SOME PROPERTIES OF NOETHERIAN DOMAINS OF DIMENSION ONE

EBEN MATLIS

Introduction. Throughout this discussion R will be an integral domain with quotient field Q and $K = Q/R \neq 0$. If A is an R-module, then A is said to be *torsion-free* (resp. *divisible*), if for every $r \neq 0 \in R$ the endomorphism of A defined by $x \to rx$, $x \in A$, is a monomorphism (resp. epimorphism). If A is torsion-free, the *rank of* A is defined to be the dimension over Q of the vector space $A \otimes_R Q$; (we note that a torsion-free R-module of rank one is the same thing as a non-zero R-submodule of Q). A will be said to be *indecomposable*, if A has no proper, non-zero, direct summands. We shall say that A has D.C.C., if A satisfies the descending chain condition for submodules. By dim R we shall mean the maximal length of a chain of prime ideals in R.

In the following remarks the ring R will be a Noetherian integral domain, unless specified otherwise.

The purpose of this paper is to study some of the properties and relationships of R-submodules of Q and R-homomorphic images of Q. With each statement concerning these objects we can associate a dual statement obtained by interchanging the words submodule and homomorphic image, torsion-free and divisible, free and injective, finitely generated and D.C.C., and the self-dual term indecomposable. We now list a series of such statements and their duals ((1′) is the dual of (1), (2′) is the dual of (2), etc.).

(1) Every proper R-submodule of Q is finitely generated.

(1′) Every proper R-homomorphic image of Q has D.C.C.

(2) No proper homomorphic image of an R-submodule of Q is torsion-free.

(2′) No proper submodule of an R-homomorphic image of Q is divisible.

(3) Every R-submodule of Q is indecomposable.

(3′) Every R-homomorphic image of Q is indecomposable.

(4) Every cyclic R-submodule of Q is free.

(4′) Every factor module of Q by a cyclic R-submodule is injective (that is, K is injective).

Statements (2), (3), and (4) are, of course, trivial for any integral domain; but their duals are highly non-trivial. One aim of this discussion is to find necessary and sufficient conditions for the validity of these statements and for their equivalence.

A most interesting feature of these statements is the close relationship they

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bear to each other and to the condition that R is a local ring of dimension one. In Theorem 1 we prove that (1′) is equivalent to R being semi-local and $\dim R = 1$. This gives us a well-known theorem of I. S. Cohen as a corollary. Condition (1) is much more restrictive, for in Theorem 2 we prove that (1) and (2′) are equivalent and imply (3′), R local, and $\dim R = 1$. However, if we replace (1) by a weaker finiteness condition on the proper R-submodules of Q, then Theorem 3 shows that we do get a condition for local rings that is equivalent to $\dim R = 1$.

We now consider the statement:

(5) $\mathrm{Ext}_R{}^1(Q, S) = 0$ for every R-submodule S of Q.

In Theorem 4 we show that (5) is equivalent to R being a complete, local domain of dimension one; and if R is complete and local, then (5) is equivalent to (1), (1′), (2′), and (3′). As a corollary we obtain a special case of a theorem of M. Nagata. The strength of condition (5) is not too surprising when we compare it with the fact that if R is any integral domain, then R is a maximal valuation ring if and only if $\mathrm{Ext}_R{}^1(A, S) = 0$ for every torsion-free R-module A and every R-submodule S of Q [Theorem A4].

Now in **(8)** we also studied the condition that every R-homomorphic image of Q is injective, which, in the case of a Noetherian domain, we showed is equivalent to R being a Dedekind ring. Condition (4′) is not equivalent to R being a Dedekind ring, but does imply $\dim R = 1$. In Thoerems 5 and 6 we study this condition and some of its implications. Finally, we show the relationship of some of the listed conditions to the existence of indecomposable, torsion-free R-modules of rank two.

To facilitate matters for the reader we shall provide in the Appendix a list of some of the theorems we shall find necessary, together with indications of where their proofs may be found. In the text a reference to Theorem A1 will mean a reference to Theorem A1 of the Appendix, etc.

Definitions and notations. Let A be an R-module. Then:

(1) $0(A) = \{r \in R \mid rA = 0\}$.

(2) If I is any ideal of R, then $\mathrm{Ann}_A(I) = \{x \in A \mid Ix = 0\}$.

(3) $E(A)$ is the injective envelope of A (see **(7)**).

(4) $\mathrm{Hd}_R A$ is the projective dimension of A as an R-module.

(5) A is said to be reduced, if A has no proper, non-zero, divisible submodules.

(6) An ideal I of a ring R is said to be irreducible, if it is not an intersection of two strictly larger ideals of R.

(7) Let L be a multiplicatively closed subset of R. Then R_L is the ring of quotients of R with respect to L, and $A_L = A \otimes_R R_L$.

If L is the complement of a prime ideal P of R, then we shall define R_P to be R_L, and A_P to be A_L.

1. K has D.C.C.

THEOREM 1. *Let R be a Noetherian integral domain. Then the following are equivalent:*

(1) *R is semi-local and* $\dim R = 1$.

(2) *K has* D.C.C.

If one of these conditions is satisfied, then every proper R-homomorphic image of Q has D.C.C.

