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SIMPLE STABLY PROJECTIONLESS C^*-ALGEBRAS ARISING AS CROSSED PRODUCTS

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ABSTRACT. A large class of simple stably projectionless C^*-algebras are shown to arise as crossed products of simple purely infinite C^*-algebras by trace scaling one-parameter automorphism groups. The Elliott invariant is computed for this class.

The program proposed by George A. Elliott for the classification of stable nuclear simple C^*-algebras (see [El2] for its current form) has seen considerable progress in the classification of infinite C^*-algebras and finite C^*-algebras with nontrivial projections. In this article we investigate the class of finite C^*-algebras which are stably projectionless. The first example of a C^*-algebra in this class was constructed by Blackadar in [B1]. Our examples consist of crossed products of the form $O_n \rtimes \mathbb{R}$ studied by the first author in [Ks] as well as analogs of the construction in [B1] involving inductive limits of mapping tori of simple C^*-algebras with trace scaling automorphisms.

There is in each case a natural trace scaling action by a one-parameter automorphism group for which the crossed product is purely infinite. It follows that a C^*-algebra with such a trace scaling action must be projectionless (see [Co2]). Tensoring such an example with any finite simple C^*-algebra (*i.e.* one with a trace) yields another such example. The ultimate objective of this work is to understand in what sense an analog of Takesaki's continuous decomposition of type III factors holds for simple nuclear purely infinite C^*-algebras (see Remark 4.5) and thereby, perhaps, to determine which values of the Elliott invariant occur for simple stably projectionless C^*-algebras arising as crossed products of purely infinite C^*-algebras by one-parameter automorphism groups.

The organization of this paper is as follows. In Section 1 we deal with preliminary matters including a brief review of basic facts concerning one-parameter automorphism groups, traces and the mapping torus. In Section 2 the mapping torus construction is presented; it is shown in Theorem 2.4 that this construction gives rise to simple stably projectionless C^*-algebras with arbitrary 2-divisible K-theory and "unique" trace—moreover, these examples are all stably isomorphic to crossed products of purely infinite C^*-algebras by one-parameter automorphism groups. By tensoring with Thomsen's examples (see [Th]) one obtains C^*-algebras for which the cone of traces may be identified with the cone over an arbitrary metrizable Choquet simplex (see Corollary 2.5). In Section 3 the relationship between KMS states on a unital C^*-algebra for a given one-parameter automorphism group and traces on the crossed product is discussed (see Theorem 3.2);

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the analogous relationship is well known in Tomita-Takesaki theory. It follows that if the C^*-algebra is simple and there is a KMS state at some non-zero inverse temperature then the crossed product is stably projectionless (see Corollary 3.4). Finally in Section 4 we return to the original example which was the starting point for this work: we show in the case considered, where $O_n \rtimes \mathbb{R}$ is simple and has a trace, that the trace is unique up to scalar multiple.

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1. **Preliminaries.**

1.1. *Notation and conventions.* Let $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{T}$ denote the sets of integers, integers mod n, rational numbers, real numbers, complex numbers and those of modulus one. Denote by $\mathcal{K}$ the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space. The algebra of $n \times n$ matrices over the complex numbers is denoted M_n; let M_{n^∞} denote the UHF algebra given as the infinite tensor product

$$M_{n^\infty} = \bigotimes_{j=1}^{\infty} M_n.$$

Given a C^*-algebra A, a strongly continuous homomorphism $\alpha: \mathbb{R} \longrightarrow \mathrm{Aut}(A)$ will be referred to as a *one-parameter automorphism group* and the pair (A, α) will be referred to as an $\mathbb{R}$-*dynamical system*. Denote the associated crossed product $A \rtimes_\alpha \mathbb{R}$; there is a dense subalgebra which may be identified with $L^1(\mathbb{R}, A)$ with the usual product and involution (see [Pd, Section 7.6]). The dual action $\hat{\alpha}$ yields another $\mathbb{R}$-dynamical system for which the restriction to $L^1(\mathbb{R}, A)$ is given by the formula: $\big(\hat{\alpha}_s(f)\big)(t) = e^{its}f(t)$.

Given an $\mathbb{R}$-dynamical system (A, α) there is a strictly continuous unitary group $t \longmapsto u_t \in M(A \rtimes_\alpha \mathbb{R})$ and a natural embedding $A \longrightarrow M(A \rtimes_\alpha \mathbb{R})$ so that for every $a \in A$ and $t \in \mathbb{R}$ one has $\alpha_t(a) = u_t a u_t^*$. Moreover, $A \rtimes_\alpha \mathbb{R}$ is generated by elements of the form $\lambda(f)a$ where $\lambda(f) = \int_{\mathbb{R}} f(t)u_t \, dt$ for $f \in L^1(\mathbb{R})$ and $a \in A$. Let H denote the unbounded multiplier that generates the group u_t (see [Co1, App. 5]); one has $u_t = e^{itH}$ and $\lambda(f) = \hat{f}(-H)$ where $\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)e^{-its} \, dt$. Let ϵ_s denote the character on $\mathbb{R}$ given by $\epsilon_s(t) = e^{its}$ (regarded as a function with values in the complex numbers).

By a *trace* on a (not necessarily unital) C^*-algebra A is meant a densely defined lower semicontinuous trace. Note that each such trace is defined on the *Pedersen ideal* $\mathcal{J}(A)$ (see [Pd, Proposition 5.6.7]); recall that $\mathcal{J}(A)$ is the minimal dense ideal in A.

DEFINITION 1.2. Let (A, α) be an $\mathbb{R}$-dynamical system; a trace τ on A is said to have the *scaling property* with respect to α if there is β so that $\tau \circ \alpha_t = e^{-\beta t}\tau$ for all t and $a \in \mathcal{J}(A)$ (we will refer to β as the *scaling parameter*). It is said to have the *nontrivial scaling property* if $\beta \neq 0$. A C^*-algebra A is said to be *stably projectionless* if $A \otimes \mathcal{K}$ contains no nontrivial projections.

REMARK 1.3. As Connes observed in [Co2, pp. 586–587] a C^*-algebra with a faithful trace that has the nontrivial scaling property with respect to some one-parameter automorphism group is stably projectionless. For if τ and α are as above and $p \in A$ is a projection, then p is homotopic to $\alpha_1(p)$; hence $\tau(p) = \tau(\alpha_1(p))$, so by the scaling property $\tau(p) = 0$ and thus $p = 0$. This observation is similar to that of Blackadar in [B1] (*cf.* Fact 2.3.1). Note that a trace on a simple C^*-algebra is necessarily faithful.

Given an automorphism γ of a C^*-algebra C let M_γ denote the *mapping torus* for the pair (C, γ) (see [B2, Section 10.3]):

$$\begin{aligned} M_\gamma &:= \{f \in C([0,1], C) \mid f(1) = \gamma(f(0))\} \\ &\cong \{f : \mathbb{R} \longrightarrow C \mid f(t+1) = \gamma(f(t)),\ \forall t \in \mathbb{R}\}. \end{aligned}$$

REMARK 1.4. There are several facts concerning M_γ which may be needed below:

1. There is a natural one-parameter automorphism group $\alpha: \mathbb{R} \longrightarrow \mathrm{Aut}(M_\gamma)$ given by $\alpha_t(f)(s) = f(s-t)$ for $s, t \in \mathbb{R}$ and $f \in M_\gamma$.

2. The crossed products $C \rtimes_\gamma \mathbb{Z}$ and $M_\gamma \rtimes_\alpha \mathbb{R}$ are strong Morita equivalent (see [G, Theorem 17]).

