu)(F)\backslash O_2(\mu)(\mathbb{A})} \varphi(g)\theta_\phi(g,h)\mathrm{d}g\mathrm{d}h$$

is nonzero.

Remark. We note by the comments in the introduction that we expect that the automorphic modules π which have a pole at $s = 1$ probably come from automorphic induction for $\mathrm{GL}_3(F_\mu)$ into GL_6. We have not yet checked directly that such forms have in fact the required period (as given in Theorem 3.3) to be nonvanishing. One possible way to check this is by use of some version of a relative trace formula identity relating generic forms in GL_6 with the coperiod condition given in Theorem 3.3 with generic forms in $\mathrm{GL}_3(F_\mu)$. We also note here that if the finite set S (in Theorem 3.3) is enlarged to S' then the new partial L function $L^{S'}$ is multiplied by the inverse of the L-factor at a finite number of places $(S' - S)$. Thus it is possible that $L^{S'}$ may not have a pole (the extra finite places may cancel the pole of L_S by a local zero). However, it is expected that the local components π_v of π in $S' - S$ are tempered and thus the local factor $L_v(\pi_v, \Lambda^3 \otimes \chi_v, s)$ is holomorphic for $\mathrm{Re}(s) > 0$.

4. On The Nonvanishing of the Partial L-Function at $s = 1/2$

We keep the same notations as in Section 3. In the section we will relate the nonvanishing of $L^S(\pi, \Lambda^3 \otimes \chi, 1/2)$ with a nonvanishing of certain periods. As in Section 3 we shall apply the Siegel–Weil formula as stated in [K-R].

We shall assume that $\omega_\pi\chi^2 = 1$. Let $\theta_\phi(h)$ denote the theta function of $\widetilde{\mathrm{Sp}_{24}}$. In this case we have,

PROPOSITION 4.1 ([K-R] Theorem 4.10). *If $\mathrm{Val}_{s=1/2}E(g, f_s, \chi, s)$ is nonzero, then depending on the choice of f_s, it equals*

$$\int_{O_4(D)(F)\backslash O_4(D)(\mathbb{A})} \theta_\phi(g,h)\mathrm{d}h \qquad \text{or} \qquad \int_{O_{2,2}(F)\backslash O_{2,2}(\mathbb{A})} \theta_{\Delta\phi}(g,h)\mathrm{d}h$$

for some choice of $\phi \in S(\mathbb{A}^{12})$. Here $O_4(D)$ defines the orthogonal group in 4 variables associated to the norm form of a quaternion algebra D/F and $O_{2,2}$ is the split orthog-

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

onal form. Also $\Delta\phi$ is some regularization needed so that the above integral will converge.

Using Proposition 4.1 we may relate the nonvanishing of $L^S(\pi, \Lambda^3 \otimes \chi, 1/2)$ with the nonvanishing of certain periods.

We note here that if $\omega_\pi \chi^2 = 1$ then the L function $L(\pi, \Lambda^3 \otimes \chi, s)$ is a self symmetric L function. In particular this means $L(\pi, \Lambda^3 \otimes \chi, s) = L(\pi^\vee, \Lambda^3 \otimes \chi^{-1}, s)$ provided that $\omega_\pi \chi^2 = 1$. Then we have

THEOREM 4.2. *Suppose that $\omega_\pi \chi^2 = 1$. If $L^S(\pi, \Lambda^3 \otimes \chi, 1/2) \neq 0$ then there is a choice of data such that at least one of the following two periods is nonzero. Either*

$$\int_{\mathrm{Sp}_6(F)\backslash \mathrm{Sp}_6(\mathbb{A})} \int_{O_4(D)(F)\backslash O_4(D)(\mathbb{A})} \varphi(g)\theta_\phi(g, h)\mathrm{d}g\mathrm{d}h \tag{4.1}$$

or

$$\int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} \varphi\left(u\begin{pmatrix} g & & \\ & g & \\ & & g \end{pmatrix}\right)\psi_U(u)\mathrm{d}u\mathrm{d}g\,. \tag{4.2}$$

Here U is defined by

$$U = \left\{\begin{pmatrix} I & X & Y \\ & I & Z \\ & & I \end{pmatrix} : X, Y, Z \in M_2\right\},$$

where I is the 2×2 identity matrix and M_2 the group of all 2×2 matrices. Also ψ_U is defined as $\psi_U(u) = \psi(tr(X + Z))$.