Proof. (2) $\Rightarrow$ (1). Assume K has D.C.C. Let $P \neq 0$ be a prime ideal of R, and let $b \neq 0 \in P$. Let $x = 1/b + R, x \in K$. Then Rx has a composition series, and so there exist maximal ideals $M_1, \ldots, M_n$ of R such that if $I = M_1 M_2 \ldots M_n$, then $Ix = 0$. Thus $I \subset bR \subset P$, and so P is equal to one of the M_i's and is maximal. Thus $\dim R = 1$. Now by Theorem A7 we have $K = \sum \oplus K_{M_\alpha}$, where the M_α's range over all of the maximal ideals of R. Since K has D.C.C., there can only be a finite number of components in this direct sum, and so R is semi-local.

(1) $\Rightarrow$ (2). Assume R is semi-local and $\dim R = 1$. Let $M_1, \ldots, M_n$ be the maximal ideals of R. Then by Theorem A7 we have $K = K_{M_1} \oplus \ldots \oplus K_{M_n}$. Since a finite direct sum of modules has D.C.C. if and only if each direct summand has D.C.C., and since K_{M_i} has D.C.C. over R_{M_i} if and only if it has D.C.C. over R, we can assume that R is a local ring of dimension one.

Let M be the maximal ideal of R. Then it is easy to see that since $\dim R = 1$, $M^{-1}/R \neq 0$ and K is an essential extension of M^{-1}/R (see **(6)**). Thus $E(K) = E(M^{-1}/R)$. Now M^{-1}/R is a finite direct sum of copies of R/M, and so $E(M^{-1}/R)$ is a finite direct sum of copies of $E(R/M)$. Thus K has D.C.C. by Theorem A8.

If K has D.C.C., let $T \neq 0$ be any proper R-submodule of Q. Choose $x \neq 0 \in T$. Then we have an exact sequence:

$$Q/Rx \to Q/T \to 0.$$

Since $Q/Rx \cong K$, Q/T has D.C.C.

In order to show the connection of Theorem 1 with earlier results we derive the following theorem of I. S. Cohen **(3**, Theorem 4**)** as a corollary.

COROLLARY (*Theorem of I. S. Cohen*). *Let R be a Noetherian domain such that* $\dim R = 1$. *Let $S \neq Q$ be any ring between R and Q. Then S is Noetherian,* $\dim S = 1$, *and if R is semi-local, so is S. Furthermore, if A is any non-zero ideal of S, then S/A has a composition series as an R-module.*

Proof. Let $I = A \cap R$. Then to prove the theorem it is sufficient to prove that S/SI has a composition series as an R-module. Now by Theorem A7,

$$S/SI = \sum \oplus S_{M_\alpha}/S_{M_\alpha} I_{M_\alpha},$$

where M_α ranges over all maximal ideals of R. Since I is contained in only a finite number of maximal ideals of R, the above direct sum has only a finite number of non-zero components. Since $S_{M_\alpha}/S_{M_\alpha} I_{M_\alpha}$ has a composition series as an R-module if and only if it has a composition series as an R_{M_α}-module, we can assume that R is a local ring with maximal ideal M. By Theorem 1, S/SI has D.C.C. over R. Since there exists an integer $n > 0$ such that $M^n \subset I$, we have $M^n(S/SI) = 0$. Thus S/SI has a composition series over R.

PROPOSITION 1. *Let R be a Noetherian, local domain such that* dim $R = 1$. *Let M be the maximal ideal of R, and D an R-module with* D.C.C. *Then D is a divisible R-module if and only if $MD = D$.*

Proof. Of course, if D is divisible, then $MD = D$. Conversely, assume that $MD = D$. Then by Theorem A9 there exists an element $s \neq 0 \in M$ such that $sD = D$. Let $b \neq 0 \in M$, and let L be the multiplicative system consisting of the powers of s. Then $R_L = Q$, and so there exists $r \in R$ such that $1/b = r/s^k$ for some integer $k > 0$. Hence we have $D = s^kD = b(rD)$, and so D is divisible.

THEOREM 2. *Let R be a Noetherian integral domain. Then the following are equivalent:*

(1) *The integral closure of R is a valuation ring, finitely generated over R.*

(2) *Every proper R-submodule of Q is finitely generated.*

(3) *No R-homomorphic image of Q has a proper, non-zero, divisible submodule.*

If any of these conditions hold, then R is local, dim $R = 1$, *and the following condition holds:*

(4) *Every R-homomorphic image of Q is indecomposable. Conversely, if condition* (4) *holds, then R is local,* dim $R = 1$, *and the integral closure of R is a discrete valuation ring.*

Proof.

(1) $\Rightarrow$ (2). Let V be the integral closure of R, and let S be any proper R-submodule of Q. Since V is finitely generated over R, there exists $r \neq 0 \in R$ such that $rV \subset R$; and so $rVS \subset S$. Thus $VS \neq Q$, and since V is a discrete valuation ring, VS is a finitely generated V-module. It follows that VS is a finitely generated R-module; and since $S \subset VS$, S is a finitely generated R-module.

(2) $\Rightarrow$ (3). This is an immediate consequence of the fact that a divisible module cannot be finitely generated.

Clearly (3) implies (4). Hence, assume (4). Suppose that dim $R > 1$. Then by Theorem A10 there exist two distinct minimal prime ideals P_1, P_2 of R. Let L be the complement of $P_1 \cup P_2$; then R_L has two maximal ideals and dim $R_L = 1$. By Theorem A7, Q/R_L decomposes into a direct sum. This contradicts the hypothesis, and so dim $R = 1$. By the corollary to Theorem 1 we know that every ring between R and Q is Noetherian and has dimension one. A repetition of the previous argument shows that they must all be local. Thus R is local, and if V is the integral closure of R, it follows that V is a discrete valuation ring.