3. $K_i(M_\gamma) \cong K_{i+1}(C \rtimes_\gamma \mathbb{Z})$ for $i \in \mathbb{Z}_2$ (see [Co1, Corollary 6]).

4. If there is an isomorphism of C with $B \rtimes_\beta \mathbb{T}$ which identifies γ with the generator of the automorphism group dual to β then by [B2, Proposition 10.3.2] one has $M_\gamma \cong B \rtimes_\beta \mathbb{R}$, where β is here regarded as a periodic one-parameter automorphism group. Hence, by Takai duality (see [Pd, Theorem 7.9.3]):

$$C \rtimes_\gamma \mathbb{Z} \cong B \otimes \mathcal{K} \cong M_\gamma \rtimes_\alpha \mathbb{R}$$

(since α is dual to β regarded as an action of $\mathbb{R}$).

2. The mapping torus construction.

The mapping torus construction outlined below is a modification of Blackadar's construction [B1] of a simple stably projectionless C^*-algebra as an inductive limit of mapping tori of simple C^*-algebras with trace scaling automorphisms; we use twisted double embeddings rather than the twice around embeddings used in [B1].

The following proposition is essentially contained in [Rø1, Theorem 3.6]:

PROPOSITION 2.1. *For any countable abelian groups G_0, G_1 and $\lambda > 1$ there is a stable simple $A\mathbb{T}$-algebra C of real rank zero with unique (up to scalar multiple) trace φ and an automorphism γ on C so that $\varphi(\gamma(c)) = \lambda^{-1}\varphi(c)$ for all $c \in \mathcal{J}(C)$ and $K_i(C \rtimes_\gamma \mathbb{Z}) \cong G_i$ for $i \in \mathbb{Z}_2$.*

This follows immediately from the proof of [Rø1, Theorem 3.6]. The theorem is stated for unital C and corner endomorphism γ, but by [Pa] there is a embedding of C into $C \otimes \mathcal{K}$ and an automorphism of $C \otimes \mathcal{K}$ which "extends" γ in the obvious sense (the extension of the trace to $C \otimes \mathcal{K}$ is scaled by the automorphism in like manner); moreover, the crossed product by the endomorphism is embedded into the full crossed product in a way compatible with this embedding.

REMARK 2.2. By [Rø1, Theorem 3.1] the resulting crossed product, $C \rtimes_\gamma \mathbb{Z}$, is simple and purely infinite.

FACT 2.3. With notation as above we record some useful properties of the associated mapping torus M_γ:

1. M_γ has no nontrivial projections (see [B1]); indeed, if $p = p(t)$ is a projection in M_γ then p(0) is homotopic to $p(1)$ so $\varphi\big(p(0)\big) = \varphi\big(p(1)\big) = \lambda^{-1}\varphi\big(p(0)\big)$. Hence, $p = 0$.
2. $\mathrm{Prim}(M_\gamma) \cong \mathbb{T}$ since every primitive ideal is the kernel of some point evaluation.
3. For any $f \in \mathcal{J}(M_\gamma)$ the function $t \longmapsto \lambda^t\varphi\big(f(t)\big)$ is continuous and periodic and thus may be regarded as an element in $C(\mathbb{T})$. Moreover, for any measure μ on $\mathbb{T}$ there is a trace τ_μ on M_γ so that $\tau_\mu(f) = \int_{\mathbb{T}} \lambda^t\varphi\big(f(t)\big)\, d\mu(t)$ for all $f \in \mathcal{J}(M_\gamma)$.
4. Let μ be normalized Haar measure on $\mathbb{T}$ with associated trace τ_μ, then a straightforward calculation reveals:

$$\tau_\mu\big(\alpha_t(f)\big) = \lambda^t\tau_\mu(f)$$

for $t \in \mathbb{R}$ and $f \in \mathcal{J}(M_\gamma)$.

Fix an irrational $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and consider the embedding $\sigma\colon M_\gamma \longrightarrow M_2 \otimes M_\gamma$ given by:

$$\sigma(f)(t) = \begin{pmatrix} f(t) & 0 \\ 0 & f(t-\theta) \end{pmatrix}$$

for $t \in \mathbb{R}$ and $f \in M_\gamma$. Set $A_n = M_{2^n} \otimes M_\gamma$ for $n \geq 0$ and define $\sigma_n\colon A_n \longrightarrow A_{n+1}$ by $\sigma_n(x \otimes f) = x \otimes \sigma(f)$ for $x \in M_{2^n}$ and $f \in M_\gamma$. One notes that $\mathcal{J}(A_n) = M_{2^n} \otimes \mathcal{J}(M_\gamma)$ and hence $\sigma_n\big(\mathcal{J}(A_n)\big) \subset \mathcal{J}(A_{n+1})$.

THEOREM 2.4. *With notation as above $A = \varinjlim A_n$ is a simple stably projectionless C^*-algebra with unique (up to scalar multiple) trace τ. Let $\alpha\colon \mathbb{R} \longrightarrow \mathrm{Aut}(A)$ be the one-parameter automorphism group which leaves A_n invariant and which restricts to the canonical action (cf. Remark 1.4.1) for each n; then $A \rtimes_\alpha \mathbb{R}$ is simple and purely infinite and τ has the scaling property with respect to α:*

$$\tau\big(\alpha_t(f)\big) = \lambda^t\tau(f).$$

Moreover, $K_i(A) \cong G_i \otimes \mathbb{Z}[\frac{1}{2}]$ for $i \in \mathbb{Z}_2$.

PROOF. Note first that A is simple by [BK, Proposition 1.3], since rotation by θ defines a minimal homeomorphism on $\mathbb{T} \cong \mathrm{Prim}(M_\gamma)$ (see Remark 2.3.2). The canonical one-parameter automorphism group acting on M_γ is extended to A_n in the obvious way; it is clear that these actions are compatible with the above embeddings and thus define an action of $\mathbb{R}$ on A which we again denote by α. Note that $(\sigma_n)_* = \sigma_* = 2(\mathrm{id}_{M_\gamma})_*$ as elements of $\mathrm{End}\big(K_i(M_\gamma)\big)$ for $i \in \mathbb{Z}_2$ (here we have identified $K_i(A_n) = K_i(M_\gamma)$ in the natural way); thus:

$$K_i(A) = K_i(\varinjlim A_n) \cong \varinjlim K_i(A_n) \cong K_i(M_\gamma) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \cong G_{i+1} \otimes \mathbb{Z}\left[\frac{1}{2}\right]$$

for $i \in \mathbb{Z}_2$. Since $M_\gamma \rtimes_\alpha \mathbb{R}$ and $C \rtimes_\gamma \mathbb{Z}$ are strong Morita equivalent and $C \rtimes_\gamma \mathbb{Z}$ is simple and purely infinite [Rø1, Theorem 3.1] it follows that

$$A_n \rtimes_\alpha \mathbb{R} = (M_{2^n} \otimes M_\gamma) \rtimes_\alpha \mathbb{R} \cong M_{2^n} \otimes (M_\gamma \rtimes_\alpha \mathbb{R})$$

is simple and purely infinite. Hence

$$A \rtimes_\alpha \mathbb{R} = \varinjlim A_n \rtimes_\alpha \mathbb{R}$$

is simple and purely infinite. It remains to define a trace on each A_n in a consistent way, to show that the resulting trace is unique (up to scalar multiple) and then to show it has the scaling property with respect to α. For $x \otimes f \in M_{2^n} \otimes M_\gamma = A_n$ define:

$$\tau_n(x \otimes f) := \frac{\mathrm{Tr}(x)}{(1+\lambda^\theta)^n} \int_{\mathbb{T}} \lambda^t \varphi\big(f(t)\big)\, dt,$$

where Tr denotes the usual (unnormalized) trace on M_{2^n}. To check compatibility, that is, $\tau_n = \tau_{n+1} \circ \sigma_n$, it suffices to show that:

$$\int_{\mathbb{T}} \lambda^t \varphi\big(f(t-\theta)\big)\, dt = \lambda^\theta \int_{\mathbb{T}} \lambda^t \varphi\big(f(t)\big)\, dt,$$

but this follows from the translation invariance of Haar measure. By the compatibility of the traces τ_n one may define a tracial function on the union $\bigcup_n \mathcal{J}(A_n)$ which forms a dense subalgebra. Let τ denote the unique lower semicontinuous extension of this tracial function to $\mathcal{J}(A)$ and observe that it is a trace and that it has the scaling property with respect to α since each τ_n has.