Proof. It follows from Proposition 4.1 that if $L^S(\pi, \Lambda^3 \otimes \chi, 1/2) \neq 0$ then either (4.1) is nonzero for some choice of data or that

$$\int_{\mathrm{Sp}_6(F)\backslash \mathrm{Sp}_6(\mathbb{A})} \int_{O_{2,2}(F)\backslash O_{2,2}(\mathbb{A})} \varphi(g)\theta_{\Delta\phi}(g, h)\mathrm{d}g\mathrm{d}h \tag{4.3}$$

is nonzero for some choice of data. To prove the Theorem we have to show that if (4.2) is zero for all $\varphi \in \pi$ then (4.3) is zero for all choice of data. To do so we need to consider another Eisenstein series on Sp_6. Let Q be the maximal parabolic subgroup of Sp_6 whose Levi part is $\mathrm{GL}_2 \times \mathrm{SL}_2$. Consider the induced representation $I(s) = \mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{Sp}_6(\mathbb{A})} \delta_Q^s$ and for $F_s \in I(s)$ let $E_1(g, F_s)$ denote the corresponding Eisenstein series. It follows from [K-R] Lemma 5.5.6, that there is a point $s = s_0$ and a constant $c \neq 0$ such that

$$\int_{O_{2,2}(F)\backslash O_{2,2}(\mathbb{A})} \theta_{\Delta\phi}(g, h)\mathrm{d}h = c\mathrm{Val}_{s=s_0} E_1(g, F_s)\ .$$

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

To prove our Theorem, it is enough to show that if (4.2) is zero for all choice of data then the integral

$$\int_{\mathrm{Sp}_6(F)\backslash \mathrm{Sp}_6(\mathbb{A})} \varphi(g)E_1(g, F_s)\mathrm{d}g \tag{4.4}$$

is zero for all $\varphi \in \pi$, $F_s \in I(s)$ and $s \in \mathbb{C}$. Let V denote the unipotent radial of Q. In matrices

$$V = \left\{ \begin{pmatrix} I & X & Y \\ & I & X^* \\ & & I \end{pmatrix} : X, Y \in M_2 \right\}$$

and X^* and Y is such that the above matrix is in Sp_6. Unfolding (4.4) we obtain

$$\int_{\mathrm{GL}_2(F)\times \mathrm{SL}_2(F)V(\mathbb{A})\backslash \mathrm{Sp}_6(\mathbb{A})} \int_{V(F)\backslash V(\mathbb{A})} \varphi\left[\begin{pmatrix} I & X & Y \\ & I & X^* \\ & & I \end{pmatrix} g\right] F_s(g)\mathrm{d}v\mathrm{d}g.$$

The group $\mathrm{GL}_2 \times \mathrm{SL}_2$ is embedded in Sp_6 (and GL_6) as

$$(g, h) \to \begin{pmatrix} g & & \\ & h & \\ & & g^* \end{pmatrix} g \in \mathrm{GL}_2\ ,\ \ h \in \mathrm{SL}_2$$

and g^* is such that the above matrix is in Sp_6. Let $V_1 \supset V$ be the unipotent subgroup of GL_6 defined by

$$V_1 = \{ \begin{pmatrix} I & X & Y \\ & I & X^* \\ & & I \end{pmatrix}\ ,\ \ X, Y \in M_2 \}.$$

The group GL_2 as defined above acts on V_1/V by scalar of the determinant and hence the above integral equals

$$\begin{aligned} &\int_{\mathrm{GL}_2(F)\times \mathrm{SL}_2(F)V(\mathbb{A})\backslash \mathrm{Sp}_6(\mathbb{A})} \int_{V_1(F)\backslash V_1(\mathbb{A})} \varphi\left[\begin{pmatrix} I & X & Y \\ & I & X^* \\ & & I \end{pmatrix} g\right] F_s(g)\mathrm{d}v_1\mathrm{d}g + \\ &+ \int_{\mathrm{SL}_2(F)\times \mathrm{SL}_2(F)V(\mathbb{A})\backslash \mathrm{Sp}_6(\mathbb{A})} \int_{V_1(F)\backslash V_1(\mathbb{A})} \varphi\left[\begin{pmatrix} I & X & Y \\ & I & X^* \\ & & I \end{pmatrix} g\right] \widetilde{\psi}(Y)F_s(g)\mathrm{d}v_1\mathrm{d}g\,. \end{aligned} \tag{4.5}$$