Let N be the maximal ideal of V; then there exists $x \in V$ such that $N = Vx$.

Now if M is the maximal ideal of R, then V/N has D.C.C. over R/M by Theorem 1, and thus V/N is finitely generated over the field R/M. Let $v_1, \ldots, v_n$ be representatives of these generators in V, and let $R' = R[x, v_1, \ldots, v_n]$. By the above remarks R' is a Noetherian, local ring with maximal ideal M'. We have $V = VM' + R'$, and hence $M'(V/R') = V/R'$. But V/R' has D.C.C. over R' by Theorem 1, and so it follows from Proposition 1 that V/R' is divisible over R', *a fortiori* divisible over R. If we now assume condition (3), then $V = R'$, and so (1) is true.

Remarks. It is an open question whether condition (4) of the previous theorem is equivalent to the other three conditions. As the theorem shows, the finiteness of condition (2) is too strong to characterize local domains of dimension one. Now if $A \neq 0$ is any finitely generated module over a local ring R with maximal ideal M, then $MA \neq A$. If we replace condition (2) by this weaker condition on proper R-submodules of Q, then the following theorem shows that this does characterize local domains of dimension one.

THEOREM 3. *Let R be a Noetherian, local domain with maximal ideal M. Then* $\dim R = 1$ *if and only if $MS \neq S$ for every non-zero, reduced, torsion-free R-module S of finite rank.*

Proof. If $\dim R > 1$, take a non-zero prime ideal $P \neq M$. Then we have $MR_P = R_P$. Conversely, assume that $\dim R = 1$.

Case I: Rank $S = 1$. Suppose $MS = S$. Choose $x \neq 0 \in S$. Then $M(S/Rx) = S/Rx$. Since S/Rx has D.C.C. by Theorem 1, it follows from Proposition 1 that S/Rx is divisible. Let $a \neq 0 \in R$; then we have $S = aS + Rx$. Therefore, $S/aS \cong Rx/(aS \cap Rx)$ is finitely generated. However, $M(S/aS) = S/aS$; and so $S = aS$. Thus S is divisible. This contradicts the assumption that S is reduced, and thus $MS \neq S$.

Case II: Rank $S > 1$. Suppose rank $S = n$, and make the induction hypothesis that the theorem is true for modules of smaller rank. We note that for any R-module A, $A \otimes_R R/M \cong A/MA$. Now S has a reduced, torsion-free submodule T of rank $n - 1$ such that S/T is torsion-free of rank 1. We have an exact sequence:

$$\mathrm{Tor}_1^R(S/T, R/M) \to T \otimes_R R/M \to S \otimes_R R/M \to S/T \otimes_R R/M \to 0.$$

If S/T is reduced, then $S/T \otimes_R R/M \neq 0$ by Case I, and so we have $S \otimes_R R/M \neq 0$. On the other hand, if S/T is not reduced, then $S/T \cong Q$, and we have $\mathrm{Tor}_1{}^R(S/T, R/M) = 0 = S/T \otimes_R R/M$. Hence we have $T \otimes_R R/M \cong S \otimes_R R/M$; and since $T \otimes_R R/M \neq 0$ by induction, the proof of the theorem is complete.

2. $\mathrm{Ext}_R{}^1(Q, S) = 0$.

PROPOSITION 2. *Let R be a Noetherian, local domain with maximal ideal M, $E = E(R/M)$, and completion $\bar{R}$. Then the following are equivalent:*

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(1) *If $\bar{I}$ is any non-zero ideal of $\bar{R}$, then $\bar{I} \cap R \neq 0$.*
(2) *E has no proper, faithful R-submodules.*
(3) *E has no proper, non-zero, divisible R-submodules.*

Proof.
(1) $\Rightarrow$ (2). Suppose that B is a proper, faithful R-submodule of E. By Theorem A7, B is also an $\bar{R}$-module. Let $\bar{I} = \{\bar{r} \in \bar{R} \mid \bar{r}B = 0\}$. Since $B \neq E$, $\bar{I} \neq 0$ by Theorem A2. However, since B is faithful as an R-module, $\bar{I} \cap R = 0$. This contradiction shows that E has no proper, faithful R-submodules.

(2) $\Rightarrow$ (3). This is trivial, since a divisible module is, *a fortiori*, faithful.

(3) $\Rightarrow$ (1). Suppose that $\bar{I}$ is an ideal of $\bar{R}$ such that $\bar{I} \cap R = 0$. Let $\bar{P}$ be an ideal of $\bar{R}$ such that $\bar{I} \subset \bar{P}$, and such that $\bar{P}$ is maximal with respect to the property that $\bar{P} \cap R = 0$. Since R is an integral domain, it is easily verified that $\bar{P}$ is a prime ideal of $\bar{R}$. Let B be the annihilator of $\bar{P}$ in E. Then by Theorem A1 B, considered as an $\bar{R}/\bar{P}$-module, is injective. Since $\bar{R}/\bar{P}$ is an integral domain, B is a divisible $\bar{R}/\bar{P}$-module. But $\bar{P} \cap R = 0$, and so B is divisible as an R-module. Hence by assumption $B = 0$. Thus $\bar{P} = 0$, and so $\bar{I} = 0$.

