Forum of Mathematics, Sigma (2023), **Vol. 11**:e110 1–18
doi:10.1017/fms.2023.97

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Gluing approximable triangulated categories

Jesse Burke [1], Amnon Neeman [2] and Bregje Pauwels [3,4]

[1] Abyss Solutions, 477 Pitt Street, Haymarket, NSW, 2000, Australia; E-mail: j.burke@abysssolutions.com.au.
[2] Dipartimento di Matematica "F. Enriques", Università degli Studi di Milano, Via Cesare Saldini 50, Milano, 20133, Italy; E-mail: amnon.neeman@unimi.it.
[3] School of Mathematical and Physical Sciences, Macquarie University, 12 Wally's Walk, Macquarie Park, NSW, 2109, Australia; E-mail: bregje.pauwels@mq.edu.au.
[4] Sydney Mathematical Research Institute, University of Sydney, Sydney NSW 2006, Australia.

Received: 29 January 2023; **Revised:** 1 August 2023; **Accepted:** 14 September 2023

2020 Mathematics Subject Classification: *Primary* – 18G80; *Secondary* – 14F08, 18G35

Abstract
Given a bounded-above cochain complex of modules over a ring, it is standard to replace it by a projective resolution, and it is classical that doing so can be very useful.

Recently, a modified version of this was introduced in triangulated categories other than the derived category of a ring. A triangulated category is *approximable* if this modified procedure is possible. Not surprisingly this has proved a powerful tool. For example: the fact that $\mathsf{D}_{\mathsf{qc}}(X)$ is approximable when X is a quasi compact, separated scheme led to major improvements on old theorems due to Bondal, Van den Bergh and Rouquier.

In this article, we prove that, under weak hypotheses, the recollement of two approximable triangulated categories is approximable. In particular, this shows many of the triangulated categories that arise in noncommutative algebraic geometry are approximable. Furthermore, the lemmas and techniques developed in this article form a powerful toolbox which, in conjunction with the groundwork laid in [16], has interesting applications in existing and forthcoming work by the authors.

Contents

1. Introduction

Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. We can consider two objects $x,\ y \in \mathcal{D}$ as "close together" if there exists an exact triangle $x \longrightarrow y \longrightarrow z$ in $\mathcal{D}$ with $z \in \mathcal{D}^{\leq -n}$ for some large n. This intuitive definition of distance was used in [16] to define two new notions, assuming $\mathcal{D}$ has coproducts and is compactly generated by a single object G. The first is the property of (weak) approximability: $\mathcal{D}$ is *approximable* if every object in $\mathcal{D}^{\leq 0}$ can be approximated by objects that are finitely built out of arbitrary coproducts of certain shifts of G. That is, every object in $\mathcal{D}^{\leq 0}$ has a sequence of "simpler" objects converging to it. The second is a pair of subcategories $\mathcal{D}^b_c \subseteq \mathcal{D}^-_c \subseteq \mathcal{D}$. Here, $\mathcal{D}^-_c$ denotes the full subcategory with objects that can be approximated by compact objects, and $\mathcal{D}^b_c$ is $\mathcal{D}^-_c \cap \mathcal{D}^b$, where $\mathcal{D}^b$ is the bounded part of the given t-structure.

The two notions are connected: if $\mathcal{D}$ is approximable and G satisfies certain finiteness conditions, then every finite cohomological functor, respectively, locally finite cohomological functor, on the compact objects $\mathcal{D}^c$ of $\mathcal{D}$ is represented by an object of $\mathcal{D}^b_c$, respectively, $\mathcal{D}^-_c$, by [16, Theorem 0.3]. In this paper, we prove that approximability and the two distinguished subcategories may be glued along a recollement. To be precise, we let $\mathcal{D}_F$ and $\mathcal{D}_U$ be triangulated categories with a single compact generator and let $\mathcal{D}$ be a compactly generated triangulated category. We assume there is a recollement

$$\mathcal{D}_F \underset{\overset{\longleftarrow}{i^!}}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{i_*}}} \mathcal{D} \underset{\overset{\longleftarrow}{j_*}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{j^*}}} \mathcal{D}_U,$$

which implies that $\mathcal{D}$ has a compact generator G.

The following are Proposition 4.3 and Theorem 5.1, respectively.

Proposition. *If $\mathcal{D}_U$ is weakly approximable, $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-n}G, G) = 0$ for $n \gg 0$, and j^*G is in $(\mathcal{D}_U)^-_c$, then there are equalities:*

$$\mathcal{D}^-_c = \left\{X \in \mathcal{D} \,|\, i^*X \in (\mathcal{D}_F)^-_c \text{ and } j^*X \in (\mathcal{D}_U)^-_c\right\},$$
$$\mathcal{D}^b_c = \left\{X \in \mathcal{D} \,|\, i^*X \in (\mathcal{D}_F)^-_c,\ j^*X \in (\mathcal{D}_U)^b_c \text{ and } i^!X \in (\mathcal{D}_F)^+\right\}.$$

The point of this proposition is an explicit description of the subcategories $\mathcal{D}^-_c$ and $\mathcal{D}^b_c$ of $\mathcal{D}$ in terms of the gluing functors of the recollement and of the corresponding subcategories of $\mathcal{D}_F$ and $\mathcal{D}_U$. The notation that enters these formulas will be explained in the lead-up to the proposition.

Theorem. *If $\mathcal{D}_F$ and $\mathcal{D}_U$ are approximable and $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-n}G, G) = 0$ for $n \gg 0$, then $\mathcal{D}$ is approximable.*

Approximability is a very useful and natural notion in the study of the triangulated categories of algebraic geometry. If $\mathcal{D} = \mathsf{D}_{\mathsf{qc}}(X)$ is the derived category of quasi-coherent sheaves on a scheme X, then $\mathcal{D}$ is approximable if X is quasi-compact and separated [16, Example 3.6], and if we further assume X Noetherian, the subcategories $\mathcal{D}^b_c$ and $\mathcal{D}^-_c$ are $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$ and $\mathsf{D}^-_{\mathsf{coh}}(X)$, respectively. The approximability property of the category $\mathsf{D}_{\mathsf{qc}}(X)$ is used in [15] and [16] to significantly generalize foundational results of Bondal and Van den Bergh [3], and Rouquier [23]; in particular, to mixed characteristic. For an overview of the power of approximability, we refer the reader to the survey article [14], as well as to the ICM proceedings [13] which surveys the applications obtained to date.

We expect that approximability will also be useful in noncommutative geometry. We mean noncommutative algebraic geometry in the sense of [22], where a *noncommutative scheme* is a small pretriangulated dg-category with a classical generator, and the derived category of the scheme is the derived category of the dg-category. In [3, Theorem 3.1.1(ii)], it was proved that if X is a quasi-compact quasi-separated scheme, then $\mathsf{D}_{\mathsf{perf}}(X)$ has a classical generator. In this way, the classical commutative schemes are considered within the world of noncommutative schemes.

A central construction in noncommutative algebraic geometry is the gluing $\mathcal{A}$ of two small dg-categories $\mathcal{B}, \mathcal{C}$ along a $\mathcal{C}$-$\mathcal{B}$-bimodule [(22, 24)]. In the final section of this paper, we show that if $\mathcal{B}$ and $\mathcal{C}$ have approximable derived categories, and the bimodule is cohomologically bounded above, then the glued dg-category $\mathcal{A}$ also has an approximable derived category. Furthermore, we describe how $\mathsf{D}(\mathcal{A})^-_c$ is built from gluing $\mathsf{D}(\mathcal{B})^-_c$ and $\mathsf{D}(\mathcal{C})^-_c$. This result significantly expands the known examples of approximable triangulated categories in noncommutative algebraic geometry. Even if $\mathsf{D}(\mathcal{B})$ and $\mathsf{D}(\mathcal{C})$ are “commutative”, that is, equivalent to the derived categories of quasi-coherent sheaves on a commutative scheme, the glued category $\mathsf{D}(\mathcal{A})$ is rarely commutative.

https://doi.org/10.1017/fms.2023.97 Published online by Cambridge University Press

Many interesting classes of noncommutative schemes are constructed via gluing. In [9], Kuznetsov and Lunts construct categorical resolutions of separated schemes of finite type over a field k of characteristic zero by gluing derived categories of smooth varieties. This notion of categorical resolutions was defined in [10] as a generalization of Van den Bergh's notion of noncommutative crepant resolution [25]. Gluing of DG-categories and categorical resolutions are also used in [5] to show that $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$ is homotopically finitely presented whenever X is a separated scheme of finite type over a field k of characteristic zero. For other places where gluing is used in noncommutative geometry see [4, 8, 11, 21].

In addition to the main results listed above, the lemmas and techniques developed in this paper form a toolbox which is powerful in its own right. Already, Lemma 3.6 in this paper is used in [17, Proposition 3.2], whereas Lemma 3.9 is used to prove [17, Lemma 3.1] as well as [18, Proposition 4.8]. More applications will appear in forthcoming work of the second author together with Alberto Canonaco and Paolo Stellari, as well as in forthcoming work by the third author.

2. Notation and definitions

Throughout, $\mathcal{D}$ denotes a triangulated category with coproducts. We recall some standard notation, see [16, Definition 0.21].

2.1. Let $\mathcal{A}$, $\mathcal{B}$ be full subcategories of $\mathcal{D}$, and define the following full subcategories:

a) $\mathcal{A} * \mathcal{B}$ has for objects all the $x \in \mathcal{D}$, such that there exists an exact triangle $a \longrightarrow x \longrightarrow b$ in $\mathcal{D}$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$.
b) $\mathrm{Add}(\mathcal{A})$ has for objects all coproducts of objects of $\mathcal{A}$.
c) $\mathrm{smd}(\mathcal{A})$ has for objects all direct summands of objects of $\mathcal{A}$.

2.2. Let G be an object in $\mathcal{D}$, and suppose $n \in \mathbb{N}$ and $A \leq B \in \mathbb{Z} \cup \{-\infty, +\infty\}$. We define the following full subcategories:

a) If $A, B \in \mathbb{Z}$, then $G[A,B] \subset \mathcal{D}$ has objects $\{\Sigma^{-i}G \mid A \leq i \leq B\}$. Similarly, $G(-\infty, B] \subset \mathcal{D}$ has objects $\{\Sigma^{-i}G \mid i \leq B\}$. We define $G[A,\infty)$ and $G(-\infty,\infty)$ analogously.
b) $\mathrm{Coprod}_n(G[A,B])$ is defined inductively on the integer n by setting

$$\begin{aligned}\mathrm{Coprod}_1(G[A,B]) &= \mathrm{Add}(G[A,B]) \ \text{ and}\\ \mathrm{Coprod}_{n+1}(G[A,B]) &= \mathrm{Coprod}_1(G[A,B]) * \mathrm{Coprod}_n(G[A,B]).\end{aligned}$$

c) $\mathrm{Coprod}(G[A,B])$ is the smallest full subcategory $\mathcal{S} \subset \mathcal{D}$, closed under coproducts, with $\mathcal{S} * \mathcal{S} \subset \mathcal{S}$ and with $G[A,B] \subset \mathcal{S}$.
d) $\langle G \rangle_n^{[A,B]} = \mathrm{smd}(\mathrm{Coprod}_n(G[A,B]))$.
e) $\langle G \rangle^{[A,B]} = \mathrm{smd}(\mathrm{Coprod}(G[A,B]))$.