In both cases $Y \in M_2$ and if $Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$ then $\widetilde{\psi}(Y) = \psi(y_1 - y_4)$. Consider the first summand in (4.5). It equals

$$\sum_v \int \int_{M_2(F)\backslash M_2(\mathbb{A})} \varphi\left[\begin{pmatrix} I & X+T & Y \\ & I & X^* \\ & & I \end{pmatrix} g\right] v(T) F_s(g) \mathrm{d}T \mathrm{d}v_1 \mathrm{d}g \,.$$

Here $T \in M_2$, v is summed over all characters of T and v_1 and g are integrated as before. The group $\mathrm{GL}_2(F) \times \mathrm{SL}_2(F)$ acts on the group characters of T with three orbit characterized by the rank of T. The contribution from the trivial orbit is zero since we obtain as an integral

$$\int_{M_2(F)\times M_2(F)\backslash M_2(\mathbb{A})\times M_2(\mathbb{A})} \varphi\left[\begin{pmatrix} I & T & Y \\ & I & 0 \\ & & I \end{pmatrix}\right] \mathrm{d}T \mathrm{d}Y \,.$$

By cuspidality of φ this is zero. For the rank one orbit we choose as representative the character $\psi_T\begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} = \psi(t_3)$. It is not hard to check that the unipotent group $\left\{\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix}\right\}$ $\subset \mathrm{SL}_2$ is in the stabilizer of ψ_T inside SL_2. We thus obtain

$$\int_{M_3(F)\backslash M_3(\mathbb{A})} \varphi\left[\begin{pmatrix} I & Z \\ & I \end{pmatrix}\right] \mathrm{d}Z$$

as an inner integral and hence, by cuspidality, we get zero contribution. Finally, for the rank two case, we choose $\psi_T\begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} = \psi(t_1 + t_4)$. The stabilizer in $\mathrm{GL}_2 \times \mathrm{SL}_2$ is SL_2^{Δ} i.e. the group SL_2 embedded diagonally. Hence the contribution to the first summand of (4.5) from this orbit is

$$\int_{\mathrm{SL}_2^{\Delta}(F)V(\mathbb{A})\backslash \mathrm{Sp}_6(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} \varphi\left[\begin{pmatrix} I & T & Y \\ & I & X \\ & & I \end{pmatrix} g\right] \psi(\mathrm{tr}T)\widetilde{\psi}(X) F_s(g) \mathrm{d}u \mathrm{d}g \,.$$

Thus we obtain as an inner integration

$$\int_{\mathrm{SL}_2^{\Delta}(F)\backslash \mathrm{SL}_2^{\Delta}(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} \varphi\left[\begin{pmatrix} I & T & Y \\ & I & X \\ & & I \end{pmatrix}\begin{pmatrix} h & & \\ & h & \\ & & h^* \end{pmatrix} g\right] \psi(\mathrm{tr}T)\widetilde{\psi}(X) \mathrm{d}u \mathrm{d}h \,.$$

Denote $\gamma = \mathrm{diag}\{1, 1, 1, 1, -1, 1\}$. Since, for $g \in \mathrm{SL}_2$, $g^* = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} g \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$, the above integral equals

$$\int \varphi\left[\begin{pmatrix} I & T & Y \\ & I & X \\ & & I \end{pmatrix} \gamma \begin{pmatrix} h & & \\ & h & \\ & & h \end{pmatrix} \gamma g\right] \psi(\mathrm{tr}T)\widetilde{\psi}(X) \mathrm{d}u \mathrm{d}h \,.$$

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press

Conjugating γ to the left and changing variables in X we obtain

$$\int \varphi\left[\begin{pmatrix} I & T & Y \\ & I & X \\ & & I \end{pmatrix}\begin{pmatrix} h & & \\ & h & \\ & & h \end{pmatrix}\gamma g\right]\psi(\mathrm{tr}(T+X))\mathrm{d}u\mathrm{d}h\,.$$

Thus we obtain (4.2) as an inner integral which is zero by our assumption.