Two cases where the conditions of Proposition 2 are true are the following:
(i) *R is a complete domain.* For then (1) is trivially satisfied.
(ii) Dim $R = 1$ *and $\bar{R}$ is an integral domain.* For if $\bar{M}$ is the maximal ideal of $\bar{R}$, then $\bar{M} \cap R = M$. Since $\bar{M}$ is the only non-zero prime ideal of $\bar{R}$, condition (1) is fulfilled.

PROPOSITION 3. *Let R be a complete, Noetherian, local domain with maximal ideal M and $E = E(R/M)$. Then* dim $R = 1$ *if and only if every proper R-submodule of E is finitely generated.*

Proof. Suppose that dim $R = 1$, and let $B \neq 0$ be a proper submodule of E. Then $0(B)$ is an M-primary ideal, and so $0(B) = I_1 \cap \ldots \cap I_n$, where the I_j's are irreducible, M-primary ideals. By Theorem A1 there exist elements $x_1, \ldots, x_n \in E$ such that $I_j = 0(x_j)$. By Theorem A2,

$$B = \mathrm{Ann}_E(0(B)) = \mathrm{Ann}_E\left(\bigcap_{j=1}^{n} I_j\right) = \sum_{j=1}^{n} \mathrm{Ann}_E(I_j) = \sum_{j=1}^{n} Rx_j.$$

Thus B is finitely generated.

Conversely, suppose that every proper R-submodule of E is finitely generated. Let $I \neq 0$ be an ideal of R. Then $\mathrm{Ann}_E(I)$ is a proper submodule of E by Theorem A2. Hence there exist elements $x_1, \ldots, x_n \in E$ such that

$$\mathrm{Ann}_E(I) = \sum_{j=1}^{n} Rx_j.$$

Then by Theorem A2 again, we have

$$I = 0(\mathrm{Ann}_E(I)) = 0\left(\sum_{j=1}^{n} Rx_j\right) = \bigcap_{j=1}^{n} 0(x_j).$$

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By Theorem A1 each $0(x_j)$ is an irreducible, M-primary ideal. Thus I is an M-primary ideal, Since I was an arbitrary non-zero ideal of R, it follows that M is the only non-zero, prime ideal of R, and so $\dim R = 1$.

PROPOSITION 4. *Let R be an arbitrary integral domain, and let $S \neq Q$ be a torsion-free R-module of rank one. Then the following are equivalent:*

(1) $\mathrm{Hom}_R(Q, Q/S) \cong Q$.

(2) $\mathrm{Ext}_R^1(Q, S) = 0$.

(3) $S \cong \mathrm{Ext}_R^1(K, S)$.

If any of these conditions hold, then Q/S is an indecomposable R-module.

Proof. Since $\mathrm{Hom}_R(Q, S) = 0$ and $\mathrm{Hom}_R(Q, Q) \cong Q$, we have an exact sequence

$$0 \to Q \to \mathrm{Hom}_R(Q, Q/S) \to \mathrm{Ext}_R^1(Q, S) \to 0.$$

From this sequence it follows immediately that (1) and (2) are equivalent. Since $\mathrm{Hom}_R(R, S) \cong S$, we also have an exact sequence:

$$0 \to S \to \mathrm{Ext}_R^1(K, S) \to \mathrm{Ext}_R^1(Q, S) \to 0.$$

Since $\mathrm{Ext}_R^1(Q, S)$ is a Q-module, the equivalence of (2) and (3) follows readily from this sequence and considerations of rank.

Now assume that $\mathrm{Ext}_R^1(Q, S) = 0$, and suppose that we have a direct sum decomposition: $Q/S = T/S \oplus U/S$, where T, U are proper R-submodules of Q that contain S. Then $T + U = Q$ and $T \cap U = S$. Thus we have an exact sequence:

$$0 \to S \to T \oplus U \to Q \to 0.$$

Since $\mathrm{Ext}_R^1(Q, S) = 0$, we have $T \oplus U \cong S \oplus Q$. Thus we have a non-zero projection of Q into one of the modules T or U. This is impossible, since both T and U are reduced. Therefore, Q/S is indecomposable.

Remark. We note that if R is any integral domain, then by Theorem A5 we have $\mathrm{Ext}_R^1(Q, R) = 0$ if and only if $\mathrm{Hom}_R(K, K) \cong R$.

COROLLARY. *Let R be an arbitrary integral domain, and V a ring between R and Q. Let T be a torsion-free V-module of rank one. Then $\mathrm{Ext}_V^1(Q, T) = 0$ if and only if $\mathrm{Ext}_R^1(Q, T) = 0$.*

Proof. Using Theorem A11 we have

$$\mathrm{Hom}_V(Q, Q/T) \cong \mathrm{Hom}_V(Q \otimes_R V, Q/T) \cong \mathrm{Hom}_R(Q, \mathrm{Hom}_V(V, Q/T)) \cong \mathrm{Hom}_R(Q, Q/T).$$

Then the corollary follows from Proposition 4.

PROPOSITION 5. *Let R be a Noetherian, local domain with maximal ideal M and $E = E(R/M)$. Then R is complete and $\dim R = 1$ if and only if $Q \cong \mathrm{Hom}_R(Q, E)$.*

Proof. If A is an R-module, we will denote by A^* the module $A^* = \operatorname{Hom}_R(A, E)$.