2.3 [2, Definition 1.3.1]. A *t-structure* on a triangulated category $\mathcal{D}$ is a pair of full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ satisfying

1. $\Sigma\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0} \subset \Sigma\mathcal{D}^{\geq 0}$.
2. $\mathrm{Hom}_{\mathcal{D}}(\Sigma\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) = 0$.
3. For every $X \in \mathcal{D}$, there exists an exact triangle $Y \longrightarrow X \longrightarrow Z$ with $Y \in \Sigma\mathcal{D}^{\leq 0}$ and $Z \in \mathcal{D}^{\geq 0}$.

2.4 [2, Section 1.3]. Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. For every integer $n \in \mathbb{Z}$, we set $\mathcal{D}^{\leq n} := \Sigma^{-n}\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq n} := \Sigma^{-n}\mathcal{D}^{\geq 0}$. The inclusion $\mathcal{D}^{\leq n} \subset \mathcal{D}$ has a right adjoint which we denote $(-)^{\leq n}$, whereas $\mathcal{D}^{\geq n} \subset \mathcal{D}$ has a left adjoint which we denote by $(-)^{\geq n}$. For every $X \in \mathcal{D}$, the counit and unit of these adjunctions fit into an exact triangle

$$X^{\leq n} \longrightarrow X \longrightarrow X^{\geq n+1}.$$

This is, up to unique isomorphism, the only triangle $Y \longrightarrow X \longrightarrow Z$ with $Y \in \mathcal{D}^{\leq n}$ and $Z \in \mathcal{D}^{\geq n+1}$. We write $\mathcal{H}^n : \mathcal{D} \longrightarrow \mathcal{D}$ for the functor taking X to $(X^{\leq n})^{\geq n}$.

2.5. A *recollement* is a diagram of exact functors between triangulated categories,

$$\mathcal{D}_F \underset{\overset{\longleftarrow}{i^!}}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{i_*}}} \mathcal{D} \underset{\overset{\longleftarrow}{j_*}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{j^*}}} \mathcal{D}_U,$$

with the following properties:

1. each functor is left adjoint to the one below it;
2. $j^* i_* = 0$;
3. there exist natural transformations $d : i_* i^* \to \Sigma j_! j^*$ and $d' : j_* j^* \to \Sigma i_* i^!$, such that for any $X \in \mathcal{D}$, the following are exact triangles:

$$j_! j^* X \xrightarrow{\epsilon} X \xrightarrow{\eta} i_* i^* X \xrightarrow{d} \Sigma j_! j^* X$$
$$i_* i^! X \xrightarrow{\epsilon} X \xrightarrow{\eta} j_* j^* X \xrightarrow{d'} \Sigma i_* i^! X,$$

   where η and ϵ are the (co)unit maps of the adjunctions;
4. $i_*, j_!, j_*$ are fully faithful.

We note that i^*, i_*, $j_!$, j^* preserve coproducts, since they are left adjoints. Since i^* and $j_!$ have right adjoints preserving coproducts, [19, Theorem 5.1] informs us that i^* and $j_!$ respect compact objects.

2.6 [(2, 1.4.10)]. Consider a recollement as above. Given t-structures $(\mathcal{D}_F^{\le 0}, \mathcal{D}_F^{\ge 0})$ and $(\mathcal{D}_U^{\le 0}, \mathcal{D}_U^{\ge 0})$ on $\mathcal{D}_F$ and $\mathcal{D}_U$, respectively, the *glued t-structure on* $\mathcal{D}$ has aisle

$$\mathcal{D}^{\le 0} := \{X \in \mathcal{D} \mid i^* X \in \mathcal{D}_F^{\le 0} \text{ and } j^* X \in \mathcal{D}_U^{\le 0}\}$$

and coaisle

$$\mathcal{D}^{\ge 0} := \{X \in \mathcal{D} \mid i^! X \in \mathcal{D}_F^{\ge 0} \text{ and } j^* X \in \mathcal{D}_U^{\ge 0}\}.$$

2.7 [(2, 1.3.16)]. Let $\mathcal{D}_1, \mathcal{D}_2$ be triangulated categories endowed with t-structures. A functor $f : \mathcal{D}_1 \to \mathcal{D}_2$ is *right t-exact* if $f(\mathcal{D}_1^{\le 0}) \subset \mathcal{D}_2^{\le 0}$, and *left t-exact* if $f(\mathcal{D}_1^{\ge 0}) \subset \mathcal{D}_2^{\ge 0}$. We say f is *t-exact* if f is left and right t-exact.

Consider a recollement as in 2.5, and suppose $\mathcal{D}$ has a t-structure glued from t-structures on $\mathcal{D}_F$ and $\mathcal{D}_U$. By [2, 1.3.17(iii)], the functors i^*, $j_!$ are right t-exact, i_*, j^* are t-exact, and $i^!$, j_* are left t-exact.

2.8. Let $\mathcal{D}$ be a triangulated category with coproducts. An object $G \in \mathcal{D}$ is *compact* if the functor $\mathrm{Hom}_{\mathcal{D}}(G, -)$ respects coproducts. We call G a compact generator for $\mathcal{D}$ if for all $X \in \mathcal{D}$, we have that $\mathrm{Hom}_{\mathcal{D}}(G, \Sigma^i X) = 0$ for all $i \in \mathbb{Z}$ implies $X \cong 0$.

2.9 [16, Definition 0.21]. A triangulated category $\mathcal{D}$ with coproducts is *weakly approximable* if there exists a compact generator G, a t-structure $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$, and an integer $A > 0$, such that the following hold:

(1) $\Sigma^A G \in \mathcal{D}^{\le 0}$ and $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-A} G, \mathcal{D}^{\le 0}) = 0$.
(2) For every $X \in \mathcal{D}^{\le 0}$, there exists an exact triangle $E \to X \to D$ with $E \in \langle G \rangle^{[-A,A]}$ and $D \in \mathcal{D}^{\le -1}$.

If Properties (1) and (2) hold for a compact generator G and integer A, then for every compact generator H, there is an integer A_H for which they hold, by [16, Proposition 2.6].

The category $\mathcal{D}$ is *approximable* if we can choose $A > 0$, such that, moreover,

(3) in the exact triangle $E \to X \to D$ above, we may assume $E \in \langle G \rangle_A^{[-A,A]}$.

If Properties (1), (2), and (3) hold for a compact generator G and integer A, then for every compact generator H, there is an integer A_H for which they hold, by [16, Proposition 2.6].

https://doi.org/10.1017/fms.2023.97 Published online by Cambridge University Press

3. The preferred *t*-structure

In this section, we start developing the necessary machinery, some of which has since demonstrated its usefulness in other contexts. We begin by reminding the reader of some definitions.

Definition 3.1 [16, Definition 0.10]. Let $\mathcal{D}$ be a triangulated category. Two t-structures $(\mathcal{D}_1^{\le 0}, \mathcal{D}_1^{\ge 0})$ and $(\mathcal{D}_2^{\le 0}, \mathcal{D}_2^{\ge 0})$ on $\mathcal{D}$ are *equivalent* if there exists an integer $A > 0$ with $\mathcal{D}_1^{\le -A} \subset \mathcal{D}_2^{\le 0} \subset \mathcal{D}_1^{\le A}$.

Lemma 3.2. *The gluing of t-structures along a recollement is stable under equivalence. That is, consider a recollement as in 2.5 and suppose $(\mathcal{D}_F^{\le 0}, \mathcal{D}_F^{\ge 0})$, $(\mathcal{D'}_F^{\le 0}, \mathcal{D'}_F^{\ge 0})$ are equivalent t-structures on $\mathcal{D}_F$ and $(\mathcal{D}_U^{\le 0}, \mathcal{D}_U^{\ge 0})$, $(\mathcal{D'}_U^{\le 0}, \mathcal{D'}_U^{\ge 0})$ are equivalent t-structures on $\mathcal{D}_U$. Then the t-structure on $\mathcal{D}$ obtained by gluing $(\mathcal{D}_F^{\le 0}, \mathcal{D}_F^{\ge 0})$, $(\mathcal{D}_U^{\le 0}, \mathcal{D}_U^{\ge 0})$ is equivalent to the t-structure obtained by gluing $(\mathcal{D'}_F^{\le 0}, \mathcal{D'}_F^{\ge 0})$, $(\mathcal{D'}_U^{\le 0}, \mathcal{D'}_U^{\ge 0})$.*

Proof. Write $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$, $(\mathcal{D'}^{\le 0}, \mathcal{D'}^{\ge 0})$ for the t-structures on $\mathcal{D}$ obtained by gluing $(\mathcal{D}_F^{\le 0}, \mathcal{D}_F^{\ge 0})$, $(\mathcal{D}_U^{\le 0}, \mathcal{D}_U^{\ge 0})$ and $(\mathcal{D'}_F^{\le 0}, \mathcal{D'}_F^{\ge 0})$, $(\mathcal{D'}_U^{\le 0}, \mathcal{D'}_U^{\ge 0})$, respectively. If $A > 0$ is an integer such that $\mathcal{D}_F^{\le -A} \subset \mathcal{D'}_F^{\le 0} \subset \mathcal{D}_F^{\le A}$ and $\mathcal{D}_U^{\le -A} \subset \mathcal{D'}_U^{\le 0} \subset \mathcal{D}_U^{\le A}$, then $\mathcal{D}^{\le -A} \subset \mathcal{D'}^{\le 0} \subset \mathcal{D}^{\le A}$. □

Definition 3.3 [1, Theorem A.1]. Given a compact object G of $\mathcal{D}$, the t-structure *generated by G*, denoted $(\mathcal{D}_G^{\le 0}, \mathcal{D}_G^{\ge 0})$, has aisle $\mathcal{D}_G^{\le 0} = \langle G \rangle^{(-\infty, 0]}$.

Remark 1. Since the category $\mathrm{Coprod}(G(-\infty, 0])$ is closed under positive suspensions and coproducts, it contains all direct summands of its objects. This shows $\mathcal{D}_G^{\le 0} = \langle G \rangle^{(-\infty, 0]} = \mathrm{Coprod}(G(-\infty, 0]))$.

Suppose now that $\mathcal{D}$ is compactly generated by the compact object G. If H is also a compact generator for $\mathcal{D}$, then [16, Lemma 0.9] provides a positive integer A with $H \in \langle G \rangle_A^{[-A, A]}$ and $G \in \langle H \rangle_A^{[-A, A]}$. This shows $\mathcal{D}_H^{\le -A} \subset \mathcal{D}_G^{\le 0} \subset \mathcal{D}_H^{\le A}$, hence the t-structures generated by G and H are equivalent. Thus, the next definition does not depend on the choice of compact generator:

Definition 3.4. If $\mathcal{D}$ is compactly generated by the compact object G, then the *preferred equivalence class of t-structures* is the equivalence class containing the t-structure $(\mathcal{D}_G^{\le 0}, \mathcal{D}_G^{\ge 0})$ generated by G.

