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A WEIBULL LIMIT FOR THE RELIABILITY OF A CONSECUTIVE k-WITHIN-m-OUT-OF-n SYSTEM

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Abstract

A consecutive k-within-m-out-of-n system consists of n identical and stochastically independent components arranged on a line. The system will fail if and only if within m consecutive components, there are at least k failures. Let T_n be the system's lifetime. Then, under quite general conditions we prove that there is a positive constant a such that the random variable $n^{1/ka}T_n$ converges to a Weibull distribution as $n \to \infty$.

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1. Introduction

Consecutive k-within-m-out-of-n systems were introduced by Griffith (1986), as a generalization of consecutive k-out-of-n systems (see for example Tong (1985)). Essentially, in a different context, the mathematical equivalent of a consecutive k-within-m-out-of-n system, was studied much earlier (see for example, Saperstein (1973), (1975)).

A consecutive k-within-m-out-of-n system consists of n linearly ordered components. The n components are identical and their failure times are stochastically independent. The system will fail if and only if there are m consecutive components which include among them, at least k failed components. Let T be the time of failure of an individual component and T_n be the time of failure of the system. Let q be the failure distribution of an individual component, i.e., $q(t) = \Pr\{T \geqq t\}$, for $t \geqq 0$. We are interested in studying the failure distribution of the system, $\Pr\{T_n \leqq t\}$.

As Saperstein (1973), (1975) and Griffith (1986) explain, it is very difficult to compute the failure distribution of the system.

Our main result is the following limit theorem.

Theorem. Let $m \geqq k \geqq 2$, and

$$q(t) = \lambda^a t^a + o(t^a)$$

where a, λ are positive real constants. Then

$$\Pr\{n^{(1/ka)}T_n \leqq t\} \to 1 - \exp\left(-(\lambda t)^{ak} \sum_{j=k}^{m} \binom{j-2}{k-2}\right) \quad \text{as} \quad n \to \infty.$$

For the special case of a consecutive k-out-of-n system, this theorem has been proved by Papastavridis (1987); the proof was based on the generating function of the failure distribution, which was quite manageable for this special case.

Remark. The assumption that we made on the distribution q is a very mild one. Really, since $q(0) = 0$, assuming the existence of a Taylor expansion of q, around zero, in a right neighborhood of zero, implies the assumption of the theorem.

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2. Proof of theorem

The proof of our theorem is based on Theorem 2, p. 399 of Barbour and Eagleson (1984). We consider the set

$$A = \{(i-j+1, i-j+2, \cdots, i-1, i) : 1 \leqq i-j+1, i \leqq n, k \leqq j \leqq m\}$$

which consists of j-tuples of numbers $1, 2, \cdots, n$ which represent the component of the system in their ordering. Following the notation of Barbour and Eagleson (1984) p. 399, let $J = (i-j+1, \cdots, i) \in A$ and let X_J be the random variable which has the value 1 if and only if components $(i-j+1)$ and i are failed and there are k failures among components, $i-j+1, \cdots, i$. In all other cases, X_J has the value 0. Let the random variables $X = \sum_{J \in A} X_J$. Clearly, the system fails if and only if X is greater than 0. Let $t_n = tn^{-(1/ka)}$, where $t \geqq 0$. We consider the system in the time interval from 0 to t_n. Clearly

$$p_J = EX_J = q(t_n)^k (1-q(t_n))^{j-k} \binom{j-2}{k-2} \tag{1}$$

and

$$EX = (n-m+1) q(t_n)^k \sum_{j=k}^{m} (1-q(t_n))^{j-k} \binom{j-2}{k-2} + \sum p_J$$

where in the second summation J ranges over j-tuples of the numbers $1, 2, \cdots, m$, only.

It is easy to see that

$$q(t_n)^k = (\lambda t)^{ak}/n + o(1/n) \tag{2}$$

so, as $n \to \infty$ we have from (1)

$$EX \to (\lambda t)^{ak} \sum_{j=k}^{m} \binom{j-2}{k-2} \tag{3}$$

because the summand $\sum p_J$ clearly goes to 0. By Theorem 2, p. 399 of Barbour and Eagleson (1984), the difference

$$\Pr\{T_n \leqq t_n\} - (1 - \exp(-EX)) \tag{4}$$

is bounded absolutely by

$$\min(1, 1/EX)\left(\sum p_J^2 + \sum (p_J p_K + EX_J X_K)\right) \tag{5}$$

where the first summation ranges over $J \in A$ and the second one ranges over $J, K \in A$ with $J \neq K$ and J and K have at least one common component. Clearly we have

$$p_J p_K = (\lambda t)^{2ak}/n^2 + o(1/n^2), \quad \text{for} \quad J, K \in A$$

and for $J, K \in A$, $J \neq K$ and having at least one common component, we have

$$EX_J X_K = O(n^{-1-(1/k)}).$$

That means that quantity (5) goes to 0 as $n \to \infty$. So (3) and (4) prove our theorem.

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