The uniqueness of τ follows by a calculation involving the Fourier coefficients of a family of measures. By Fact 2.3.3 each trace on the mapping torus arises from a measure on $\mathbb{T}$. Thus any trace on A arises from a sequence of measures μ_n corresponding to the traces on A_n subject to a compatibility condition. Given a sequence of probability measures μ_n, the associated traces on A_n will be compatible if:

$$\int_{\mathbb{T}} \lambda^t \varphi\big(f(t)\big)\, d\mu_n(t) = \frac{1}{1+\lambda^\theta} \int_{\mathbb{T}} \lambda^t \varphi\big(f(t) + f(t-\theta)\big)\, d\mu_{n+1}(t)$$

for $f \in \mathcal{J}(M_\gamma)$. That is, for any $n \geq 0$ and $g \in C(\mathbb{T})$ one must have:

$$\int_{\mathbb{T}} g(t)\, d\mu_n(t) = \frac{1}{1+\lambda^\theta} \int_{\mathbb{T}} g(t) + \lambda^\theta g(t-\theta)\, d\mu_{n+1}(t).$$

Setting $\widehat{\mu_n}(k) = \int_{\mathbb{T}} e^{-2\pi i k t}\, d\mu_n(t)$, it follows that:

$$\widehat{\mu_n}(k) = \frac{1}{1+\lambda^\theta}\Big(\widehat{\mu_{n+1}}(k) + \lambda^\theta e^{2\pi i k\theta}\widehat{\mu_{n+1}}(k)\Big)$$

for all $k \in \mathbb{Z}$ and $n \geq 0$. Hence one obtains for all $k \in \mathbb{Z}$ and $m, n \geq 0$:

$$\widehat{\mu_n}(k) = \Big(\frac{1+\lambda^\theta e^{2\pi i k\theta}}{1+\lambda^\theta}\Big)^m \widehat{\mu_{n+m}}(k).$$

Since $|\widehat{\mu_l}(k)| \leq 1$ for all k and l, and

$$\left|\frac{1+\lambda^\theta e^{2\pi i k\theta}}{1+\lambda^\theta}\right| < 1$$

for $k \neq 0$, it follows that (recall that μ_n is a probability measure):

$$\widehat{\mu_n}(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $n \geq 0$. Thus μ_n is normalized Haar measure. ∎

COROLLARY 2.5. *Given two countable abelian groups D_0, D_1 for which $D_i \cong D_i \otimes \mathbb{Z}[\frac{1}{2}]$, $\lambda > 1$, and a metrizable Choquet simplex X, there is a simple stably projectionless C^*-algebra A such that $K_i(A) \cong D_i$ for $i \in \mathbb{Z}_2$ and for which the cone of traces may be identified with CX, the cone over X; moreover, there is a one-parameter automorphism group, $\alpha\colon \mathbb{R} \longrightarrow \mathrm{Aut}(A)$, so that $A \rtimes_\alpha \mathbb{R}$ is simple and purely infinite and if τ is any trace on A we have:*

$$\tau\big(\alpha_t(f)\big) = \lambda^t \tau(f).$$

PROOF. Let C be the C^*-algebra constructed as in the above theorem with $K_*(C) \cong D_*$, one-parameter automorphism group $\gamma\colon \mathbb{R} \longrightarrow \mathrm{Aut}(C)$, trace ω (unique up to scalar multiple) and $\lambda > 1$; then ω has the nontrivial scaling property with respect to γ with scaling parameter $-\ln\lambda$. By [Th, Theorem 3.9] there is a simple inductive limit of interval algebras B for which the simplex of traces is affinely homeomorphic to X and $K_*(B) \cong \mathbb{Z}[\frac{1}{2}] \oplus 0$. Now set $A = B \otimes C$ (note that both B and C are nuclear so there is a unique cross-norm), then A is simple and by the Künneth formula (see [S] or [B1, Theorem 23.1.3]) one has: $K_i(A) \cong D_i \otimes \mathbb{Z}[\frac{1}{2}] \cong D_i$. Given a trace τ on A there is a constant r and a trace φ on B such that $\tau(b \otimes c) = r\varphi(b)\omega(c)$ (and every functional of this form is a trace); define $\alpha\colon \mathbb{R} \longrightarrow \mathrm{Aut}(A)$, by $\alpha_t(b \otimes c) = b\gamma_t(c)$. One checks that every trace on A has the required scaling property with respect to this one-parameter group. Thus A is stably projectionless. Since $C \rtimes_\gamma \mathbb{R}$ is simple, nuclear and purely infinite, it follows that

$$C \rtimes_\gamma \mathbb{R} \cong (C \rtimes_\gamma \mathbb{R}) \otimes \mathcal{O}_\infty$$

(*cf.* [Kr, Corollary 22]); hence one has

$$A \rtimes_\alpha \mathbb{R} = (B \otimes C) \rtimes_{\mathrm{id}\otimes\gamma} \mathbb{R} \cong B \otimes (C \rtimes_\gamma \mathbb{R}) \cong (A \rtimes_\alpha \mathbb{R}) \otimes \mathcal{O}_\infty.$$

Thus $A \rtimes_\alpha \mathbb{R}$ is simple and purely infinite (*cf.* [Kr, Corollary 22]). ∎

REMARK 2.6. For any n one can modify the mapping torus construction used in the above theorem to obtain examples with arbitrary n-divisible K-theory and essentially unique trace; further, tensoring with the appropriate AI-algebra as in the above corollary gives a simple stably projectionless C^*-algebra with the same K-theory and cone of traces identified with the cone over an arbitrary metrizable Choquet simplex.

REMARK 2.7. Using [BEH] one may also construct examples of simple stably projectionless C^*-algebras for which there are uncountably many extreme traces with a one-parameter automorphism group which scales each by a different factor. Consider the abelian group $G = \mathbb{Z}[x, x^{-1}, (1-x)^{-1}]$ and let S be a non-empty compact subset of $(0, 1)$; then G becomes a simple dimension group when endowed with the order $p > 0$ if $p(s) > 0$ for all $s \in S$. There is up to isomorphism a unique simple stable AF algebra C with this dimension group. The set of extremal traces of C may be identified with S by means of a map $s \in S \longmapsto \omega_s$ for which one has $(\omega_s)_*(p) = p(s)$ for all $p \in G \cong K_0(C)$. Furthermore, there is a $\gamma \in \mathrm{Aut}(C)$ so that $\gamma_*(p) = px(1-x)^{-1}$ for all $p \in G$. It follows that $C \rtimes_\gamma \mathbb{Z}$ is simple and that for any element $c \in \mathcal{J}(C)$ one has $\omega_s\big(\gamma(c)\big) = s(1-s)^{-1}\omega_s(c)$.