The first part of the following lemma follows from [16, Remark 0.20], and the second part follows from [16, Observation 0.12(ii) and Lemma 2.8]:

Lemma 3.5. *Suppose $\mathcal{D}$ has a compact generator G with $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-n} G, G) = 0$ for $n \gg 0$. Assume $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ is a t-structure in the preferred equivalence class. Then*

1. *there exists an integer $C > 0$, such that $\mathrm{Hom}_{\mathcal{D}}(G, \mathcal{D}^{\le -C}) = 0$;*
2. *for all compact objects $F, H \in \mathcal{D}$, we have $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-m} F, H) = 0$ for $m \gg 0$.*

In particular, the vanishing hypothesis on G is independent of the choice of compact generator.

Recall that a t-structure $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ on $\mathcal{D}$ is *nondegenerate* if

$$\bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\le n} = 0 \quad \text{and} \quad \bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\ge n} = 0.$$

Lemma 3.6. *Suppose $\mathcal{D}$ has a compact generator G with $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-n} G, G) = 0$ for $n \gg 0$. Then every t-structure in the preferred equivalence class is nondegenerate.*

Proof. Being nondegenerate is clearly stable under equivalence, so it suffices to show the t-structure $(\mathcal{D}_G^{\le 0}, \mathcal{D}_G^{\ge 0})$ generated by G is nondegenerate. Let X be a nonzero object in $\mathcal{D}$. Since G is a compact generator for $\mathcal{D}$, there exists a nonzero morphism $\Sigma^l G \to X$ for some $l \in \mathbb{Z}$. It follows that $X \notin \mathcal{D}_G^{\ge -l+1} = (\mathcal{D}_G^{\le -l})^{\perp}$. Moreover, by the first part of Lemma 3.5, there exists $C > 0$, such that $\mathrm{Hom}_{\mathcal{D}}(G, \mathcal{D}_G^{\le -C}) = 0$. Hence, $\mathrm{Hom}_{\mathcal{D}}(\Sigma^l G, \mathcal{D}_G^{\le -C-l}) = 0$, and we can conclude that $X \notin \mathcal{D}_G^{\le -C-l}$. □

https://doi.org/10.1017/fms.2023.97 Published online by Cambridge University Press

The next lemma is exactly [16, Proposition 2.6]. It tells us that once we know a category to be (weakly) approximable, any compact generator and any t-structure in the preferred equivalence class will fulfil the approximability criteria in Definition 2.9.

Lemma 3.7. *Let $\mathcal{D}$ be weakly approximable. Suppose G is a compact generator for $\mathcal{D}$ and that $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ is any t-structure in the preferred equivalence class. Then there exists $A > 0$, such that Conditions (1) and (2) in Definition 2.9 hold for the compact generator G and the t-structure $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$. If $\mathcal{D}$ is moreover approximable, then the integer A can be chosen to further satisfy Condition (3) in Definition 2.9.*

Lemma 3.8. *Let $\mathcal{D}$ be weakly approximable. Suppose G is a compact generator for $\mathcal{D}$ and $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ is a t-structure in the preferred equivalence class. Assume the integer $A > 0$ is chosen to satisfy Conditions (1) and (2) in Definition 2.9. If $X \in \mathcal{D}$ is an object, such that* $\operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, X) = 0$ *for all i in the interval $-A \le i \le A$, then $\mathcal{H}^0(X) = 0$.*

Proof. The t-structure gives us an exact triangle $X^{\le 0} \to X \to X^{\ge 1}$, and we may choose a triangle $E \to X^{\le 0} \to D$ with $E \in \langle G \rangle^{[-A,A]}$ and $D \in \mathcal{D}^{\le -1}$. We are assuming that ${}^{\perp}X$ contains $\{\Sigma^i G, -A \le i \le A\}$, and hence, ${}^{\perp}X$ must contain $\langle G \rangle^{[-A,A]}$. Therefore, the composite $E \to X^{\le 0} \to X$ must vanish, and the map $X^{\le 0} \to X$ must factor as $X^{\le 0} \to D \to X$. Applying the functor $\mathcal{H}^0$, we deduce that the isomorphism $\mathcal{H}^0(X^{\le 0}) \to \mathcal{H}^0(X)$ factors through $\mathcal{H}^0(D) = 0$. □

The following lemma turns out to be surprisingly useful. It has already been used in [17, 18], as well as in forthcoming work by the authors.

Lemma 3.9. *Let $\mathcal{D}$ be weakly approximable. Suppose G is a compact generator for $\mathcal{D}$ and $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ is a t-structure in the preferred equivalence class. Assume the integer $A > 0$ is chosen to satisfy Conditions (1) and (2) in Definition 2.9. Suppose $X \in \mathcal{D}$ is an object, such that* $\operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, X) = 0$ *for all i in the interval $-2A \le i \le 2A$. Then there exists in $\mathcal{D}$ a triangle*

$$W \xrightarrow{\varphi} X \xrightarrow{\psi} Y,$$

such that, for each $i \in \mathbb{Z}$, each of the abelian groups $\operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, W)$ *and* $\operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, Y)$ *is either zero or isomorphic to* $\operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, X)$*, with the precise ranges in which each happens given in (i) and (ii) below. More fully, the triangle satisfies:*

(i) *For the object $W \in \mathcal{D}$, we have that* $\operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, W)$ *vanishes when $i \le 2A$, while the map*

$$\operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, W) \xrightarrow{\operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, \varphi)} \operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, X)$$

is an isomorphism for all $i \ge -2A$.

(ii) *For the object $Y \in \mathcal{D}$, we have that* $\operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, Y)$ *vanishes when $i \ge -2A$, while the map*

$$\operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, X) \xrightarrow{\operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, \psi)} \operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, Y)$$

is an isomorphism for all $i \le 2A$.

(iii) *Any triangle $W \to X \to Y$ satisfying (i) and (ii) is canonically isomorphic to the triangle $X^{\le i} \to X \to X^{\ge i+1}$ for any i such that $-A-1 \le i \le A$.*

(iv) *If we strengthen the assumption on X to say that* $\operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, X) = 0$ *for all $i \le 2A$, then X must belong to $\mathcal{D}^{\le -A-1}$.*

(v) *If we strengthen the assumption on X to say that* $\operatorname{Hom}_{\mathcal{D}}(\Sigma^i G, X) = 0$ *for all $i \ge -2A$, then X must belong to $\mathcal{D}^{\ge A+1}$.*

Proof. With the hypotheses as in the current lemma, Lemma 3.8 guarantees the vanishing of $\mathcal{H}^i(X)$ for all $-A \le i \le A$. The triangles $X^{\le i-1} \to X^{\le i} \to \mathcal{H}^i(X)$ then tell us that $X^{\le A}$ belongs to $\mathcal{D}^{\le -A-1}$. Consider the triangle $W \xrightarrow{\varphi} X \xrightarrow{\psi} Y$ with $W := X^{\le A} \in \mathcal{D}^{\le -A-1}$ and $Y := X^{\ge A+1} \in \mathcal{D}^{\ge A+1}$.

The integer A is such that $\mathrm{Hom}_{\mathcal{D}}(G, \mathcal{D}^{\le -A}) = 0$, and hence, $\mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, W) \cong \mathrm{Hom}_{\mathcal{D}}(G, \Sigma^{-i} W)$ vanishes for all $i \le 1$. But we also know that $\Sigma^A G \in \mathcal{D}^{\le 0}$, and hence, $\mathrm{Hom}_{\mathcal{D}}(G, \mathcal{D}^{\ge A+1}) = 0$. Therefore, $\mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, Y) \cong \mathrm{Hom}_{\mathcal{D}}(G, \Sigma^{-i} Y)$ must vanish for all $i \ge 0$. It follows that, for every integer $i \in \mathbb{Z}$,

$$\begin{array}{llcl} \text{either} & \mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, W) & = 0 = & \mathrm{Hom}_{\mathcal{D}}(\Sigma^{i-1} G, W) \\ \text{or} & \mathrm{Hom}_{\mathcal{D}}(\Sigma^{i+1} G, Y) & = 0 = & \mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, Y) \end{array}$$

Thus, in the exact sequence

$$(\dagger\dagger) \qquad \begin{array}{ccccc} \mathrm{Hom}_{\mathcal{D}}(\Sigma^{i+1} G, Y) & & & & \\ \downarrow & & & & \\ \mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, W) & \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, \varphi)} & \mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, X) & \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, \psi)} & \mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, Y) \\ & & & & \downarrow \\ & & & & \mathrm{Hom}_{\mathcal{D}}(\Sigma^{i-1} G, W), \end{array}$$

we have that, for every $i \in \mathbb{Z}$, either the map $\mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, \varphi)$ or the map $\mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, \psi)$ is an isomorphism. More precisely: the map $\mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, \psi)$ is an isomorphism when $i \le 1$, while the map $\mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, \varphi)$ is an isomorphism when $i \ge 0$.

We have already proved the vanishing of $\mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, W)$ for all $i \le 0$ and the vanishing of $\mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, Y)$ for all $i \ge 0$. For $1 \le i \le 2A$, we have isomorphisms

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, W) & \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, \varphi)} & \mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, X) & = & 0 \\ \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-i} G, Y) & \xleftarrow{\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-i} G, \psi)} & \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-i} G, X) & = & 0 \end{array}$$

which extend the vanishing statements of (i) and (ii) to the full range asserted. And the assertions in (i) and (ii), about the ranges in which $\mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, \varphi)$ and $\mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, \psi)$ are isomorphisms, follow from the vanishing statements combined with the long exact sequence $(\dagger\dagger)$ above.

Now for (iv) and (v): the proof of (i) and (ii) produced an explicit triangle $W \to X \to Y$, with $W \in \mathcal{D}^{\le -A-1}$ and with $Y \in \mathcal{D}^{\ge A+1}$, and showed that this particular choice of triangle satisfies (i) and (ii). If we strengthen the assumption on X as in (iv), then $\mathrm{Hom}_{\mathcal{D}}(\Sigma^i G, Y) = 0$ for all $i \in \mathbb{Z}$, and as G is a compact generator, this forces $Y = 0$. This in turn means that $W \to X$ must be an isomorphism, but by construction, W belongs to $\mathcal{D}^{\le -A-1}$. Under the hypotheses of (v), we would deduce that $W = 0$, the map $X \to Y$ is an isomorphism, but by construction, Y belongs to $\mathcal{D}^{\ge A+1}$.