Suppose that R is complete and $\dim R = 1$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & R & \to & Q & \to & K & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & R^{**} & \to & Q^{**} & \to & K^{**} & \to & 0. \end{array}$$

Since K has D.C.C. by Theorem 1, we have by Theorem A2 that the maps $R \to R^{**}$ and $K \to K^{**}$ are isomorphisms. It follows that the map $Q \to Q^{**}$ is an isomorphism. Since Q^* is a Q-module, it follows from consideration of rank that $Q \cong Q^*$.

Now suppose that $Q \cong Q^*$. Now we have $R^{**} \subset Q^{**}$; but since $Q \cong Q^{**}$ by assumption and $R^{**} \cong \bar{R}$, the completion of R, by Theorem A1, we actually have $\bar{R} \subset Q$. However, as is well known, $\bar{R} \cap Q = R$. Thus $R = \bar{R}$; that is, R is complete.

Suppose $\dim R > 1$. Then by Theorem A10 we can find two distinct minimal prime ideals P_1, P_2 of R. Let L be the complement of $P_1 \cup P_2$; and let $C = \operatorname{Hom}_R(R_L, E)$. Then C is an injective R_L-module **(2**, Prop. 2.6.1a**)**. Using Theorem A11 we have

$$\begin{aligned} \operatorname{Hom}_{R_L}(C, C) = \operatorname{Hom}_{R_L}(C, \operatorname{Hom}_R(R_L, E)) &\cong \operatorname{Hom}_R(C \otimes_{R_L} R_L, E) \\ &\cong \operatorname{Hom}_R(C, E) \cong R_L^{**}. \end{aligned}$$

Since $R_L^{**} \subset Q^{**} \cong Q$, R_L^{**} is R_L-indecomposable. Therefore, C is an indecomposable injective R_L-module. Now we have

$$\operatorname{Hom}_{R_L}(R_L/P_1R_L, C) \cong \operatorname{Hom}_R(R_L/P_1R_L, E) \neq 0.$$

Thus, since P_1R_L is a maximal ideal of R_L, C has an element of order P_1R_L; and thus C is the injective envelope over R_L of R_L/P_1R_L. Similarly, C is the injective envelope over R_L of R_L/P_2R_L. Thus by Theorem A1, $P_1R_L = P_2R_L$; and so $P_1 = P_2$. Since P_1, P_2 were chosen distinct, this contradiction shows that $\dim R = 1$.

The unusual strength of the condition $Q \cong Q^*$ will be investigated further in Proposition 6, but first we need a lemma.

LEMMA 1. *Let V be a valuation ring with quotient field Q, maximal ideal N, and $C = E(V/N)$. Then V is a maximal valuation ring if and only if* $\operatorname{Hom}_V(C, C) \cong V$; *and in this case $C \cong Q/N$.*

Proof. If V is a maximal valuation ring, then $\operatorname{Hom}_V(C, C) \cong V$ by Theorem A3. Conversely, assume that $\operatorname{Hom}_V(C, C) \cong V$. We will show first that $C \cong Q/N$. Now C is the injective envelope of Q/N by **(8**, Prop. 1**)**; and, therefore, $Q/N \subset C$. Let $x \neq 0 \in C$, and choose $a \neq 0 \in 0(x)$. Let $y = 1/a + N \in Q/N$; then $0(y) = aN \subset 0(x)$. Hence we have a map $f: Ry \to Rx$ such that $f(y) = x$. Then f can be extended to an endomorphism

$g: C \to C$. But g is multiplication by an element $v \in V$. Thus $x = g(y) = vy \in Q/N$. Hence $C \subset Q/N$, and so $C = Q/N$.

We next show that Q/V is injective. Let A be any V-module. Since w. gl. dim. $V = 1$ by **(4**, Theorem 2**)**, we have $\mathrm{Tor}_2{}^V(A, V/N) = 0$. Thus, since $V/N \cong \mathrm{Hom}_V(V/N, C)$, we have by Theorem A11,

$$\mathrm{Ext}_V{}^2(A, V/N) \cong \mathrm{Ext}_V{}^2(A, \mathrm{Hom}_V(V/N, C)) \\ \cong \mathrm{Hom}_V(\mathrm{Tor}_2{}^V(A, V/N), C) = 0.$$

Hence the injective dimension of V/N is one. Therefore, since $Q/N \cong C$ is injective, we conclude from the exact sequence:

$$0 \to V/N \to Q/N \to Q/V \to 0$$

that Q/V is injective. Thus by Theorem A3, V is an almost maximal valuation ring. Since $\mathrm{Hom}_V(Q/N, Q/N) \cong V$, it follows from Theorem A3 that V is a maximal valuation ring.

PROPOSITION 6. *Let R be a local domain (not necessarily Noetherian) with maximal ideal M and $E = E(R/M)$. Then the following are equivalent:*

(1) *$Q \cong \mathrm{Hom}_R(Q, E)$.*

(2) *If V is any valuation ring between R and Q, then V is a maximal valuation ring; and if $C = \mathrm{Hom}_R(V, E)$, then C is the injective envelope over V of V/N, where N is the maximal ideal of V.*

Proof.