One proceeds as above by forming the inductive limit of $M_{2^n} \otimes M_\gamma$ which we again denote by A together with the associated one-parameter automorphism group denoted by α; note that both A and $A \rtimes_\alpha \mathbb{R}$ are simple. For each $s \in S$ we construct a trace τ_s on A starting with ω_s. Observe that τ_s has the scaling property with respect to α:

$$\tau_s\big(\alpha_t(a)\big) = \Big(\frac{1-s}{s}\Big)^t \tau_s(a).$$

Thus A is stably projectionless if $S \neq \{\frac{1}{2}\}$. Further, by [Rø2, Theorem 2.1] it follows that $C \rtimes_\gamma \mathbb{Z}$ and hence $A \rtimes_\alpha \mathbb{R}$ is purely infinite if and only if $\frac{1}{2} \notin S$. Note that if $\frac{1}{2} \in S$ but $S \neq \{\frac{1}{2}\}$, then A is stably projectionless but $A \rtimes_\alpha \mathbb{R}$ is not infinite (since it has a trace).

Using the construction of [BEK] in place of [BEH] one may obtain a simple stably projectionless C^*-algebra together with a one-parameter automorphism group for which the cone of traces scaled by a given factor can be fairly arbitrary. In both [BEH] and [BEK] the traces on C were shown to give rise to KMS-states on $C \rtimes_\gamma \mathbb{Z}$ at various inverse temperatures (related to the factors discussed above) for the dual automorphism group regarded as a periodic one-parameter automorphism group.

3. KMS **states and traces.**

DEFINITION 3.1. Let (A, α) be an $\mathbb{R}$-dynamical system and β a real number; then a state ω on A is said to satisfy the KMS *condition for α at inverse temperature β* or, briefly, to be an (α, β)-KMS *state* if

$$\omega\big(b\alpha_{i\beta}(a)\big) = \omega(ab)$$

for all $a, b \in A$ with a entire for α (see [BEK] or [Pd, Section 8.12]). Note that if ω is a KMS state for α and $\beta \neq 0$ then ω is α-invariant.

The following theorem may be well known to specialists. It is closely related to the construction of the dual trace in [Ta, Lemma 8.2] where the role played by the scaling property is critical in obtaining the continuous decomposition of type III factors; an analogous dual result for $\mathbb{Z}$-actions may be found in [BEK, Lemma 6.3]. As we were unable to find a precise reference, we provide a proof.

THEOREM 3.2. *Let (A, α) be an $\mathbb{R}$-dynamical system with A unital; let T denote the cone of traces on $A \rtimes_\alpha \mathbb{R}$, and set*

$$T_\beta = \{\tau \in T \mid \tau \circ \hat{\alpha}_s = e^{-\beta s}\tau, s \in \mathbb{R}\}$$

where $\beta \in \mathbb{R}$ and $\hat{\alpha}$ is the dual action by $\hat{\mathbb{R}} = \mathbb{R}$. Then for every α-invariant (α, β)-KMS state ω on A (note that the requirement that ω be α-invariant is redundant if $\beta \neq 0$) there is a trace $\tau = \tau_\omega \in T_\beta$ such that:

$$\tau_\omega\big(\lambda(f)a\big) = \omega(a)\int_{\mathbb{R}} \hat{f}(s)e^{-\beta s}\,ds,$$

for all $f \in L^1(\mathbb{R})$ with $\hat{f} \in C_c(\mathbb{R})$ and $a \in A$. Moreover, for every $\tau \in T_\beta$ there is an (α, β)-KMS state ω on A (necessarily α-invariant) so that

$$\tau\big(\lambda(f)a\big) = \tau\big(\lambda(f)\big)\omega(a),$$

for all $f \in L^1(\mathbb{R})$ with $\hat{f} \in C_c(\mathbb{R})$ and $a \in A$; further, there is a constant c_τ so that $\tau = c_\tau\tau_\omega$.

PROOF. We first show that traces on the crossed product with the scaling property with respect to the dual automorphism group arise from KMS states. Fix a nonzero $\tau \in T_\beta$; for nonzero $f \in L^1(\mathbb{R})$ with $\hat{f} \in C_c(\mathbb{R})^+$, it follows from the scaling property that $\tau(f) > 0$. Given such an f define a state ω_f on A by

$$\tau\big(\lambda(f)a\big) = \tau\big(\lambda(f)\big)\omega_f(a);$$

note that if $a \geq 0$ then $\tau\big(\lambda(f)a\big) = \tau\big(\lambda(f)^{1/2}a\lambda(f)^{1/2}\big) \geq 0$ so ω_f is positive. By the scaling property one has:

$$e^{-\beta s}\tau\big(\lambda(f)\big) = \tau\Big(\hat{\alpha}_s\big(\lambda(f)\big)\Big) = \tau\big(\lambda(\epsilon_s f)\big);$$

and for any $a \in A$ one has:

$$e^{-\beta s}\tau\big(\lambda(f)a\big) = \tau\big(\lambda(\epsilon_s f)a\big) = \tau\big(\lambda(\epsilon_s f)\big)\omega_{\epsilon_s f}(a) = e^{-\beta s}\tau\big(\lambda(f)\big)\omega_{\epsilon_s f}(a);$$

it follows that $\omega_f = \omega_{\epsilon_s f}$. For any $\epsilon > 0$ there is a $\delta > 0$ such that if $|s - t| < \delta$, one has

$$|\hat{f}(s) - \hat{f}(t)| < \epsilon.$$

Choose $h_\delta \in L^1(\mathbb{R})$ with $\widehat{h_\delta} \in C_c(\mathbb{R})^+$ such that $\operatorname{supp}\widehat{h_\delta} \subset [-\delta, \delta]$ and

$$\sum_{n \in \mathbb{Z}} \widehat{h_\delta}(s - n\delta) = 1,$$

for all $s \in \mathbb{R}$. Let $g_\delta \in L^1(\mathbb{R})$ be defined by

$$\widehat{g_\delta}(s) = \sum_{n \in \mathbb{Z}} \hat{f}(n\delta)\widehat{h_\delta}(s - n\delta).$$

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Then $\widehat{g_\delta} \in C_c(\mathbb{R})$ and $\|\hat{f} - \widehat{g_\delta}\| < \epsilon$; further with $\omega_\delta = \omega_{h_\delta}$, one has:

$$\tau\big(\lambda(g_\delta)a\big) = \sum_{n\in\mathbb{Z}} \hat{f}(n\delta)\tau\big(\lambda(\epsilon_{n\delta}h_\delta)\big)\omega_\delta(a) = \Big(\sum_{n\in\mathbb{Z}} \hat{f}(n\delta)e^{-\beta n\delta}\Big)\tau\big(\lambda(h_\delta)\big)\omega_\delta(a)$$

for all $a \in A$. Since

$$\big|\tau\big(\lambda(f)a\big) - \tau\big(\lambda(g_\delta)a\big)\big| \leq \|a\|\tau\big(|\lambda(f - g_\delta)|\big)$$

and $\operatorname{supp}(\hat{f} - \widehat{g_\delta})$ is bounded as $\delta \downarrow 0$, it follows that $\tau(\lambda(g_\delta)a) \longrightarrow \tau\big(\lambda(f)a\big)$; hence, as $\delta \downarrow 0$, $\tau\big(\lambda(h_\delta)\big)/\delta$ converges to a constant $c > 0$ and ω_δ has a weak* limit point ω. Moreover,

$$\tau\big(\lambda(f)a\big) = c\,\omega(a)\int_{\mathbb{R}} \hat{f}(s)e^{-\beta s}\,ds.$$