Finally, assume we are given a triangle $W \to X \to Y$, as in the hypotheses. By (iv), we have $W \in \mathcal{D}^{\le -A-1}$, and by (v), we have $Y \in \mathcal{D}^{\ge A+1}$, and (iii) now follows immediately. □

Part (iv) and (v) of the above lemma show that:

Corollary 3.10. *Let $\mathcal{D}$ be a weakly approximable category, let G be a compact generator, and let $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ be a t-structure in the preferred equivalence class.*

1. *If* $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-n} G, X) = 0$ *for $n \gg 0$, then $X \in \mathcal{D}^{\le m}$ for some integer m.*
2. *If* $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{n} G, X) = 0$ *for $n \gg 0$, then $X \in \mathcal{D}^{\ge m}$ for some integer m.*

Lemma 3.11. *Consider a recollement as in Definition 2.5. Suppose $\mathcal{D}_U$ is weakly approximable with compact generator G_U, and $\mathcal{D}$ has a compact generator G', such that* $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-n} G', G') = 0$ *for $n \gg 0$.*

We can then find a compact generator G of $\mathcal{D}$, such that the t-structure on $\mathcal{D}$ generated by G is equal to the gluing of the t-structure on $\mathcal{D}_F$ generated by i^*G and the t-structure on $\mathcal{D}_U$ generated by G_U.

Proof. Write $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ for the t-structure on $\mathcal{D}_U$ generated by G_U. We note that $j_!G_U$ is compact in $\mathcal{D}$ because the functor $j_!$ preserves compactness (see 2.5). The second part of Lemma 3.5 now shows

$$\mathrm{Hom}_{\mathcal{D}_U}(\Sigma^{-n}G_U, j^*G') \cong \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-n}j_!G_U, G') = 0$$

for $n \gg 0$. It follows that $j^*G' \in \mathcal{D}_U^{\leq m}$ for some $m > 0$ by Corollary 3.10. Now let

$$G := \Sigma^m G' \oplus j_!G_U,$$

which is clearly a compact generator for $\mathcal{D}$. We write $(\mathcal{D}_F^{\leq 0}, \mathcal{D}_F^{\geq 0})$ for the t-structure on $\mathcal{D}_F$ generated by i^*G and $(\mathcal{D}_{\mathrm{gl}}^{\leq 0}, \mathcal{D}_{\mathrm{gl}}^{\geq 0})$ for the t-structure on $\mathcal{D}$ obtained by gluing $(\mathcal{D}_F^{\leq 0}, \mathcal{D}_F^{\geq 0})$ and $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$. It remains to show that

$$\mathcal{D}_G^{\leq 0} = \mathcal{D}_{\mathrm{gl}}^{\leq 0},$$

where $(\mathcal{D}_G^{\leq 0}, \mathcal{D}_G^{\geq 0})$ is the t-structure on $\mathcal{D}$ generated by G. First, we note that $i^*G \in \mathcal{D}_F^{\leq 0}$ and $j^*G = \Sigma^m j^*G' \oplus G_U \in \mathcal{D}_U^{\leq 0}$. Using that the functors i^* and j^* commute with coproducts, this shows $i^*\mathcal{D}_G^{\leq 0} \subset \mathcal{D}_F^{\leq 0}$ and $j^*\mathcal{D}_G^{\leq 0} \subset \mathcal{D}_U^{\leq 0}$. Hence,

$$\mathcal{D}_G^{\leq 0} \subset \mathcal{D}_{\mathrm{gl}}^{\leq 0}.$$

On the other hand, we clearly have $j_!G_U \in \mathcal{D}_G^{\leq 0}$, and thus $j_!\mathcal{D}_U^{\leq 0} \subset \mathcal{D}_G^{\leq 0}$, because the functor $j_!$ commutes with coproducts. Since i_*i^*G fits in an exact triangle

$$j_!j^*G \longrightarrow G \longrightarrow i_*i^*G$$

with $j_!j^*G$, $G \in \mathcal{D}_G^{\leq 0}$, we see that $i_*i^*G \in \mathcal{D}_G^{\leq 0}$. It follows that $i_*\mathcal{D}_F^{\leq 0} \subset \mathcal{D}_G^{\leq 0}$. Now, let $X \in \mathcal{D}_{\mathrm{gl}}^{\leq 0}$ and consider the exact triangle

$$j_!j^*X \longrightarrow X \longrightarrow i_*i^*X.$$

By the above, $j_!j^*X$, $i_*i^*X \in \mathcal{D}_G^{\leq 0}$, so $X \in \mathcal{D}_G^{\leq 0}$. We conclude that $\mathcal{D}_G^{\leq 0} = \mathcal{D}_{\mathrm{gl}}^{\leq 0}$. □

Combining the above lemma with Lemma 3.2, we get:

Corollary 3.12. *Consider a recollement as in Definition 2.5. Suppose $\mathcal{D}_U$ is weakly approximable and $\mathcal{D}$ has a compact generator G, such that $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-n}G, G) = 0$ for $n \gg 0$. Then the t-structure on $\mathcal{D}$ obtained by gluing t-structures on $\mathcal{D}_F$ and $\mathcal{D}_U$, both of which are in the preferred equivalence class, is in the preferred equivalence class.*

4. Gluing $\mathcal{D}_c^-$

We remind the reader of the subcategories $\mathcal{D}_c^b$ and $\mathcal{D}_c^-$, and investigate their behavior under recollements. For examples of these subcategories and their applications, we refer back to the Introduction, or to [14] for an extensive discussion.

Definition 4.1 [16, Definition 0.16]. Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Recall that we write $\mathcal{D}^+ := \bigcup_{m>0} \mathcal{D}^{\geq -m}$, $\mathcal{D}^- := \bigcup_{m>0} \mathcal{D}^{\leq m}$, and $\mathcal{D}^b = \mathcal{D}^+ \cap \mathcal{D}^-$. The full subcategory $\mathcal{D}_c^-$ has for objects all $X \in \mathcal{D}$, such that, for any integer $m > 0$, there exists an exact triangle $E \longrightarrow X \longrightarrow E'$ with E compact and $E' \in \mathcal{D}^{\leq -m}$. The subcategory $\mathcal{D}_c^b$ is defined to be $\mathcal{D}_c^- \cap \mathcal{D}^b$.

https://doi.org/10.1017/fms.2023.97 Published online by Cambridge University Press

Remark 2. Observe that equivalent t-structures yield equal $\mathcal{D}^+$, $\mathcal{D}^-$, $\mathcal{D}^b$, $\mathcal{D}^-_c$, $\mathcal{D}^b_c$. If $\mathcal{D}$ has a single compact generator, form the subcategories $\mathcal{D}^+$, $\mathcal{D}^-$, $\mathcal{D}^b$, $\mathcal{D}^-_c$, $\mathcal{D}^b_c$ corresponding to the preferred equivalence class of t-structures. These are intrinsic.

Remark 3. Suppose $\mathcal{D}$ has a compact generator G, such that $\operatorname{Hom}_{\mathcal{D}}(\Sigma^{-n}G, G) = 0$ for $n \gg 0$, and endow $\mathcal{D}$ with a t-structure in the preferred equivalence class. By [16, Proposition 0.19 and Remark 0.20], the subcategories $\mathcal{D}^b_c \subseteq \mathcal{D}^-_c$ are thick triangulated subcategories of $\mathcal{D}^-$.

If $\mathcal{D}$ is weakly approximable, then the above applies: one easily sees that $\mathcal{D}$ has a compact generator G with $\operatorname{Hom}_{\mathcal{D}}(\Sigma^{-n}G, G) = 0$ for $n \gg 0$.

Lemma 4.2. *Consider a recollement as in Definition 2.5. Suppose $\mathcal{D}_U$ is weakly approximable, and $\mathcal{D}$ has a compact generator G, such that* $\operatorname{Hom}_{\mathcal{D}}(\Sigma^{-n}G, G) = 0$ *for* $n \gg 0$. *Then the intrinsic categories $(\mathcal{D}_U)^-_c$, $(\mathcal{D}_U)^b_c$, $\mathcal{D}^-_c$, $\mathcal{D}^b_c$, $(\mathcal{D}_F)^-_c$, and $(\mathcal{D}_F)^b_c$, corresponding to the preferred equivalence class of t-structures as in Remark 2, are all thick triangulated subcategories of (respectively) $\mathcal{D}_U$, $\mathcal{D}$, and $\mathcal{D}_F$.*

Proof. For the categories $(\mathcal{D}_U)^-_c$, $(\mathcal{D}_U)^b_c$, $\mathcal{D}^-_c$, and $\mathcal{D}^b_c$, the assertion is immediate from Remark 3. What we will prove is that the category $\mathcal{D}_F$ has a compact generator H with $\operatorname{Hom}_{\mathcal{D}_F}(\Sigma^{-n}H, H) = 0$ for $n \gg 0$.

Choose for $\mathcal{D}_F$, $\mathcal{D}_U$ t-structures in the preferred equivalence class, and glue them to form a t-structure on $\mathcal{D}$. By Corollary 3.12, the glued t-structure on $\mathcal{D}$ is in the preferred equivalence class. Pick a compact generator $G \in \mathcal{D}$.

Lemma 3.5(1) allows us to choose an integer $C > 0$ with $\operatorname{Hom}_{\mathcal{D}}(G, \mathcal{D}^{\le -C}) = 0$. As i_* is t-exact, we have $i_*\mathcal{D}_F^{\le -C} \subset \mathcal{D}^{\le -C}$, and hence

$$\operatorname{Hom}_{\mathcal{D}_F}(i^*G, \mathcal{D}_F^{\le -C}) \cong \operatorname{Hom}_{\mathcal{D}}(G, i_*\mathcal{D}_F^{\le -C}) = 0.$$

But the t-structure on $\mathcal{D}_F$ is in the preferred equivalence class, and [16, Observation 0.20(ii)] informs us that the compact object $i^*G \in \mathcal{D}_F$ must be contained in $\mathcal{D}_F^-$. Hence, we may choose an integer $B > 0$ with $i^*G \in \mathcal{D}_F^{\le B}$.