(1) $\Rightarrow$ (2). If A is any R-module, we let $A^* = \mathrm{Hom}_R(A, E)$. Let V be any valuation ring between R and Q, and let N be the maximal ideal of V. Let $C = \mathrm{Hom}_R(V, E)$ and $\Lambda = \mathrm{Hom}_V(C, C)$. Then by a repetition of the type of argument used in the second part of Proposition 5 we can show that C is the injective envelope over V of V/N, and that $\Lambda \cong V^{**}$. Thus $V \subset \Lambda \subset Q$. By **(6**, Prop. 2.6**)** Λ is a local ring; denote its maximal ideal by P. Then to prove that $V = \Lambda$, it is sufficient to prove that $N \subset P$. However, if $v \in N$, then there exists $x \in C$ such that $vx = 0$, and so $v \in P$. Thus $V = \Lambda$. It now follows from Lemma 1 that V is a maximal valuation ring.

(2) $\Rightarrow$ (1). Let V be a valuation ring between R and Q. Let N be the maximal ideal of V and $C = \mathrm{Hom}_R(V, E)$. Then by assumption V is a maximal valuation ring, and C is the injective envelope over V of V/N. By Theorem A3 we have $C \cong Q/N$. Now $\mathrm{Ext}_V{}^1(Q, N) = 0$ by Theorem A4, and so $\mathrm{Hom}_V(Q, Q/N) \cong Q$ by Proposition 4. Thus, using Theorem A11 we have

$$\mathrm{Hom}_R(Q, E) \cong \mathrm{Hom}_R(Q \otimes_V V, E) \cong \mathrm{Hom}_V(Q, \mathrm{Hom}_R(V, E)) \\ \cong \mathrm{Hom}_V(Q, Q/N) \cong Q.$$

We note that if R is a Noetherian, local domain, then it follows from Propositions 5 and 6 that R is complete and $\dim R = 1$ if and only if condition (2) of Proposition 6 is satisfied.

Theorem 4. *Let R be a Noetherian integral domain. Then the following are equivalent:*

(1) *R is a complete local ring and* $\dim R = 1$.

(2) *Every proper R-submodule of Q is finitely generated and* $\mathrm{Hom}_R(K, K) \cong R$.

(3) $\mathrm{Ext}_R{}^1(Q/S, R) \neq 0$ *for every proper, non-zero, R-submodule S of Q and* $\mathrm{Ext}_R{}^1(Q, R) = 0$.

(4) $\mathrm{Ext}_R{}^1(Q, S) = 0$ *for every R-submodule S of Q.*

(5) *Every R-homomorphic image of Q is indecomposable and* $\mathrm{Hom}_R(K, K) \cong R$.

(6) *R is a local ring*, $\dim R = 1$, *and* $\mathrm{Hom}_R(K, K) \cong R$.

Proof.

(1) $\Rightarrow$ (2). We will denote the maximal ideal of R by M, and let $E = E(R/M)$. If A is any R-module, we let $A^* = \mathrm{Hom}_R(A, E)$. By Proposition 5 we have $Q \cong Q^*$; and, of course, we have $R^* = E$. Thus we have an exact sequence:

$$0 \to K^* \to Q \to E \to 0,$$

and so $E \cong Q/K^*$. By Theorem 1, K has D.C.C. Therefore, by Theorem A2, K^* is finitely generated. Hence, since K^* is an R-submodule of Q, K^* is isomorphic to an ideal I of R. Thus we have $E \cong Q/K^* \cong Q/I$, and so K is a homomorphic image of E. It now follows from Proposition 3 that every proper submodule of K is finitely generated. It is an immediate consequence that every proper R-submodule of Q is finitely generated. By Theorem A5 we have $\mathrm{Hom}_R(K, K) \cong R$.

(2) $\Leftrightarrow$ (3). Let S be a proper, non-zero R-submodule of Q. Then we have an exact sequence:

$$0 \to \mathrm{Hom}_R{}^1(S, R) \to \mathrm{Ext}_R{}^1(Q/S, R) \to \mathrm{Ext}_R(Q, R).$$

Since $\mathrm{Hom}_R(K, K) \cong R$ is equivalent to $\mathrm{Ext}_R{}^1(Q, R) = 0$ by Theorem A5, we have, assuming either (2) or (3), that $\mathrm{Hom}_R(S, R) \cong \mathrm{Ext}_R{}^1(Q/S, R)$. The equivalence of (2) and (3) now follows from the fact that $\mathrm{Hom}_R(S, R) \neq 0$ if and only if S is finitely generated.

(2) $\Rightarrow$ (4). This follows immediately from Theorem A5.

(4) $\Rightarrow$ (5). This is an immediate consequence of Proposition 4 and Theorem A5.

(5) $\Rightarrow$ (6). This follows directly from Theorem 2.

(6) $\Rightarrow$ (1). If $\bar{R}$ is the completion of R, we can define an operation of $\bar{R}$ on K as follows. Let $x \in K$ and $\bar{r} \in \bar{R}$. Then there exists a Cauchy sequence $\{r_n\}$ of elements $r_n \in R$ such that $r_n \to \bar{r}$. Let M be the maximal ideal of R. Then there exists an integer $k > 0$ such that $M^k x = 0$, and such that $r_n - r_m \in M^k$, whenever $n, m \geqslant k$. We define $\bar{r}x = r_k x$. It is easily verified that this definition makes K into an $\bar{R}$-module. Suppose $\bar{r}K = 0$. Let $a \neq 0 \in M$. Then there exists an integer $N > 0$ such that $r_n \in Ra$ for all $n > N$; hence $r_n \to 0$, and so $\bar{r} = 0$. Thus we have $\bar{R} \subset \mathrm{Hom}_R(K, K)$. Since $\mathrm{Hom}_R(K, K) \cong R$ by assumption, it follows that R is complete.