For $f_k \in L^1(\mathbb{R})$ with $\hat{f_k} \in C_c(\mathbb{R})$ and $a_k \in A$ for $1 \leq k \leq m$, let $f \in L^1(\mathbb{R}, A)$ be defined by $f(t) = \sum_{k=1}^m f_k(t)a_k$. Setting

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-its}f(t)\,dt = \sum_{k=1}^{m} \hat{f_k}(s)a_k$$

yields an element in $C_c(\mathbb{R}, A)$; regarding f as an element of $A \rtimes_\alpha \mathbb{R}$ one has:

$$\tau(f) = \tau\Big(\int_{\mathbb{R}} u_t f(t)\,dt\Big) = c\int_{\mathbb{R}} e^{-\beta s}\omega\big(\hat{f}(s)\big)\,ds.$$

We show that this formula holds for all $f \in L^1(\mathbb{R}, A)$ with $\operatorname{supp}\hat{f}$ compact. Given such an f there is a $g \in L^1(\mathbb{R})$ with $\hat{g} \in C_c(\mathbb{R})^+$ and $\hat{g} = 1$ on $\operatorname{supp}\hat{f}$; since $L^1(\mathbb{R}, A) \cong L^1(\mathbb{R})\otimes A$ with projective cross-norm, there is a sequence $\{f_n\}$ in the algebraic tensor product $L^1(\mathbb{R}) \odot A$ such that $f_n \longrightarrow f$ in $L^1(\mathbb{R}, A)$. Note that $g * f_n$ is an element of the above form and $g * f_n \longrightarrow g * f = f$ where $*$ is the usual convolution. Since $g * f_n = \lambda(g)f_n$, one has:

$$\tau\big(\lambda(g)^{1/2}f_n\lambda(g)^{1/2}\big) = \tau\big(\lambda(g)f_n\big) = c\int_{\mathbb{R}} e^{-\beta s}\omega\big(\hat{g}(s)\hat{f_n}(s)\big)\,ds.$$

The right hand side converges to

$$c\int_{\mathbb{R}} e^{-\beta s}\omega\big(\hat{f}(s)\big)\,ds$$

and since $x \longmapsto \tau\big(\lambda(g)^{1/2}x\lambda(g)^{1/2}\big)$ is bounded, the left hand side converges to

$$\tau\big(\lambda(g)^{1/2}f\lambda(g)^{1/2}\big) = \tau(f).$$

Thus one obtains

$$\tau(f) = c\int_{\mathbb{R}} e^{-\beta s}\omega\big(\hat{f}(s)\big)\,ds$$

for all $f \in L^1(\mathbb{R}, A)$ with $\operatorname{supp}\hat{f}$ compact.

We now show that ω is an (α,β)-KMS state. Let $f \in L^1(\mathbb{R})$ with $\hat{f} \in C_c(\mathbb{R})^+$ and $\tau\big(\lambda(f)\big) = 1$, and let $a, b \in A$ where a has compact α-spectrum. Then

$$\begin{aligned}\omega\big(b\alpha_{i\beta}(a)\big) &= \tau\big(\lambda(f)b\alpha_{i\beta}(a)\big) = \tau\big(\alpha_{i\beta}(a)\lambda(f)b\big)\\ &= \tau\Big(\int_{\mathbb{R}} f(t)u_t\alpha_{i\beta-t}(a)b\,dt\Big)\\ &= c\int_{\mathbb{R}} e^{-\beta s}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-irs}h(r)\,dr\,ds\end{aligned}$$

where $h(t) = \omega\big(\alpha_{i\beta-t}(a)b\big)f(t)$; since $\hat{h} \in C_c(\mathbb{R})$, one may define $h(z)$ for $z \in \mathbb{C}$ by $h(z) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{izs}\hat{h}(s)\,ds$. Continuing with our calculation we obtain:

$$\begin{aligned}\omega\big(b\alpha_{i\beta}(a)\big) &= c\int_{\mathbb{R}} e^{-\beta s}\hat{h}(s)\,ds = c\sqrt{2\pi}h(i\beta) = c\sqrt{2\pi}\omega(ab)f(i\beta)\\ &= c\omega(ab)\int_{\mathbb{R}} e^{-\beta s}\hat{f}(s)\,ds = \omega(ab)\tau\big(\lambda(f)\big)\\ &= \omega(ab);\end{aligned}$$

this formula may then be extended by continuity to the case that a is entire. It follows that ω is an (α,β)-KMS state. One checks that ω is α-invariant.

Conversely, let ω be an α-invariant (α,β)-KMS state; for $f \in L^1(\mathbb{R}, A)$ with $\hat{f} \in C_c(\mathbb{R}, A)$ the dual weight is defined by:

$$\tilde{\omega}(f) = \int_{\mathbb{R}} \omega\big(\hat{f}(s)\big)\,ds;$$

if $\beta = 0$ this is already a trace and we set $\tau_\omega(f) = \tilde{\omega}(f)$; otherwise, following the proof of Lemma 8.2 in [Ta] we set $\tau_\omega(f) = \tilde{\omega}(e^{\beta H}f)$. Noting that the first is a special case of the second and rewriting we obtain:

$$\tau_\omega(f) = \tilde{\omega}(e^{\beta H}f) = \int_{\mathbb{R}} e^{-\beta s}\omega\big(\hat{f}(s)\big)\,ds.$$

The scaling property follows by inspection. ∎

REMARK 3.3. There is a natural generalization of the above theorem to non-unital C^*-algebras formulated in terms of KMS weights rather than KMS states: for each β the map, $\omega \longmapsto \tau_\omega$, defines an affine homeomorphism from the cone of α-invariant (α,β)-KMS weights on A to T_β. The proof requires more careful attention to technical detail and as we shall not require the greater generality we omit it.

COROLLARY 3.4. *Let (A,α) be an $\mathbb{R}$-dynamical system with A unital. If A is $\mathbb{R}$-simple, in particular if A is simple, then for any α-invariant (α,β)-KMS state ω the associated trace τ_ω is faithful; hence, if A is $\mathbb{R}$-simple and there is a KMS-state for α at some nonzero inverse temperature β, then $A \rtimes_\alpha \mathbb{R}$ is stably projectionless.*

PROOF. If A is $\mathbb{R}$-simple then $A \rtimes_\alpha \mathbb{R}$ is $\hat{\mathbb{R}}$-simple (see [Pd, Proposition 7.9.6]). For any KMS state ω the associated trace τ_ω has the scaling property and hence the closure of

$$\{x \in \mathcal{J}(A \rtimes_\alpha \mathbb{R}) \mid \tau_\omega(x^*x) = 0\}$$

is an $\hat{\alpha}$-invariant ideal; thus τ_ω is faithful. If, furthermore, $\beta \neq 0$ then τ_ω has the nontrivial scaling property and it follows by Remark 1.3 that A is stably projectionless. ∎