Next consider the second summand of (4.5). Define

$$w = \begin{pmatrix} 0 & I & \\ I & 0 & \\ & & I \end{pmatrix}.$$

Conjugating by w the integral equals

$$\int_{\mathrm{SL}_2(F)\times\mathrm{SL}_2(F)V(\mathbb{A})\backslash\mathrm{Sp}_6(\mathbb{A})}\int_{V_1(F)\backslash V_1(\mathbb{A})}\varphi\left[\begin{pmatrix} I & 0 & X^* \\ & I & Y \\ & & I \end{pmatrix}\begin{pmatrix} I & & \\ X & I & \\ & & I \end{pmatrix}wg\right]\widetilde{\psi}(Y)F_s(g)\mathrm{d}v_1\mathrm{d}g\,.$$

Consider the Fourier expansion

$$\sum_{v}\int\int_{M_2(F)\backslash M_2(\mathbb{A})}\varphi\left[\begin{pmatrix} I & 0 & X^*+T \\ & I & Y \\ & & I \end{pmatrix}\begin{pmatrix} I & & \\ X & I & \\ & & I \end{pmatrix}wg\right]\psi(tr\,Y)v(T)F_s(g)\mathrm{d}T\mathrm{d}v_1\mathrm{d}g\,,$$

where v over all characters of $M_2(F)\backslash M_2(\mathbb{A})$ and v_1 and g are integrated as before. Given v we can find $L \in M_2(F)$ such that the above integral equals

$$\sum_{v}\int\int_{M_2(F)\backslash M_2(\mathbb{A})}\varphi\left[\begin{pmatrix} I & 0 & T \\ & I & Y \\ & & I \end{pmatrix}\begin{pmatrix} I & & \\ X+L & I & \\ & & I \end{pmatrix}wg\right]\widetilde{\psi}(Y)F_s(g)\mathrm{d}T\mathrm{d}X\mathrm{d}Y\mathrm{d}g\,.$$

Here we used the fact that φ is left invariant under rational points and also need a suitable change of variables. (For similar computations see in Section one the discussion involving $X(\alpha)$). Thus to prove our Theorem, it is enough to show that

$$\int\int_{M_2(F)\times M_2(F)\backslash M_2(\mathbb{A})\times M_2(\mathbb{A})}\varphi\left[\begin{pmatrix} I & & T \\ & I & Y \\ & & I \end{pmatrix}\begin{pmatrix} h & & \\ & g & \\ & & g^* \end{pmatrix}\right]\widetilde{\psi}(Y)\mathrm{d}T\mathrm{d}Y\mathrm{d}g\mathrm{d}h$$

is zero for all $\varphi \in \pi$. Here g and h are integrated over $\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})$. Applying a Fourier expansion to the above integral it equals

$$\sum_{v}\int\int_{U(F)\backslash U(\mathbb{A})}\varphi\left[\begin{pmatrix} I & Z & T \\ & I & Y \\ & & I \end{pmatrix}\begin{pmatrix} h & & \\ & g & \\ & & g^* \end{pmatrix}\right]\widetilde{\psi}(Y)v(Z)\mathrm{d}Z\mathrm{d}T\mathrm{d}Y\mathrm{d}h\mathrm{d}g.$$

As before, the group $\mathrm{SL}_2 \times \mathrm{SL}_2$ acts on the group character of Z. It is not hard to check that all orbits corresponding to the rank zero and rank one orbits contributes zero by cuspidality. As for the rank two, under the action of $SL_2(F) \times SL_2(F)$, there are infinite number of orbits. We can parametrize them by the characters

$$\psi_\alpha(Z) = \psi_\alpha\left(\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}\right) = \psi(\alpha z_1 + z_4)$$

with $\alpha \in F^*$. Thus the above equals

$$\sum_{\alpha\in F^*} \int_{S_\alpha(F)\backslash SL_2(\mathbb{A})\times SL_2(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} \varphi\left[\begin{pmatrix} I & Z & T \\ & I & Y \\ & & I \end{pmatrix}\begin{pmatrix} h & & \\ & g & \\ & & g^* \end{pmatrix}\right]\widetilde{\psi}(Y)\psi_\alpha(Z)\mathrm{d}u\mathrm{d}h\mathrm{d}g.$$