But now it's immediate that the compact generator $i^*G \in \mathcal{D}_F$ is such that $\operatorname{Hom}_{\mathcal{D}_F}(\Sigma^{-n}i^*G, i^*G) = 0$ for $n \ge B + C$. ◻

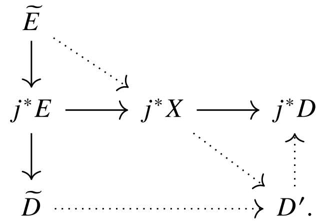
Proposition 4.3. *Consider a recollement as in Definition 2.5. Suppose $\mathcal{D}_U$ is weakly approximable, and $\mathcal{D}$ has a compact generator G, such that* $\operatorname{Hom}_{\mathcal{D}}(\Sigma^{-n}G, G) = 0$ *for* $n \gg 0$. *With $\mathcal{C}$ standing for any of $\mathcal{D}_U$, $\mathcal{D}$, or $\mathcal{D}_F$, let the categories $\mathcal{C}^-_c$, $\mathcal{C}^b_c$, and $\mathcal{C}^+$ be the intrinsic ones coming from the preferred equivalence class of t-structures (see Remark 2). If j^*G is in $(\mathcal{D}_U)^-_c$, then there are equalities:*

$$\begin{aligned}
\mathcal{D}^-_c &= \left\{X \in \mathcal{D} \mid i^*X \in (\mathcal{D}_F)^-_c \textit{ and } j^*X \in (\mathcal{D}_U)^-_c\right\},\\
\mathcal{D}^b_c &= \left\{X \in \mathcal{D} \mid i^*X \in (\mathcal{D}_F)^-_c,\ j^*X \in (\mathcal{D}_U)^b_c \textit{ and } i^!X \in (\mathcal{D}_F)^+\right\}.
\end{aligned}$$

Proof. Choose t-structures for each of $\mathcal{D}_F$, $\mathcal{D}_U$, in the preferred equivalence classes, and glue them to form a t-structure on $\mathcal{D}$. By Corollary 3.12, the glued t-structure on $\mathcal{D}$ is in the preferred equivalence class. We are assuming that, for the compact generator $G \in \mathcal{D}$, we have $j^*G \in (\mathcal{D}_U)^-_c$. As $(\mathcal{D}_U)^-_c$ is a thick triangulated subcategory which contains j^*G, it must also contain $j^*\mathcal{D}^c$, since $\mathcal{D}^c$ is the smallest thick triangulated subcategory of $\mathcal{D}$ containing G.

We first show the left sides are contained in the right sides. Let X be in $\mathcal{D}^-_c$, fix $m > 0$, and let $E \to X \to D$ be an exact triangle in $\mathcal{D}$ with $E \in \mathcal{D}^c$ and $D \in \mathcal{D}^{\le -m}$. Since $i^*D \in \mathcal{D}_F^{\le -m}$ and i^* preserves compactness (see 2.5), the exact triangle $i^*E \to i^*X \to i^*D$ shows that i^*X is in $(\mathcal{D}_F)^-_c$. Now consider the exact triangle $j^*E \to j^*X \to j^*D$ with $j^*D \in \mathcal{D}_U^{\le -m}$. By the first paragraph of the proof, $j^*E \in (\mathcal{D}_U)^-_c$, hence, we can find an exact triangle $\widetilde{E} \to j^*E \to \widetilde{D}$ with $\widetilde{E}$ in $(\mathcal{D}_U)^c$ and $\widetilde{D}$ in $\mathcal{D}_U^{\le -m}$. The octahedral axiom, applied to $\widetilde{E} \to j^*E \to j^*X$, gives an object D' and exact triangles

$\widetilde{E} \to j^*X \to D'$ and $\widetilde{D} \to D' \to j^*D$ in $\mathcal{D}_U$, such that the following diagram is commutative:



Since $\widetilde{D}$ and j^*D are in $\mathcal{D}_U^{\leq -m}$, we know D' is also in $\mathcal{D}_U^{\leq -m}$. The exact triangle $\widetilde{E} \to j^*X \to D'$ now shows that j^*X is in $(\mathcal{D}_U)_c^-$. If we assume further that X is in $\mathcal{D}_c^b = \mathcal{D}_c^- \cap \mathcal{D}^b$, then we have $j^*X \in \mathcal{D}_U^b$ and $i^!X \in (\mathcal{D}_F)^+$ because j^* is t-exact and $i^!$ is left t-exact.

We now show the right sides are contained in the left sides. Assume $i^*X \in (\mathcal{D}_F)_c^-$ and $j^*X \in (\mathcal{D}_U)_c^-$, and fix $m > 0$. By definition, there exists an exact triangle

$$E' \to i^*X \to D'$$

with $E' \in (\mathcal{D}_F)^c \subset \mathcal{D}_F^-$ and $D' \in \mathcal{D}_F^{\leq -m}$. Choose an odd integer $n > 0$ with $\Sigma^n E' \in \mathcal{D}_F^{\leq -m}$. The object $E' \oplus \Sigma^n E'$ vanishes in $K_0((\mathcal{D}_F)^c)$, and [20, Corollary 4.5.14] tells us that $E' \oplus \Sigma^n E'$ must therefore lie in the image $i^*(\mathcal{D}^c)$. And then [20, Proposition 4.4.1], applied to the composite $E' \oplus \Sigma^n E' \to E' \to i^*X$, allows us to find a morphism $E'' \to X$ in $\mathcal{D}$, with E'' in $\mathcal{D}^c$, and such that $i^*(E'' \to X)$ is isomorphic to $E' \oplus \Sigma^n E' \to E' \to i^*X$. Complete $E'' \to X$ to an exact triangle

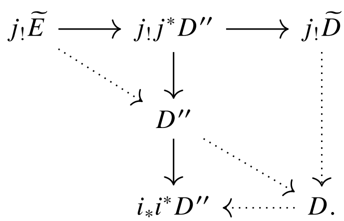
$$E'' \to X \to D''.$$

Since $i^*D'' \cong D' \oplus \Sigma^{n+1}E'$, we deduce that $i^*D'' \in \mathcal{D}_F^{\leq -m}$.

The first paragraph of the proof tells us that $j^*E'' \in (\mathcal{D}_U)_c^-$, while $j^*X \in (\mathcal{D}_U)_c^-$ by hypothesis. Thus, j^*D'' is in $(\mathcal{D}_U)_c^-$. Fix an exact triangle $\widetilde{E} \to j^*D'' \to \widetilde{D}$ with $\widetilde{E} \in (\mathcal{D}_U)^c$ and $\widetilde{D} \in \mathcal{D}_U^{\leq -m}$. We now have the following diagram

$$\begin{array}{ccccc} & & j_!\widetilde{E} & \longrightarrow & j_!j^*D'' & \longrightarrow & j_!\widetilde{D} \\ & & & & \downarrow & & \\ E'' & \longrightarrow & X & \longrightarrow & D'' & & \\ & & & & \downarrow & & \\ & & & & i_*i^*D'', & & \end{array}$$

with $E'' \in \mathcal{D}^c$, $\widetilde{E} \in (\mathcal{D}_U)^c$, and $\widetilde{D} \in \mathcal{D}_U^{\leq -m}$. Applying the octahedral axiom, to $j_!\widetilde{E} \to j_!j^*D'' \to D''$, we find an object D and exact triangles $j_!\widetilde{E} \to D'' \to D$ and $j_!\widetilde{D} \to D \to i_*i^*D''$ in $\mathcal{D}$ that fit into the following commutative diagram:



https://doi.org/10.1017/fms.2023.97 Published online by Cambridge University Press

Since $i^*D'' \in \mathcal{D}_F^{\leq -m}$ and i_* is t-exact, we know $i_*i^*D'' \in \mathcal{D}^{\leq -m}$. Since $\widetilde{D} \in \mathcal{D}_U^{\leq -m}$ and $j_!$ is right t-exact, we know $j_!\widetilde{D} \in \mathcal{D}^{\leq -m}$. Thus, $D \in \mathcal{D}^{\leq -m}$.

Next, we complete the octahedron on $X \longrightarrow D'' \longrightarrow D$ and find an object E with exact triangles $E \longrightarrow X \longrightarrow D$ and $E'' \longrightarrow E \longrightarrow j_!\widetilde{E}$ in $\mathcal{D}$, fitting into the following commutative diagram:

$$\begin{array}{ccccc} E & & \dashrightarrow & & j_!\widetilde{E} \\ \uparrow & \searrow & & & \downarrow \\ E'' & \longrightarrow & X & \longrightarrow & D'' \\ & & & \searrow & \downarrow \\ & & & & D. \end{array}$$

Note that $j_!\widetilde{E}$ is compact because $j_!$ preserves compactness. Since E'' is also compact, so is E. Hence, the exact triangle $E \longrightarrow X \longrightarrow D$ shows that X is in $\mathcal{D}_c^-$. If we now further assume that $i^!X \in (\mathcal{D}_F)^+$ and $j^*X \in (\mathcal{D}_U)^b$, then $X \in \mathcal{D}^b$ by the definition of glued t-structure. □

5. Gluing approximability

The time has come to prove the main theorem. For the reader's convenience, we recall the statement.

Theorem 5.1. *Let $\mathcal{D}_F$ and $\mathcal{D}_U$ be approximable triangulated categories, and let $\mathcal{D}$ be a compactly generated triangulated category. Assume there is a recollement*

$$\mathcal{D}_F \begin{array}{c} \xleftarrow{\ i^*\ } \\ \xrightarrow{\ i_*\ } \\ \xleftarrow{\ i^!\ } \end{array} \mathcal{D} \begin{array}{c} \xleftarrow{\ j_!\ } \\ \xrightarrow{\ j^*\ } \\ \xleftarrow{\ j_*\ } \end{array} \mathcal{D}_U.$$

Then $\mathcal{D}$ has a compact generator G. If, moreover, $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-n}G, G) = 0$ *for $n \gg 0$, then $\mathcal{D}$ is approximable.*

We will prove the theorem via a series of lemmas. If $\mathcal{D}_F$ is approximable with compact generator G_F, then there exists $A > 0$, such that for every object X in $\mathcal{D}^{\leq 0}$, we can find an exact triangle

$$E' \xrightarrow{f'} i^*X \to D'$$

in $\mathcal{D}_F$ with $E' \in \langle G_F \rangle_A^{[-A,A]}$ and $D' \in \mathcal{D}_F^{\leq -1}$. To prove the theorem, we will show how to lift a morphism $E' \xrightarrow{f'} i^*X$ in $\mathcal{D}_F$ to a morphism $E \xrightarrow{f} X$ in $\mathcal{D}$, for increasingly complicated E'.

Lemma 5.2. *Consider a recollement that satisfies the hypotheses of Corollary 3.12, and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure in the preferred equivalence class on $\mathcal{D}$. For every integer $A > 0$, there exists an integer $B > 0$, such that, if $f' : i^*K \longrightarrow i^*X$ is a morphism in $\mathcal{D}_F$ with K compact and $K, X \in \mathcal{D}^{\leq A}$, then there exists an object $Y \in \mathcal{D}^{\leq B}$ and morphisms*

$$K \xleftarrow{\ \psi\ } Y \xrightarrow{\ f\ } X,$$

such that $i^*(\psi)$ is an isomorphism in $\mathcal{D}_F$ and $f'i^*(\psi) = i^*(f)$.