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We note that if R is a complete, Noetherian, local domain, then by Theorem A5 we may omit the hypotheses $\text{Hom}_R(K, K) \cong R$ and $\text{Ext}_R{}^1(Q, R) = 0$ wherever they appear in the statement of Theorem 4. As a corollary of Theorem 4 we obtain a special case of a theorem of M. Nagata **(11**, Th. 7**)** (see also D. G. Northcott **(12**, Prop. 4**)**).

COROLLARY 1 (*special case of a theorem of M. Nagata*). *Let R be a complete, Noetherian, local domain such that* $\dim R = 1$. *Then the integral closure of R is a complete, discrete, valuation ring that is finitely generated over R.*

Proof. This follows immediately from Theorem 4 and the corollary to Proposition 4.

COROLLARY 2. *Let R be a complete, Noetherian, local domain with maximal ideal M and $E = E(R/M)$. Then* $\dim R = 1$ *if and only if* $\text{Ext}_R{}^1(E, R) \neq 0$.

Proof. Suppose $\dim R = 1$. Now by Proposition 2 there exists a proper, non-zero R-submodule T of Q such that $E \cong Q/T$. Hence $\text{Ext}_R{}^1(E, R) \neq 0$ by Theorem 4. Conversely, assume that $\text{Ext}_R{}^1(E, R) \neq 0$. Let S be a proper, non-zero, R-submodule of Q. Then by Proposition 2 there exists an R-submodule T of Q such that $S \subset T$ and such that the following sequence is exact:

$$0 \to T/S \to Q/S \to E \to 0.$$

Since $\text{Hom}_R(T/S, R) = 0$, we have $\text{Ext}_R{}^1(E, R) \subset \text{Ext}_R{}^1(Q/S, R)$, and thus $\text{Ext}_R{}^1(Q/S, R) \neq 0$. Hence $\dim R = 1$ by Theorem 4.

COROLLARY 3. *Let R be a complete, Noetherian, local domain such that* $\dim R = 1$. *Then every torsion-free R-module of finite rank is a direct sum of a finite number of copies of Q and of a finitely generated R-module.*

Proof. Let S be a torsion-free R-module of finite rank. To prove the corollary it is sufficient to assume that S is reduced, and then prove that S is finitely generated. We proceed by induction on rank S. If rank $S = 1$, then S is finitely generated by Theorem 4. Hence assume that rank $S = n > 1$, and assume the theorem true for modules of smaller rank. Now S has a submodule T of rank $n - 1$ such that S/T is torsion-free of rank 1. Thus T is finitely generated by the induction hypothesis. If S/T is not reduced, then $S/T \cong Q$, and we have $\text{Ext}_R{}^1(S/T, T) = 0$ by Theorem 4. Thus S/T is a direct summand of S, which contradicts the fact that S is reduced. Hence S/T is reduced, and so S/T is finitely generated. Thus S is finitely generated.

COROLLARY 4. *Let R be a complete, Noetherian, local domain such that* $\dim R = 1$. *Let S be a torsion-free R-module of finite rank. Then* $\text{hd}_R S < \infty$ *if and only if S is a direct sum of a free R-module and copies of Q. And in this case,* $\text{hd}_R S \leqslant 1$.

Proof. By Corollary 3, $S = T \oplus D$, where T is a finitely generated submodule of S, and D is a direct sum of copies of Q. By Theorem A6, $\text{hd}_R D \leqslant 1$.

Thus, $\text{hd}_R S < \infty$ if and only if $\text{hd}_R T < \infty$. Since f. gl. dim. $R = 1$, and since T is a submodule of a free R-module, it follows that $\text{hd}_R T < \infty$ if and only if T is free.

3. K is injective.

THEOREM 5. *Let R be a Noetherian integral domain such that K is injective. Then* $\dim R = 1$, *and* $K = \sum \oplus E(R/M_\alpha)$, *where M_α ranges over all of the maximal ideals of R, and each $E(R/M_\alpha)$ appears exactly once.*

Proof.

Case I: R is a local ring with maximal ideal M. Let A be any indecomposable, direct summand of K. Then by Theorem A1, $A = E(R/P)$, where P is a non-zero prime ideal of R. Now P^{-1}/R is contained in K, and is a finitely generated R/P-module. By Theorem A1, a copy of the quotient field of R/P is contained in $A \cap P^{-1}/R$. Thus the quotient field of R/P is finitely generated over R/P, and so is equal to R/P. Thus P is a maximal ideal, and hence $P = M$. Therefore, by Theorem A1, we see that $K = \sum \oplus E_\beta$, where each $E_\beta = E(R/M) = E$. Now

$$E \cong \text{Tor}_1^R(K, E) \cong \sum \oplus \text{Tor}_1^R(E_\beta, E).$$

Since E is indecomposable, $K \cong E$. Let P' be any non-zero, prime ideal of R. Then we have an exact sequence:

$$0 \to R_{P'} \to Q \to E \otimes_R R_{P'} \to 0.$$

Since $R_{P'} \neq Q$, we have $E \otimes_R R_{P'} \neq 0$; and thus $P' = M$. Hence $\dim R = 1$.