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REMARK 3.5. Let Γ be an r-discrete principal groupoid with left Haar system and $c \in Z^1(\Gamma, \mathbb{R})$ a continuous real-valued 1-cocycle. Let α^c denote the associated real action on $C^*(\Gamma)$ (the dense subalgebra $C_c(\Gamma)$ is α^c-invariant and for all $f \in C_c(\Gamma)$ one has $\big(\alpha_t^c(f)\big)(\gamma) = e^{itc(\gamma)}f(\gamma)$ for $\gamma \in \Gamma$ and $t \in \mathbb{R}$). Let μ be a quasi-invariant probability measure on the unit space Γ^0 so that the modular function (or Radon-Nikodym derivative) of μ is $e^{-\beta c}$ for some $\beta \in \mathbb{R}$, then μ is said to satisfy the (c, β)-KMS condition (see [Re, Definition I.3.15]); the associated state ω_μ defined by $\omega_\mu(f) = \int_{\Gamma^0} f(x)\, d\mu(x)$ for $f \in C_c(\Gamma)$ is an (α^c, β)-KMS state (see [Re, Proposition II.5.4]). The associated crossed product is the groupoid C^*-algebra of the skew-product $\Gamma(c) = \Gamma \times_c \mathbb{R}$; note that $\Gamma(c)^0 = \Gamma^0 \times \mathbb{R}$ and that $\mu \times e^{-\beta t}\, dt$ is an invariant measure. This invariant measure defines a trace on $C^*\big(\Gamma(c)\big)$ which has the scaling property with respect to the dual automomorphism group as in the above theorem.

4. **Uniqueness of the trace on $O_n \rtimes \mathbb{R}$.** For $n \geq 2$ let O_n denote the universal C^*-algebra generated by n isometries $S_1, \ldots, S_n$ satisfying the relation

$$\sum_{j=1}^{n} S_j S_j^* = 1;$$

O_n is called the *Cuntz algebra* and is known to be simple and purely infinite (see [Cu1]). Given $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, there is by universality a one-parameter automorphism group $\alpha \colon \mathbb{R} \longrightarrow \mathrm{Aut}(O_n)$ given by:

$$\alpha_t(S_j) = e^{it\lambda_j} S_j.$$

By Cuntz's calculation of the K-theory of O_n (see [Cu2]) and Connes' analog of the Thom isomorphism (see [Co1]) one may compute the K-theory of the crossed product:

$$K_0(O_n \rtimes_\alpha \mathbb{R}) = 0$$
$$K_1(O_n \rtimes_\alpha \mathbb{R}) = \mathbb{Z}_{n-1}.$$

Elliott observed that a KMS state on O_n (the existence of which is considered in [Ev] and [OP]) gives rise to a trace on $O_n \rtimes_\alpha \mathbb{R}$; this together with the above K-theory calculation allowed him to conclude that $O_n \rtimes_\alpha \mathbb{R}$ is (in some cases) simple and stably projectionless (see [El1], [Cu3]). Note that $O_n \rtimes_\alpha \mathbb{R}$ is simple if and only if either λ_j are all nonzero of the same sign and $\{\lambda_1, \ldots, \lambda_n\}$ generates $\mathbb{R}$ as a closed subgroup or $\{\lambda_1, \ldots, \lambda_n\}$ generates $\mathbb{R}$ as a closed subsemigroup (see [Ks]).

THEOREM 4.1. *With α as above, $O_n \rtimes_\alpha \mathbb{R}$ is simple and has a trace if and only if λ_j are all nonzero of the same sign and $\{\lambda_1, \ldots, \lambda_n\}$ generates $\mathbb{R}$ as a closed subgroup. In this case, $O_n \rtimes_\alpha \mathbb{R}$ is stably projectionless; moreover, the trace is unique up to scalar multiple and has the scaling property with respect to $\hat{\alpha}$ where the scaling parameter β is defined by*

$$\sum_{j=1}^{n} e^{-\beta\lambda_j} = 1.$$

PROOF. By [Ev, Proposition 2.2] this condition is necessary and sufficient for the existence of an (α,β)-KMS state, which is then necessarily unique; moreover, such a β exists precisely when all the λ_j have the same sign. By Theorem 3.2 there is a trace which has the scaling property with respect to $\hat{\alpha}$ with scaling parameter β (simplicity follows by [Ks]). It follows that $O_n \rtimes_\alpha \mathbb{R}$ is stably projectionless since $\beta \neq 0$ (see Corollary 3.4). Conversely, assume that $O_n \rtimes_\alpha \mathbb{R}$ is simple and has a trace; it remains to show that λ_j are are all nonzero of the same sign and that the trace is then completely determined (up to scalar multiple).

Let τ be a trace on $O_n \rtimes_\alpha \mathbb{R}$—note that τ is faithful and that we may assume without loss of generality that τ is factorial. Since $\lambda(f) \in \mathcal{J}(O_n \rtimes_\alpha \mathbb{R})$ for all $f \in L^1(\mathbb{R})$ with $\hat{f} \in C_c(\mathbb{R})$, there is a measure μ on $\mathbb{R}$ such that

$$\tau\big(\lambda(f)\big) = \int_{\mathbb{R}} \hat{f}(s)\, d\mu(s)$$

for all such f. Note that

$$\begin{aligned}\lambda(f)S_j &= \int_{\mathbb{R}} f(t)u_t S_j\, dt\\ &= S_j \int_{\mathbb{R}} f(t)e^{i\lambda_j t}u_t\, dt\\ &= S_j\lambda\big(\epsilon_{\lambda_j}f\big);\end{aligned}$$

thus for $p = (p_1,\ldots,p_k)$ and $S_p = S_{p_1}\cdots S_{p_k}$ one has

$$\lambda(f)S_p = S_p\lambda(\epsilon_{E(p)}f)$$

where $E(p) = \sum_{j=1}^{|p|} \lambda_{p_j}$ and $|p| = k$. Fix a nonzero element $f \in L^1(\mathbb{R})$ with $\hat{f} \in C_c(\mathbb{R})$ and define a faithful positive linear functional ω_f on the UHF subalgebra of O_n generated by elements of the form $S_pS_q{}^*$ with $|p| = |q|$ by

$$\omega_f(S_pS_q{}^*) = \tau(\lambda(f)^*\lambda(f)S_pS_q{}^*).$$

Set $F(x) = \int_{\mathbb{R}} |\hat{f}(s-x)|^2\, d\mu(s)$ and note that F is a positive continuous function defined on $\mathbb{R}$. Further one has:

$$\begin{aligned}\omega_f(S_pS_q{}^*) &= \tau\big(S_p\lambda(\epsilon_{E(p)}f)^*\lambda(\epsilon_{E(p)}f)S_q{}^*\big)\\ &= \tau\big(\lambda(\epsilon_{E(p)}f)S_q{}^*S_p\lambda(\epsilon_{E(p)}f)^*\big)\\ &= \delta_{pq}\tau\big(\lambda(\epsilon_{E(p)}f)\lambda(\epsilon_{E(p)}f)^*\big)\\ &= \delta_{pq}\int_{\mathbb{R}} \big|\hat{f}\big(s-E(p)\big)\big|^2\, d\mu(s)\\ &= \delta_{pq}F\big(E(p)\big).\end{aligned}$$

By identifying $S_pS_q{}^*$ with

$$e_{p_1q_1}\otimes e_{p_2q_2}\otimes\cdots\otimes e_{p_kq_k} \in M_{n^k}\otimes\mathbb{C}\cdot 1 \subset M_{n^\infty},$$

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it follows that ω_f is invariant under the permutation of factors of M_{n^∞}.