Proof. Choose t-structures for each of $\mathcal{D}_F$, $\mathcal{D}_U$, in the preferred equivalence classes, and glue them to form a t-structure on $\mathcal{D}$. By Corollary 3.12, the glued t-structure on $\mathcal{D}$ is in the preferred equivalence class, hence, equivalent to the given t-structure on $\mathcal{D}$. Since replacing the given t-structure by the equivalent glued one is harmless, let us assume that the t-structure on $\mathcal{D}$ is the one obtained from the gluing.

https://doi.org/10.1017/fms.2023.97 Published online by Cambridge University Press

Now choose a compact generator $G \in \mathcal{D}$ and a compact generator $H \in \mathcal{D}_U$. The object $G \oplus j_! H$ is a compact generator for $\mathcal{D}$, and Lemma 3.5 permits us to choose an integer $C > 0$ with $\mathrm{Hom}_{\mathcal{D}}(G \oplus j_! H, \mathcal{D}^{\leq -C}) = 0$. Assume also that $C \geq A$, where $A > 0$ is the integer given in the lemma. By the weak approximability of $\mathcal{D}_U$ and Lemma 3.7, we may, possibly at the cost of increasing C, assume that every object $Z \in \mathcal{D}_U^{\leq 0}$ admits an exact triangle $E \to Z \to D$ with $E \in \langle H \rangle^{[-C,C]}$ and $D \in \mathcal{D}_U^{\leq -1}$. We assert that in the lemma, we may set $B = 3C - 1$.

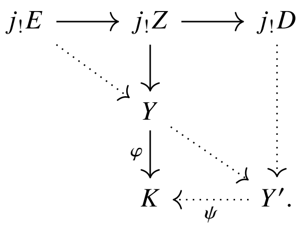
By [20, Proposition 4.4.1], we can represent $f' : i^*K \to i^*X$ in $\mathcal{D}_F$ by a roof in $\mathcal{D}$,

$$K \xleftarrow{\ \varphi\ } Y \xrightarrow{\ \widetilde{f}\ } X,$$

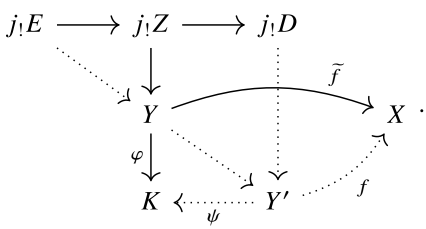
such that Y is compact, $i^*(\varphi)$ is an isomorphism in $\mathcal{D}_F$, and $f' i^*(\varphi) = i^*(\widetilde{f})$. Since $i^*(\varphi)$ is an isomorphism, we can find an exact triangle $j_! Z \to Y \xrightarrow{\varphi} K$ for some $Z \in \mathcal{D}_U$. Moreover, we know $Y \in \mathcal{D}^c \subset \mathcal{D}^-$ and $K \in \mathcal{D}^{\leq A}$. Hence, $j_! Z \in \mathcal{D}^{\leq N}$ for some N, and if $N \leq 3C - 1$, we are done. Assume, therefore, $N \geq 3C$. The isomorphism $Z \cong j^* j_! Z$ tells us that $Z \in \mathcal{D}_U^{\leq N}$. By [16, 2.2.1] (with $m = N + 1 - 3C$ and $F = \Sigma^N Z$), there is an exact triangle $E \to Z \to D$ in $\mathcal{D}_U$ with $E \in \langle H \rangle^{[2C, N+C]}$ and $D \in \mathcal{D}_U^{\leq 3C-1}$. We have the diagram

$$\begin{array}{ccccc} j_!E & \longrightarrow & j_!Z & \longrightarrow & j_!D \\ & & \big\downarrow & & \\ & & Y & & \\ & & {\scriptstyle \varphi}\big\downarrow & & \\ & & K & & \end{array}$$

which we complete to an octahedron, giving an object Y' and exact triangles $j_! E \to Y \to Y'$ and $j_! D \to Y' \xrightarrow{\psi} K$ in $\mathcal{D}$, making the following diagram commutative:



Using that $\mathrm{Hom}_{\mathcal{D}}(j_! H, \mathcal{D}^{\leq -C}) = 0$, we see that there are no nonzero maps from $j_! E \in \langle j_! H \rangle^{[2C, N+C]}$ to $X \in \mathcal{D}^{\leq A} \subset \mathcal{D}^{\leq C}$. Hence, the morphism $\widetilde{f} : Y \to X$ factors via $Y \to Y'$:



We have thus found morphisms

$$K \xleftarrow{\ \psi\ } Y' \xrightarrow{\ f\ } X,$$

https://doi.org/10.1017/fms.2023.97 Published online by Cambridge University Press

such that $i^*(\psi)$ is an isomorphism in $\mathcal{D}_F$ and $f'i^*(\psi) = i^*(f)$. Furthermore, the exact triangle $j_!D \to Y' \to K$ with $j_!D \in \mathcal{D}^{\leq 3C-1}$ and $K \in \mathcal{D}^{\leq A}$ shows $Y' \in \mathcal{D}^{\leq 3C-1}$. This proves the lemma. □

Lemma 5.3. *Consider a recollement that satisfies the hypotheses of Corollary 3.12. Choose a compact generator $G \in \mathcal{D}$, and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure in the preferred equivalence class on $\mathcal{D}$.*

For every integer $A > 0$, there exists an integer $B > 0$, such that, if $f' : E' \to i^*X$ is a morphism in $\mathcal{D}_F$ with $E' \in \mathrm{Coprod}_1(i^*G[-A, A])$ and $X \in \mathcal{D}^{\leq A}$, then there exists a morphism $f : E \to X$ in $\mathcal{D}$ with $E \in \langle G\rangle^{[-B,B]}$, such that $i^*E \cong E'$ and $i^*f \cong f'$.

If $\mathcal{D}_U$ is approximable, then the integer B may be chosen so that $E \in \langle G\rangle_B^{[-B,B]}$.

Proof. Corollary 3.12 allows us to choose t-structures for $\mathcal{D}_F$, $\mathcal{D}_U$ in the preferred equivalence classes, glue them to form a t-structure on $\mathcal{D}$, and replace the given t-structure on $\mathcal{D}$ by the equivalent glued one we have just constructed.

Now choose a compact generator $H \in \mathcal{D}_U$. The object G is a compact generator for $\mathcal{D}$, and Lemma 3.5(1) permits us to choose an integer $C > 0$ with $\mathrm{Hom}_{\mathcal{D}}(G, \mathcal{D}^{\leq -C}) = 0$. Since G is a compact generator in $\mathcal{D}$ and $j_!H$ is compact, [16, Lemma 0.9(i)] allows us to increase C so that $j_!H \in \langle G\rangle_C^{[-C,C]}$. Let $A > 0$ be the integer given in the lemma, and let $A' > 0$ be an integer with $G \in \mathcal{D}^{\leq A'}$; Lemma 5.2 permits us to produce an integer $B' > 0$ so that any morphism $i^*K \to i^*X$, with $K \in \mathcal{D}^c \cap \mathcal{D}^{\leq A+A'}$ and $X \in \mathcal{D}^{\leq A+A'}$, can be represented by a roof with $Y \in \mathcal{D}^{\leq B'}$. At the cost of possibly increasing C, assume $B' \leq C$ and $A + A' \leq C$. Finally, the weak approximability (respectively, approximability) of $\mathcal{D}_U$ and Lemma 3.7 permits us, possibly at the cost of increasing C, to assume that every object $Z \in \mathcal{D}_U^{\leq 0}$ admits an exact triangle $E \to Z \to D$ with $D \in \mathcal{D}_U^{\leq -1}$ and $E \in \langle H\rangle^{[-C,C]}$ (respectively, $E \in \langle H\rangle_C^{[-C,C]}$). We assert that in the lemma, we may set $B = \max(4C, 3C^3 + 1)$.

It suffices to show the lemma for $E' = \Sigma^n i^*G$ with $n \in [-A, A]$. Consider a morphism $f' : \Sigma^n i^*G \to i^*X$ in $\mathcal{D}_F$ with $X \in \mathcal{D}^{\leq A}$. By the choice of C, there exists an object Y in $\mathcal{D}^{\leq C}$ and morphisms

$$\begin{array}{ccc} & Y & \\ {\scriptstyle\psi}\swarrow & & \searrow{\scriptstyle\widetilde{f}} \\ \Sigma^n G & & X, \end{array}$$

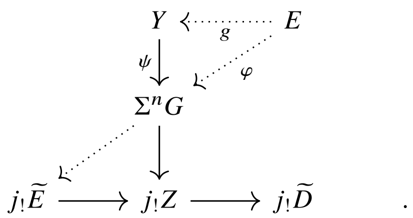
such that $i^*(\psi)$ is an isomorphism in $\mathcal{D}_F$ and $f'i^*(\psi) = i^*(\widetilde{f})$. In particular, we can find an exact triangle $Y \xrightarrow{\psi} \Sigma^n G \to j_!Z$ for some $Z \in \mathcal{D}_U$. Since $\Sigma^n G$ and Y are both in $\mathcal{D}^{\leq C}$, so is $j_!Z$. Hence, $Z \cong j^*j_!Z \in \mathcal{D}_U^{\leq C}$. By [16, 2.2.1 and 2.2.2] (with $m = 3C$ and $F = \Sigma^C Z$), there exists an exact triangle in $\mathcal{D}_U$

$$\widetilde{E} \to Z \to \widetilde{D},$$

with $\widetilde{D} \in \mathcal{D}_U^{\leq -2C}$ and $\widetilde{E} \in \langle H\rangle^{[1-3C,2C]}$, or if $\mathcal{D}_U$ is approximable with $\widetilde{E} \in \langle H\rangle_{3C^2}^{[1-3C,2C]}$. We thus get the following diagram in $\mathcal{D}$:

$$\begin{array}{ccccc} & & Y & & \\ & & \downarrow{\scriptstyle\psi} & & \\ & & \Sigma^n G & & \\ & & \downarrow & & \\ j_!\widetilde{E} & \longrightarrow & j_!Z & \longrightarrow & j_!\widetilde{D}. \end{array}$$

Since $j_!$ is right t-exact, we have $j_!\widetilde{D} \in \mathcal{D}^{\leq -2C}$. Since $\mathrm{Hom}_{\mathcal{D}}(G, \mathcal{D}^{\leq -C}) = 0$, there are no nonzero maps $\Sigma^n G \to j_!\widetilde{D}$. It follows that $\Sigma^n G \to j_!Z$ factors via $j_!\widetilde{E} \to j_!Z$. We thus find a morphism $\Sigma^n G \to j_!\widetilde{E}$, an exact triangle $E \xrightarrow{\varphi} \Sigma^n G \to j_! \widetilde{E}$, and a morphism $E \xrightarrow{g} Y$, such that the following diagram commutes:



Since $j_! H \in \langle G \rangle_C^{[-C,C]}$ and $\widetilde{E}$ is in $\langle H \rangle^{[1-3C,2C]}$ (respectively, $\widetilde{E} \in \langle H \rangle_{3C^2}^{[1-3C,2C]}$), we see that $j_! \widetilde{E} \in \langle G \rangle^{[1-4C,3C]}$ (respectively, $j_! \widetilde{E} \in \langle G \rangle_{3C^3}^{[1-4C,3C]}$). The triangle

$$E \to \Sigma^n G \to j_! \widetilde{E}$$

now shows that $E \in \langle G \rangle^{[-4C,4C]}$ (respectively, $E \in \langle G \rangle_{3C^3+1}^{[-4C,4C]}$). Finally, we let $f := \widetilde{f} g : E \to X$. Now $i^*(\varphi)$ is an isomorphism in $\mathcal{D}_F$ and

$$f' i^*(\varphi) = f' i^*(\psi) i^*(g) = i^*(\widetilde{f} g) = i^*(f). \qquad \square$$

Lemma 5.4. *Consider a recollement that satisfies the hypotheses of Corollary 3.12, and suppose $\mathcal{D}_U$ is approximable. Choose a compact generator $G \in \mathcal{D}$, and let $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ be a t-structure in the preferred equivalence class on $\mathcal{D}$.*

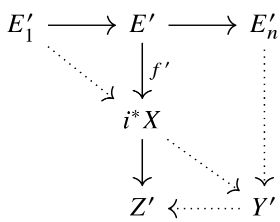
For every pair of integers $A > 0$ and $n > 0$, there exists an integer $B > 0$, such that, if $f' : E' \to i^* X$ is a morphism in $\mathcal{D}_F$ with $E' \in \mathrm{Coprod}_n(i^* G[-A, A])$ and $X \in \mathcal{D}^{\le A}$, then there exists a morphism $f : E \to X$ in $\mathcal{D}$ with $E \in \langle G \rangle_B^{[-B,B]}$, such that $i^* E \cong E'$ and $i^* f \cong f'$.