Here S_α is the stabilizer of ψ_α in $SL_2(F) \times SL_2(F)$. Thus

$$S_\alpha = \left(\begin{pmatrix} \alpha^{-1} & \\ & 1 \end{pmatrix} h \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}, h\right)$$

where $h \in SL_2$. Denote $r(\alpha) = \mathrm{diag}(\alpha, 1, 1, 1, 1, 1)$. A change of variables in Z implies that the above equals

$$\sum_{\alpha\in F^*} \int_{S_\alpha(F)\backslash SL_2(\mathbb{A})\times SL_2(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})}$$

$$\varphi\left[\begin{pmatrix} I & Z & T \\ & I & Y \\ & & I \end{pmatrix} r(\alpha)\begin{pmatrix} h & & \\ & g & \\ & & g^* \end{pmatrix}\right]\widetilde{\psi}(Y)\psi(\mathrm{tr}Z)\mathrm{d}u\mathrm{d}g\mathrm{d}h\ .$$

As before we can change variables in Y to obtain

$$\sum \int \varphi\left[\begin{pmatrix} I & Z & T \\ & I & Y \\ & & I \end{pmatrix} r(\alpha)\begin{pmatrix} h & & \\ & g & \\ & & g \end{pmatrix}\gamma\right]\psi(\mathrm{tr}(Y + Z))\mathrm{d}u\mathrm{d}g\mathrm{d}h\ .$$

Conjugating $r(\alpha)$ to the right and changing variables in h we obtain (4.2) as an inner integration. This shows that (4.4) is zero for all choice of data.

This completes the proof of the Theorem. □

Remark. We note here that if we can extend the validity of the Siegel formula (stated in Proposition 4.1) when $g \in GSp_6(\mathbb{A})$ then in formulae (4.1) and (4.2) we can replace $Sp_6(\mathbb{A})$ by $GSp_6(\mathbb{A})$ in (4.1) and $SL_2(\mathbb{A})$ by $GL_2(\mathbb{A})$ and the outer integration is given over $Z_\mathbb{A}GSp_6(F)\backslash GSp_6(\mathbb{A})$ and $Z_\mathbb{A}GL_2(F)\backslash GL_2(\mathbb{A})$. Moreover in the respective integrals (4.1) and (4.2) we must also have the character χ. In particular then (4-2) is replaced by the period

$$(\varphi) = \int_{Z_{\mathbb{A}}\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} \varphi\left(u\begin{pmatrix} g & & \\ & g & \\ & & g\end{pmatrix}\right)\psi_U(u)\chi(\det g)\mathrm{d}u\mathrm{d}g\,. \qquad (*)$$

With this period as the starting point we can make a conjecture concerning the relation of the nonvanishing of the restricted L function $L^S(\pi, \Lambda^3 \otimes \chi, 1/2)$ to the nonvanishing of the above period.

In fact for each quaternion algebra D/F we consider the group $\mathrm{GL}_3(D)$. Then we have an analogue of the group U. In fact let $U(D)$ be the upper triangular subgroup in $\mathrm{GL}_3(D)$ given by

$$\left\{\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1\end{pmatrix}\middle| x, y, z \in D\right\}$$

Then we define the Whittaker type character on $U(D)$ as given by

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1\end{pmatrix} \xrightarrow{\psi_U^D} \psi_D(x+z)\,,$$

where ψ_D is a character given on the vectorspace of the quaternion algebra D.

Thus for an automorphic cuspidal representation τ of $\mathrm{GL}_3(D)$ we consider the period ($\varphi \in \tau$)

$$(\varphi) = \int_{Z_{\mathbb{A}}D^*(F)\backslash D^*(\mathbb{A})} \int_{U_D(F)\backslash U_D(\mathbb{A})} \varphi\left(u\begin{pmatrix} g & & \\ & g & \\ & & g\end{pmatrix}\right)\psi_U^D(u)\chi(N_D(g))\mathrm{d}u\mathrm{d}^*g \qquad (*)_D$$

(assuming τ has central character ω_τ so that $\omega_\tau\chi^2 = 1$ and N_D the corresponding reduced norm on D^*).