By the following lemma we may assume that ω_f is factorial. For $i \neq j$ let σ_{ij} denote the automorphism of M_{n^∞} obtained by permuting the i-th and j-th factors (fixing all others) and let σ_i denote the embedding of M_n into the i-th factor of M_{n^∞}. Then for all $a \in M_n$ and $i \geq 0$, $\left\{\sigma_{ij}\big(\sigma_i(a)\big)\right\}_{j>i}$ forms a central sequence in M_{n^∞}; hence, $\pi_{\omega_f}\Big(\sigma_{ij}\big(\sigma_i(a)\big)\Big)$ converges weakly to $\omega_f(1)^{-1}\omega_f\big(\sigma_i(a)\big) \cdot 1$ in the GNS representation π_{ω_f}. Define a state ω_0 on M_n by $\omega_0(a) = \omega_f(1)^{-1}\omega_f\big(\sigma_i(a)\big)$ for $a \in M_n$ (note that ω_f is invariant under σ_{ij} so ω_0 is well defined); it follows that

$$\lim_{j\to\infty} \omega_f\Big(\sigma_{ij}\big(\sigma_i(a)\big)b\Big) = \omega_0(a)\omega_f(b)$$

for $a \in M_n$ and $b \in M_{n^\infty}$; hence for $a_1, \ldots, a_k \in M_n$

$$\begin{aligned}\omega_f\big(\sigma_1(a_1)\cdots\sigma_k(a_k)\big) &= \lim_{j\to\infty} \omega_f \circ \sigma_{1j}\big(\sigma_1(a_1)\cdots\sigma_k(a_k)\big)\\ &= \omega_0(a_1)\omega_f\big(\sigma_2(a_2)\cdots\sigma_k(a_k)\big)\\ &= \omega_f(1)\omega_0(a_1)\cdots\omega_0(a_k).\end{aligned}$$

It follows that ω_f is given as the infinite product, $\omega_f = \omega_f(1) \otimes_{i=1}^{\infty} \omega_0$. There are then non-negative numbers $r_1, \ldots, r_n$ for which $\sum_{j=1}^{n} r_j = 1$ and

$$\omega_f\big(S_pS_p{}^*\big) = \omega_f(1)\prod_{j=1}^{k} r_{p_j};$$

note that $0 < r_j < 1$ since ω_f is faithful and $n \geq 2$. Hence for any non-negative integers $N_1, \ldots, N_n$,

$$F\Big(\sum_{j=1}^{n} N_j\lambda_j\Big) = F(0)\prod_{j=1}^{n} r_j^{N_j}$$

(note that $\lambda_j \neq 0$) or

$$\int_{\mathbb{R}} \Big|\hat{f}\Big(s - \sum_{j=1}^{n} N_j\lambda_j\Big)\Big|^2 \, d\mu(s) = \int_{\mathbb{R}} |\hat{f}(s)|^2 \, d\mu(s) \prod_{j=1}^{n} r_j^{N_j}.$$

In particular one obtains that

$$\int_{\mathbb{R}} \Big|\hat{f}\Big(s + t - \sum_{j=1}^{n} N_j\lambda_j\Big)\Big|^2 \, d\mu(s) = \int_{\mathbb{R}} |\hat{f}(s+t)|^2 \, d\mu(s) \prod_{j=1}^{n} r_j^{N_j}$$

for all $t \in \mathbb{R}$. Replacing t by $t - \sum_{j=1}^{n} M_j$ one obtains:

$$\int_{\mathbb{R}} \Big|\hat{f}\Big(s + t - \sum_{j=1}^{n} N_j\lambda_j\Big)\Big|^2 \, d\mu(s) \prod_{j=1}^{n} r_j^{M_j} = \int_{\mathbb{R}} \Big|\hat{f}\Big(s + t - \sum_{j=1}^{n} M_j\lambda_j\Big)\Big|^2 \, d\mu(s) \prod_{j=1}^{n} r_j^{N_j}.$$

Thus for any $K_j \in \mathbb{Z}$, it follows that

$$\int_{\mathbb{R}} \Big|\hat{f}\Big(s - \sum_{j=1}^{n} K_j\lambda_j\Big)\Big|^2 \, d\mu(s) = \int_{\mathbb{R}} |\hat{f}(s)|^2 \, d\mu(s) \prod_{j=1}^{n} r_j^{K_j}$$

or, equivalently

$$F\Big(\sum_{j=1}^{n} K_j\lambda_j\Big) = F(0)\prod_{j=1}^{n} r_j^{K_j}.$$

Hence, by the continuity of F at 0, if $\sum_{j=1}^{n} K_j^{(m)}\lambda_j \longrightarrow 0$ then $\sum_{j=1}^{n} K_j^{(m)} \ln r_j \longrightarrow 0$.

CLAIM. *It follows that there is a $\tilde{\beta} \in \mathbb{R}$ such that $\ln r_j = -\tilde{\beta}\lambda_j$ or $r_j = e^{-\tilde{\beta}\lambda_j}$.*

For each pair λ_i, λ_j, if λ_i/λ_j is rational, then there are $k, l \in \mathbb{Z} \setminus \{0\}$, such that $k\lambda_i + l\lambda_j = 0$ and $k \ln r_i + l \ln r_j = 0$, that is,

$$\frac{\lambda_i}{\lambda_j} = \frac{\ln r_i}{\ln r_j}.$$

If λ_i/λ_j is irrational, there are two sequences m_k, n_k of integers such that:

$$|n_k\lambda_i - m_k\lambda_j| < \frac{|\lambda_j|}{n_k}.$$

If $\{(x_i, x_j) \in \mathbb{R}^2 \mid x_i\lambda_i + x_j\lambda_j = 0\}$ and $\{(x_i, x_j) \in \mathbb{R}^2 \mid x_i \ln r_i + x_j \ln r_j = 0\}$ are not the same, this will soon lead to a contradiction.

Since $\sum_{j=1}^{n} r_j = 1$, one must have that all $\lambda_j > 0$ or all $\lambda_j < 0$ and $\tilde{\beta} = \beta$. Since the subgroup generated by $\{\lambda_j\}_{j=1}^{n}$ is dense in $\mathbb{R}$ one obtains that

$$\int_{\mathbb{R}} |\hat{f}(s-t)|^2 \, d\mu(s) = \int_{\mathbb{R}} |\hat{f}(s)|^2 \, d\mu(s) e^{-\beta t}.$$

This implies that

$$d\mu(\cdot + t) = e^{-\beta t}\, d\mu(\cdot).$$

It follows that $e^{\beta s}\, d\mu(s)$ is translation invariant and thus proportional to Lebesgue measure. Hence

$$d\mu(s) = ce^{-\beta s}\, ds$$

for some $c \in \mathbb{R}$. Thus if $f \in L^1(\mathbb{R})$ with $\hat{f} \in C_c(\mathbb{R})^+$ and $|p| = |q|$, one has

$$\tau(\lambda(f)S_pS_q{}^*) = c\delta_{pq}e^{-\beta E(p)} \int_{\mathbb{R}} \hat{f}(s)e^{-\beta s}\, ds.$$

By linearity the above equality holds for all $f \in L^1(\mathbb{R})$ with $\hat{f} \in C_c(\mathbb{R})$.