Proof. Corollary 3.12 allows us to choose t-structures for $\mathcal{D}_F$, $\mathcal{D}_U$ in the preferred equivalence classes, glue them to form a t-structure on $\mathcal{D}$, and replace the given t-structure on $\mathcal{D}$ by the equivalent glued one we have just constructed.

The case $n = 1$ is shown in Lemma 5.3. We proceed by induction on n, and suppose the lemma holds for some $n \ge 1$. Fix $A > 0$, choose $\widetilde{B}$ as in Lemma 5.3, meaning any morphism $f' : E' \to i^* X$ in the category $\mathcal{D}_F$, with $E' \in \mathrm{Coprod}_1(i^* G[-A, A])$ and $X \in \mathcal{D}^{\le A}$, is isomorphic to $i^* f$ for some morphism $f : E \to X$ in $\mathcal{D}$ with $E \in \langle G \rangle_{\widetilde{B}}^{[-\widetilde{B},\widetilde{B}]}$. Increasing $\widetilde{B}$ if necessary, assume $\widetilde{B} \ge A$ and $G \in \mathcal{D}^{\le \widetilde{B}}$.

Using the induction hypothesis, choose B', such that, if $f' : E' \to i^* X$ is a morphism in $\mathcal{D}_F$ with $E' \in \mathrm{Coprod}_n(i^* G[-2\widetilde{B}, 2\widetilde{B}])$ and $X \in \mathcal{D}^{\le 2\widetilde{B}}$, then there exists a morphism $f : E \to X$ in $\mathcal{D}$ with $E \in \langle G \rangle_{B'}^{[-B',B']}$, such that $i^* E \cong E'$ and $i^* f \cong f'$.

Consider a morphism $f' : E' \to i^* X$ with $E' \in \mathrm{Coprod}_{n+1}(i^* G[-A, A])$ and $X \in \mathcal{D}^{\le A}$. By definition, there exists an exact triangle $E'_1 \to E' \to E'_n$ with $E'_1 \in \mathrm{Coprod}_1(i^* G[-A, A])$ and $E'_n \in \mathrm{Coprod}_n(i^* G[-A, A])$. We can complete the octahedron on $E'_1 \to E' \to i^* X$ to find exact triangles $E' \to i^* X \to Z'$, $E'_1 \to i^* X \to Y'$, and $E'_n \to Y' \to Z'$, such that the diagram

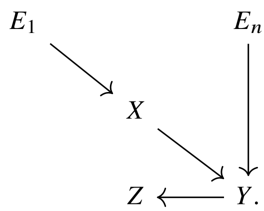


https://doi.org/10.1017/fms.2023.97 Published online by Cambridge University Press

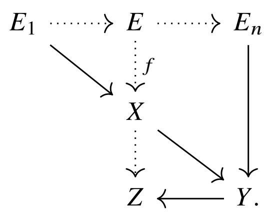
commutes. Since E'_1 is in $\mathrm{Coprod}_1(i^*G[-A,A])$ and X is in $\mathcal{D}^{\le A}$, we can apply our choices to find an exact triangle $E_1 \to X \to Y$ in $\mathcal{D}$ with $E_1 \in \langle G\rangle_{\widetilde{B}}^{[-\widetilde{B},\widetilde{B}]}$ and such that $i^*(E_1 \to X \to Y) \cong E'_1 \to i^*X \to Y'$. Note that

$$Y \in \mathcal{D}^{\le A} * \langle G\rangle_{\widetilde{B}}^{[-\widetilde{B}-1,\widetilde{B}-1]} \subseteq \mathcal{D}^{\le 2\widetilde{B}}.$$

Next, we consider the morphism $E'_n \to Y' \cong i^*Y$ with $E'_n \in \mathrm{Coprod}_n(i^*G[-A,A]) \subset \mathrm{Coprod}_n(i^*G[-2\widetilde{B},2\widetilde{B}])$ and $Y \in \mathcal{D}^{\le 2\widetilde{B}}$. By our choices of integers, there exists an exact triangle $E_n \to Y \to Z$ in $\mathcal{D}$ with $E_n \in \langle G\rangle_{B'}^{[-B',B']}$ and such that $i^*(E_n \to Y \to Z) \cong E'_n \to Y' \to Z'$. We now have the following diagram:



Completing the octahedron, we find an object E and exact triangles $E_1 \to E \to E_n$ and $E \xrightarrow{f} X \to Z$ in $\mathcal{D}$, so that the following diagram is commutative:



Since there are isomorphisms $i^*(X \to Y) \cong i^*X \to Y'$ and $i^*(Y \to Z) \cong Y' \to Z'$, there is an isomorphism $i^*f \cong f'$. The lemma follows since

$$E \in \langle G\rangle_{\widetilde{B}}^{[-\widetilde{B},\widetilde{B}]} * \langle G\rangle_{B'}^{[-B',B']} \subset \langle G\rangle_{\widetilde{B}+B'}^{[-\widetilde{B}-B',\widetilde{B}+B']}.$$

□

Lemma 5.5. *Consider a recollement that satisfies the hypotheses of Corollary 3.12, and suppose $\mathcal{D}_U$ is approximable. Choose a compact generator $G \in \mathcal{D}$, and let $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ be a t-structure in the preferred equivalence class on $\mathcal{D}$.*

For any integer $A > 0$, there exists an integer $B > 0$, such that, if $f' : E' \to i^*X$ is a morphism in $\mathcal{D}_F$ with $E' \in \langle i^*G\rangle_A^{[-A,A]}$ and $X \in \mathcal{D}^{\le A}$, then there exists a morphism $f : E \to X$ in $\mathcal{D}$ with $E \in \langle G\rangle_B^{[-B,B]}$, such that $i^*E \cong E'$ and $i^*f \cong f'$.

Proof. Fix $A > 0$. In [15, Corollary 1.12], it is shown that

$$\langle i^*G\rangle_A^{[-A,A]} \subset \mathrm{Coprod}_{2A}(i^*G[-A-1,A]) \subset \mathrm{Coprod}_{2A}(i^*G[-2A,2A]).$$

By Lemma 5.4, there exists $B > 0$, such that if $f' : E' \to i^*X$ is a morphism in $\mathcal{D}_F$ with $E' \in \mathrm{Coprod}_{2A}(i^*G[-2A,2A])$ and $X \in \mathcal{D}^{\le 2A}$, then there exists $E \in \langle G\rangle_B^{[-B,B]}$ and a morphism $f : E \to X$ in $\mathcal{D}$ with $i^*E \cong E'$ and $i^*f \cong f'$. □

Now that we understand how to lift morphisms in $\mathcal{D}_F$ to $\mathcal{D}$, we proceed with the proof of the theorem:

https://doi.org/10.1017/fms.2023.97 Published online by Cambridge University Press

Proof of Theorem 5.1. Let G_F and G_U be compact generators of $\mathcal{D}_F$ and $\mathcal{D}_U$, respectively. By [20, Theorem 4.4.9], there exists a compact object G' in $\mathcal{D}$, such that G_F is a direct summand of i^*G'. It follows that $G := G' \oplus j_!G_U$ is a compact generator for $\mathcal{D}$.

Suppose now that $\mathrm{Hom}_{\mathcal{D}}(\Sigma^n G, G) = 0$ for $n \gg 0$; we're in the situation of Corollary 3.12, allowing us to choose t-structures for $\mathcal{D}_F$, $\mathcal{D}_U$ in the preferred equivalence classes, glue them to form a t-structure on $\mathcal{D}$, and we're guaranteed that the glued t-structure on $\mathcal{D}$ is in the preferred equivalence class. Choose a compact generator $H \in \mathcal{D}_U$, and choose an integer $A > 0$ so that $G \in \mathcal{D}^{\leq A}$, $\mathrm{Hom}_{\mathcal{D}}(G, \mathcal{D}^{\leq -A}) = 0$, and $j_!H \in \langle G\rangle_A^{[-A,A]}$. Recalling that $\mathcal{D}_U$ and $\mathcal{D}_F$ are approximable and remembering Lemma 3.7, we may, by increasing A if necessary, also assume that every object $X \in \mathcal{D}_U^{\leq 0}$ admits an exact triangle $E \to X \to D$ with $E \in \langle H\rangle_A^{[-A,A]}$ and $D \in \mathcal{D}_U^{\leq -1}$, and every object $X \in \mathcal{D}_F^{\leq 0}$ admits an exact triangle $E \to X \to D$ with $E \in \langle i^*G\rangle_A^{[-A,A]}$ and $D \in \mathcal{D}_F^{\leq -1}$. By Lemma 5.5, there exists an integer $B > 0$, such that, if $f' : E' \to i^*X$ is any morphism in $\mathcal{D}_F$ with $E' \in \langle i^*G\rangle_A^{[-A,A]}$ and $X \in \mathcal{D}^{\leq A}$, then there exists a morphism $f : E \to X$ in $\mathcal{D}$ with $E \in \langle G\rangle_B^{[-B,B]}$, such that $i^*f \cong f'$.