Thus there is an analogue of the Gross Prasad conjecture for this example. We know that since $\mathrm{GL}_3(D)$ is an inner form of $\mathrm{GL}_6(F)$, this implies that there exists an automorphic functorial lifting between $\mathrm{GL}_3(D)$ and $\mathrm{GL}_6(F)$. In particular we know that an irreducible cuspidal module σ of $\mathrm{GL}_3(D)(\mathbb{A})$ lifts to an irreducible cuspidal automorphic σ' of GL_6. (The modules σ and σ' agree at all the places $\mathrm{GL}_3(D_v) \cong \mathrm{GL}_6$). Moreover at the ramified places (where $\mathrm{GL}_3(D_v) \neq \mathrm{GL}_6$) there is a local character identity between σ_v and σ'_v. In any case given π automorphic cuspidal in $\mathrm{GL}_6(\mathbb{A})$ (with central character ω_π satisfying $\omega_\pi\chi^2 = 1$) we let π^D be a cuspidal automorphic module on $\mathrm{GL}_3(D(\mathbb{A}))$ which lifts to π. (π^D may not exist).

Then the analogue of the Gross-Prasad conjecture is the following statement.

CONJECTURE. $L^S(\pi, \Lambda^3 \otimes \chi, 1/2) \neq 0$ *if and only if there exists a unique quaternion algebra D so that*

(i) $(*)_D(\varphi) \neq 0$ *for some* $\varphi \in \pi^D$;
(ii) $(*)_{D'}(\varphi') = 0$ *for all* $\varphi' \in \pi^{D'}$ *with* $D' \neq D$ *(here* D' *may in fact be the split form* $M_{2,2}(F)$ *and the period* $(*)_D$ *is given as above).*

In this case, a more quantitative version of the conjecture is expected which relates the special value $L^S(\pi, \Lambda^3 \otimes \chi, 1/2)$ with the finite positive sum of terms of the form $|(*)_D(\varphi_i)|^2$ (where φ_i runs over a basis of an appropriate finite dimensional subspace of π).

Remark. We note here that when $\omega_\pi\chi^2 \neq 1$ but $(\omega_\pi\chi^2)^2 = 1$ we can also relate the nonvanishing of $L^S(\pi, \Lambda^3 \otimes \chi, 1/2)$ to the nonvanishing of a certain period. Specifically the period condition will involve a Siegel formula relating Sp_3 to $\mathrm{O}(3, 1)$ (where the quadratic character $\omega_\pi\chi^2$ dictates the choice of $\mathrm{O}(3, 1)$).

References

[A-C] Arthur, J. and Clozel, L.: *Simple Algebras, Base Change and the Advanced Theory of the Trace Formulas,* Ann. of Math. Stud. 120, Princeton Univ. Press, 1989.

[B] Brion, M.: Invariante d'un sous-groupe unipotent maximal d'un groupe semi-simple, *Ann. Inst. Fourier (Grenoble)* **33**(1) (1983), 1–27.

[C-S] Casselman, W. and Shalika, J.: The Unramified Principal Series of p-adic groups II: The Whittaker function, *Compositio Math.* **41** (1980), 207–231.

[G] Ginzburg, D.: On spin L-functions for orthogonal groups, *Duke Math. J.* **77** (1995), 753–798.

[J-R] Jacquet, H. and Rallis, S.: Symplectic periods, *J. reine angew. Math.* **423** (1992), 175–197.

[J-S] Jacquet, H. and Shalika, J.: Exterior square L-functions, In: L. Closel and J. S. Milne (eds), *Automorphic Forms, Shimura Varieties and L-functions,* Vol. 2, Academic Press, New York, 1990.

[K-R] Kudla, S. and Rallis, S.: A regularized Siegel-Weil formula: the first term identity, *Ann. of Math.* **140** (1994), 1–80.

[M-W] Moeglin, C. and Waldspurger, J.-L.: *Spectral Decomposition of Eisenstein Series,* Cambridge Tracts in Math. 113, Cambridge Univ. Press, 1995.

[S] Soudry, D.: On the Archimedean theory of Rankin–Selberg convolutions for $\mathrm{SO}(2\ell + 1) \times \mathrm{GL}_n$, *Ann. Sci. Ecole Norm. Sup.* **28** (1995), 161–224.

https://doi.org/10.1023/A:1002461508749 Published online by Cambridge University Press