CLAIM. *If $|p| \neq |q|$, then $\tau\big(\lambda(f)S_pS_q{}^*\big) = 0$.*

Suppose $|p| > |q|$ (the case $|p| < |q|$ follows similarly). Then

$$\tau\big(\lambda(f)S_pS_q{}^*\big) = \tau(S_p\lambda\big(\epsilon_{E(p)}f)S_q{}^*\big) = \tau\big(\lambda(\epsilon_{E(p)}f)S_q{}^*S_p\big);$$

since $|p| > |q|$, either $S_q{}^*S_p = 0$ or there is a multi-index p' so that $S_q{}^*S_p = S_{p'}$. In the first case one has $\tau\big(\lambda(f)S_pS_q{}^*\big) = 0$ and in the second, by the above calculation one has

$$\tau\big(\lambda(f)S_pS_q{}^*\big) = \tau\big(\lambda(\epsilon_{E(p)}f)S_{p'}\big) = \tau\big(u_t\lambda(\epsilon_{E(p)}f)S_{p'}u_t^*\big) = e^{itE(p')}\tau(\lambda(\epsilon_{E(p)}f)S_{p'}).$$

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Since $E(p') \neq 0$, it follows that $\tau\big(\lambda(f)S_pS_q{}^*\big) = 0$ in the second case as well.

Hence τ is completely determined, up to a constant multiple, on the set of elements of the form $\lambda(f)S_pS_q{}^*$ where $\hat{f} \in C_c(\mathbb{R})$ and p, q are multi-indices, and hence on elements of the form $\lambda(f)x$ with $x \in O_n$ and $\hat{f} \in C_c(\mathbb{R})$. By the linearity and lower semicontinuity of the trace this determines τ. ∎

REMARK 4.2. In [KK] it is shown that if $\{\lambda_1, \ldots, \lambda_n\}$ generates $\mathbb{R}$ as a closed subsemigroup then the crossed product $O_n \rtimes \mathbb{R}$ is purely infinite (it is already known to be simple by [Ks]). It is further shown that if the crossed product of a purely infinite simple C^*-algebra by a one parameter automorphism group is simple then it is stable and either traceless or projectionless.

LEMMA 4.3. *With notation as above, if τ is factorial then ω_f is factorial.*

PROOF. Let $(\pi_\tau, \mathfrak{H}_\tau)$ denote the tracial representation and $\mathfrak{K}$ denote the closure of $M_{n^\infty}\lambda(f)^*$ in $\mathfrak{H}_\tau$. Then ω_f is factorial exactly when $\mathcal{M} = \pi_\tau(M_{n^\infty})''|_{\mathfrak{K}}$ is a factor. Suppose that $\mathcal{M} \cap \mathcal{M}' \neq \mathbb{C} \cdot 1$. Since ω_f is invariant under S_∞, the group of permutations on the factors of M_{n^∞} (which are all inner), each element of $\mathcal{M} \cap \mathcal{M}'$ is fixed under this action. Thus we can assume that the central decomposition of ω_f is supported by positive linear functionals of norm $\|\omega_f\|$ invariant under S_∞. We may further assume that these functionals are invariant under $\alpha_t|_{M_{n^\infty}} = \operatorname{Ad} u_t|_{M_{n^\infty}}$, since ω_f is diagonal in the sense that $\omega_f(S_pS_q{}^*) = 0$ if $p \neq q$. Indeed, if u is a unitary in the C^*-subalgebra generated by $S_pS_p{}^*$, then, since $\lambda(f)$ and u commute, one has

$$\omega_f(uxu^*) = \omega_f(x),$$

for $x \in M_{n^\infty}$, and

$$\omega_f(ES_pS_q{}^*) = \omega_f(uES_pS_q{}^*u^*) = \omega_f(EuS_pS_q{}^*u^*)$$

for such u, if $E \in \mathcal{M} \cap \mathcal{M}'$. Then it easily follows that $\omega_f(ES_pS_q{}^*) = 0$ if $p \neq q$. We may further assume that these functionals are invariant under the shift ρ defined by

$$\rho(x) = \sum_{i=1}^{n} S_jxS_j{}^*,$$

for $x \in M_{n^\infty}$, since they are S_∞-invariant. It follows that each central element of $\mathcal{M}$ is invariant under the weak extensions of $\alpha_t|_{M_{n^\infty}}$ and $\rho|_{M_{n^\infty}}$. Let E be a central projection of $\pi_\tau(M_{n^\infty})''$ such that $E|_{\mathfrak{K}}$ is nontrivial. Since $E|_{\mathfrak{K}}$ is invariant under the weak extensions of $\alpha_t|_{O_n \rtimes_\alpha \mathbb{R}}$ and $\rho|_{O_n \rtimes_\alpha \mathbb{R}}$, we may assume that E is invariant under these actions. This implies that E is a central projection of $\pi_\tau(O_n \rtimes_\alpha \mathbb{R})''$ and so $E = 0$ or 1, which is a contradiction. Hence, ω_f is factorial. ∎

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REMARK 4.4. For any metrizable Choquet simplex X there is a one-parameter automorphism group $\gamma: \mathbb{R} \longrightarrow \mathrm{Aut}(\mathcal{O}_2)$ such that the crossed product, $\mathcal{O}_2 \rtimes_\gamma \mathbb{R}$, is simple and stably projectionless and for which the cone of traces is affinely homeomorphic to CX. The argument is similar to that followed in the proof of Corollary 2.5. Let α be as in Theorem 4.1 (with $n = 2$); let B be a simple inductive limit of interval algebras for which the simplex of traces is affinely homeomorphic to X and $K_*(B) \cong \mathbb{Z}[\frac{1}{2}] \oplus 0$ (*cf.* [Th, Theorem 3.9]). Define a one-parameter automorphism group $\tilde{\alpha}: \mathbb{R} \longrightarrow \mathrm{Aut}(B \otimes \mathcal{O}_2)$ by $\tilde{\alpha}_t(b \otimes c) = b \otimes \alpha_t(c)$. Since

$$(B \otimes \mathcal{O}_2) \rtimes_{\tilde{\alpha}} \mathbb{R} \cong B \otimes (\mathcal{O}_2 \rtimes_\alpha \mathbb{R})$$

one sees that the crossed product is simple and that its cone of traces is affinely homeomorphic to CX. It remains to observe that $B \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ (*cf.* [Kr, Corollary 20]); γ is obtained from $\tilde{\alpha}$ by means of this identification. Note that each trace on the crossed product has the scaling property with respect to the dual automorphism group (with scaling parameter β as in Theorem 4.1).

REMARK 4.5. The simple (stable) purely infinite C^*-algebras constructed in Theorem 2.4 and Corollary 2.5 as crossed products of simple stably projectionless C^*-algebras by one-parameter automorphism groups belong to the bootstrap class for which the Universal Coefficient Theorem of Rosenberg and Schochet holds (*cf.* [RS, 1.17]); hence the Kirchberg-Phillips classification theorem (*cf.* [Kr, Corollary C], [Ph, Theorem 4.2.4]) applies and K-theory is a complete invariant for these C^*-algebras. It follows that any simple stable purely infinite C^*-algebra in the classifiable class with n-divisible K-theory (see Remark 2.6) can be written as a crossed product $A \rtimes_\alpha \mathbb{R}$ with A simple and stably projectionless. Moreover this may be done in many different ways—given any metrizable Choquet simplex X, A may be chosen so that the trace cone of A is affinely homeomorphic to CX. Thus there does not seem to be a precise analog of Takesaki's continuous decomposition of type III von Neumann algebras (*cf.* [Ta, Section 8]) for purely infinite C^*-algebras; Takesaki's continuous decomposition arises by applying the duality theorem to the modular automorphism group. For a given purely infinite C^*-algebra there is, in general, no canonical one-parameter automorphism group; one might, however, expect that there are many actions of $\mathbb{R}$ which give rise to simple crossed products and that the flow induced on the trace cone of the crossed product by the dual automorphism group will serve as a useful invariant (see Remark 2.7 for a non-trivial instance of this invariant).

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