Let X be an object of $\mathcal{D}^{\leq 0}$, hence, $i^*X \in \mathcal{D}_F^{\leq 0}$. We find an exact triangle

$$E' \xrightarrow{f'} i^*X \to D'$$

with $E' \in \langle i^*G\rangle_A^{[-A,A]}$ and $D' \in \mathcal{D}_F^{\leq -1}$. By the choices of integers above, there exists a morphism $E \xrightarrow{f} X$ in $\mathcal{D}$ with $E \in \langle G\rangle_B^{[-B,B]}$, such that $i^*f \cong f'$. Complete f to a triangle

$$E \xrightarrow{f} X \to D''.$$

Since $E \in \langle G\rangle_B^{[-B,B]}$ and $X \in \mathcal{D}^{\leq 0}$, we have $D'' \in \mathcal{D}^{\leq A+B}$, and thus, $j^*D'' \in \mathcal{D}_U^{\leq A+B}$. By [16, 2.2.1] (with $F = \Sigma^{A+B} j^*D''$ and $m = A + B + 1$), we find an exact triangle

$$\widetilde{E} \to j^*D'' \to \widetilde{D}$$

in $\mathcal{D}_U$, such that $\widetilde{E} \in \langle H\rangle_{(A+B+1)A}^{[-A,2A+B]}$ and $\widetilde{D} \in \mathcal{D}_U^{\leq -1}$, obtaining a diagram

$$\begin{array}{ccccc}
 & & j_!\widetilde{E} & \longrightarrow j_!j^*D'' \longrightarrow & j_!\widetilde{D} \\
 & & & \downarrow & \\
E & \xrightarrow{f} & X & \longrightarrow & D'' \\
 & & & \downarrow & \\
 & & & & i_*i^*D''.
\end{array}$$

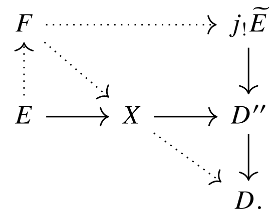
Completing the octahedron on $j_!\widetilde{E} \to j_!j^*D'' \to D''$, we find an object D with exact triangles $j_!\widetilde{E} \to D'' \to D$ and $j_!\widetilde{D} \to D \to i_*i^*D''$ in $\mathcal{D}$, that fit into the following commutative diagram:

$$\begin{array}{ccccc}
j_!\widetilde{E} & \longrightarrow & j_!j^*D'' & \longrightarrow & j_!\widetilde{D} \\
 & \searrow & \downarrow & & \vdots \\
 & & D'' & & \vdots \\
 & & \downarrow & \searrow & \downarrow \\
 & & i_*i^*D'' & \longleftarrow & D.
\end{array}$$

https://doi.org/10.1017/fms.2023.97 Published online by Cambridge University Press

We note that $j_! \widetilde{D} \in \mathcal{D}^{\leq -1}$ since $\widetilde{D} \in \mathcal{D}_U^{\leq -1}$ and $j_!$ is right t-exact. Furthermore, $i_* i^* D'' \in \mathcal{D}^{\leq -1}$, since $i^* D'' \cong D' \in \mathcal{D}_F^{\leq -1}$ and i_* is t-exact. Thus, $D \in \mathcal{D}^{\leq -1}$.

Next, we complete the octahedron on $X \to D'' \to D$, finding an object F with exact triangles $F \to X \to D$ and $E \to F \to j_! \widetilde{E}$ in $\mathcal{D}$, that fit into the following commutative diagram:



Since $E \in \langle G \rangle_B^{[-B,B]}$ and

$$j_! \widetilde{E} \in \langle j_! H \rangle_{(A+B+1)A}^{[-A,2A+B]} \subset \langle G \rangle_{(A+B+1)A^2}^{[-2A,3A+B]},$$

we see that $F \in \langle G \rangle_{(A+B+1)A^2+B}^{[-3A-B,3A+B]}$. Thus, the exact triangle $F \to X \to D$ proves the approximability of $\mathcal{D}$. □

6. Gluing dg-categories

In this section, we fix a commutative ring R, and assume everything in sight is R-linear. Recall that an *algebraic triangulated category* is a triangulated category that is equivalent to the stable category of a Frobenius exact category. By [7, Theorem 4.3], if $\mathcal{D}$ is a cocomplete algebraic triangulated category with a single compact generator, then $\mathcal{D}$ is equivalent to the derived category of a dg-algebra. Thus, determining which algebraic triangulated categories are approximable is equivalent to determining which dg-algebras have an approximable derived category.

If A is a dg-algebra, and its derived category $D(A)$ is approximable, then $H^n(A) = \operatorname{Hom}_{D(A)}(\Sigma^{-n} A, A) = 0$ for $n \gg 0$ by Lemma 3.5. By [16, Remark 3.3], if $H^n(A) = 0$ for all $n > 0$, then $D(A)$ is approximable. Below, we show that if A is an "upper triangular algebra" constructed from dg-algebras B and C, where B, C have approximable derived categories, then so does A. This gives examples of dg-algebras with approximable derived categories and cohomology in arbitrarily high degree.

Corollary 6.1. *Let B and C be dg-algebras, M a $B \otimes C^{op}$-module, and A the dg-matrix algebra of this data, that is,*

$$A = \begin{bmatrix} B & M \\ 0 & C \end{bmatrix}, \qquad d_A = \begin{bmatrix} d_B & d_M \\ 0 & d_C \end{bmatrix}.$$

Assume that M is cohomologically bounded above, that is, $H^n(M) = 0$ for $n \gg 0$, and that M has a semiprojective B-resolution that is also a $B \otimes C^{op}$-module. Then if $D(B)$ and $D(C)$ are approximable, so is $D(A)$. Moreover, $D(A)_c^-$ and $D(A)_c^b$ are glued from their analogues over B and C.

Remark 4. The condition on the resolution of M is satisfied if R is a field, or more generally, if M and B are flat over R as graded modules. Indeed, one can then construct a semiprojective B-resolution of M using the bar construction, and this resolution retains the right C-action from M.

Proof. By [12, Section 3] (see also [6]), there is a recollement:

$$D(B) \underset{\xleftarrow{\;i^!\;}}{\overset{\xleftarrow{\;i^*\;}}{\xrightarrow{\;i_*\;}}} D(A) \underset{\xleftarrow{\;j_*\;}}{\overset{\xleftarrow{\;j_!\;}}{\xrightarrow{\;j^*\;}}} D(C).$$

https://doi.org/10.1017/fms.2023.97 Published online by Cambridge University Press

Note that B and C are cohomologically bounded above, since their derived categories are approximable, and therefore A is also cohomologically bounded above. Thus, we may apply Theorem 5.1, with $A = G$, to see $D(A)$ is approximable. By [12, Lemma 3.11], $j^*(A) \cong C$, and thus is in $D(C)_c^-$, so we may apply Proposition 4.3. □

Acknowledgments. The authors would like to thank an anonymous referee for many helpful suggestions and comments that led to improvements on an earlier draft. The work of Jesse Burke was supported by the Australian Research Council (grant number DP150102313). Amnon Neeman was mostly supported by the Australian Research Council (grants numbers DP150102313 and DP200102537) but partly also by the Deutsche Forschungsgemeinschaft (SFB-TRR 358/1 2023 - 491392403). Bregje Pauwels was supported by the Australian Research Council (grants numbers DP150102313 and DP180102563).

Competing interest. The authors have no competing interest to declare.

References

[1] L. A. Tarrío, A. J. López and M. J. S. Salorio, 'Construction of t-structures and equivalences of derived categories', *Trans. Amer. Math. Soc.* **355**(6) (2003), 2523–2543.

[2] A. A. Beilinson, J. Bernstein and P. Deligne, 'Faisceaux pervers', in Analysis and topology on singular spaces, I (Luminy, 1981), *Astérisque* vol. 100 (Societe Mathematique de France, Paris, 1982), 5–171.

[3] A. Bondal and M. van den Bergh, 'Generators and representability of functors in commutative and noncommutative geometry', *Mosc. Math. J.* **3**(1) (2003), 1–36, 258.

[4] A. I. Efimov, 'Categorical smooth compactifications and generalized Hodge-to-de Rham degeneration', Preprint, 2018, arXiv:1805.09283v1.

[5] A. I. Efimov, 'Homotopy finiteness of some dg categories from algebraic geometry', Preprint, 2017, arXiv:1308.0135v2.

[6] P. Jorgensen, 'Recollement for differential graded algebras', *J. Algebra* **299**(2) (2006), 589–601.

[7] B. Keller, 'Deriving DG categories', *Ann. Sci. École Norm. Sup. (4)* **27**(1) (1994), 63–102.

[8] A. Kuznetsov, 'Semiorthogonal decompositions in algebraic geometry', in *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II* (Kyung Moon Sa, Seoul, 2014), 635–660.

[9] A. Kuznetsov and V. A. Lunts, 'Categorical resolutions of irrational singularities', *Int. Math. Res. Not. IMRN* **2015**(13) (2015), 4536–4625.

[10] V. A. Lunts, 'Categorical resolution of singularities', *J. Algebra* **323**(10) (2010), 2977–3003.

[11] V. A. Lunts, 'Categorical resolutions, poset schemes, and Du Bois singularities', *Int. Math. Res. Not. IMRN* **2012**(19) (2012), 4372–4420.

[12] D. Maycock, 'Derived equivalences of upper triangular differential graded algebras', *Comm. Algebra* **39**(7) (2011), 2367–2387.

[13] A. Neeman, 'Finite approximations as a tool for studying triangulated categories', Preprint, 2022, arXiv:2211.06587.

[14] A. Neeman, 'Approximable triangulated categories', in *Representations of algebras, geometry and physics*, *Contemporary Mathematics* 769 (American Mathematical Society, Providence, RI, 2021), 111–155.

[15] A. Neeman, 'Strong generators in $D^{\mathrm{perf}}(X)$ and $D^b_{\mathrm{coh}}(X)$', *Ann. of Math. (2)* **193**(3) (2021), 689–732.

[16] A. Neeman, 'Triangulated categories with a single compact generator and a brown representability theorem', Preprint, 2018, arXiv:1804.02240v1.

[17] A. Neeman, 'The category $[\mathcal{T}^c]^{\mathrm{op}}$ as functors on $\mathcal{T}_c^b$', Preprint, 2018, arXiv:1806.05777.

[18] A. Neeman, 'The categories $\mathcal{T}^c$ and $\mathcal{T}_c^b$ determine each other', Preprint, 2018, arXiv:1806.06471.

[19] A. Neeman, 'The Grothendieck duality theorem via Bousfield's techniques and Brown representability', *J. Amer. Math. Soc.* **9**(1) (1996), 205–236.

[20] A. Neeman, 'Triangulated categories', in *Annals of Mathematics Studies* vol 148 (Princeton University Press, Princeton, NJ, 2001), 1–449.

[21] D. O. Orlov, 'Gluing of categories and Krull-Schmidt partners', *Uspekhi Mat. Nauk* **71**(3) (2016), 594–596.

[22] D. Orlov, 'Smooth and proper noncommutative schemes and gluing of DG categories,' *Adv. Math.* **302** (2016), 59–105.

[23] R. Rouquier, 'Dimensions of triangulated categories', *J. K-Theory* **1**(2) (2008), 193–256.

[24] G. Tabuada, 'Higher K-theory via universal invariants', *Duke Math. J.* **145**(1) (2008), 121–206.

[25] M. van den Bergh, 'Non-commutative crepant resolutions', in *The Legacy of Niels Henrik Abel* (Springer, Berlin, 2004), 749–770.

https://doi.org/10.1017/fms.2023.97 Published online by Cambridge University Press