Case II: R is an arbitrary Noetherian domain. Let M be any maximal ideal of R. Then by (**2**, Ch. 6, Ex. 11), $K_M = K \otimes_R R_M \cong Q/R_M$ is an injective R_M-module. Hence by Case I, $\dim R_M = 1$. Thus $\dim R = 1$. Now by Theorem A7, $K = \sum \oplus K_{M_\alpha}$, where M_α ranges over all of the maximal ideals of R. By Case I,

$$K_{M_\alpha} \cong E(R_{M_\alpha}/M_\alpha R_{M_\alpha}) = E(R/M_\alpha).$$

Thus each $E(R/M_\alpha)$ appears exactly once.

The following corollary is a generalization of a theorem of R. J. Nunke (**13**, Cor. 7.9).

COROLLARY. *Let R be a Noetherian integral domain such that K is injective. Then R is a complete local ring and* $\dim R = 1$ *if and only if* $\text{Ext}_R{}^1(K, R) \cong R$.

Proof. Suppose $\text{Ext}_R{}^1(K, R) \cong R$. Then by Proposition 4 and Theorem A5, we have $\text{Hom}_R(K, K) \cong R$. Thus K is indecomposable, and so by Theorem 5 R is a local ring and $\dim R = 1$. By Theorem 4, R is complete. Conversely, if R is any complete local domain, then $\text{Ext}_R{}^1(Q, R) = 0$ by Theorem A5. Consequently, $\text{Ext}_R{}^1(K, R) \cong R$ by Proposition 4.

THEOREM 6. *Let* R *be a Noetherian, local domain such that* $\dim R = 1$; *and such that* $\bar{R}$, *the completion of* R, *is an integral domain. Let* M *be the maximal ideal of* R *and* $E = E(R/M)$. *Then the following are equivalent:*

(1) K *is injective.*

(2) $\mathrm{Hd}_R E = 1$.

(3) R *has an irreducible, principal ideal.*

(4) $M^{-1}/R \cong R/M$.

(5) M^{-1} *is generated by at most two elements.*

If any of the above conditions hold, then every proper R*-submodule of* Q *is finitely generated; and every principal ideal of* R *is irreducible.*

Proof.

(1) $\Rightarrow$ (2). By Theorem 5 we have $K \cong E$. By Theorem A6, $\mathrm{hd}_R K = 1$.

(2) $\Rightarrow$ (1). By assumption we have an exact sequence:

$$0 \to F_1 \to F_0 \to E \to 0,$$

where F_1, F_0 are free R-modules. Let $F_1^* = \mathrm{Hom}_R(F_1, E)$ and $F_0^* = \mathrm{Hom}_R(F_0, E)$. Since $\mathrm{Hom}_R(E, E) \cong \bar{R}$ by Theorem A1, we have an exact sequence:

$$0 \to \bar{R} \to F_0^* \to F_1^* \to 0.$$

Now F_0^*, F_1^* are injective R-modules, and so the injective dimension of $\bar{R}$, considered as an R-module, is one. Since $Q \otimes_R \bar{R}$ is R-injective, we conclude from the exact sequence:

$$0 \to \bar{R} \to Q \otimes_R \bar{R} \to K \otimes_R \bar{R} \to 0$$

that $K \otimes_R \bar{R}$ is R-injective. By Theorem 1, K has D.C.C., and so by Theorem A7 we have $K \cong K \otimes_R \bar{R}$. Thus K is injective.

(1) $\Rightarrow$ (3). Let $s \neq 0 \in M$, and let $x = 1/s + R, x \in K$. Then $0(x) = sR$. Hence, since $K \cong E$, sR is irreducible by Theorem A1. Thus every principal ideal of R is irreducible.

(3) $\Rightarrow$ (1). Suppose that $s \neq 0 \in M$, and sR is an irreducible ideal of R. By Theorem A1 there exists $x_1 \neq 0 \in E$ such that $0(x_1) = sR$. Since E is divisible, we can find $x_2 \in E$ such that $sx_2 = x_1$, and $x_3 \in E$ such that $sx_3 = x_2$, etc. Let L be the multiplicative system consisting of the powers of s. Then $R_L = Q$, and so K is generated by the set $\{1/s^k + R\}$. Define $f: K \to E$ by $f(a/s^k + R) = ax_k$, $a \in R$. It is easily verified that f is a monomorphism. Thus $K \subset E$, and so by Proposition 2, $K = E$.

(1) $\Rightarrow$ (4). We have $\mathrm{Hom}_R(R/M, K) \cong \mathrm{Ann}_K(R/M) = M^{-1}/R$. If $K \cong E$, then $\mathrm{Hom}_R(R/M, K) \cong R/M$ by Theorem A1. Therefore, $M^{-1}/R \cong R/M$.

(4) $\Rightarrow$ (1). M^{-1}/R is the socle of K. Since R has dimension one, it is easily seen that K is an essential extension of M^{-1}/R. Thus $E(K) = E(M^{-1}/R)$. Hence, if $M^{-1}/R \cong R/M$, then $E(K) = E$. But then $K = E$ by Proposition 2.

(4) $\Rightarrow$ (5). If $M^{-1}/R \cong R/M$, take $u \in M^{-1}$, $u \notin R$. Then M^{-1} will be generated by 1, u.

(5) $\Rightarrow$ (4). Suppose M^{-1} is generated by at most two elements. If $MM^{-1} = R$, then M is projective, hence principal, and so we have $M^{-1}/R \cong R/M$. Hence assume $MM^{-1} \neq R$. Then $MM^{-1} = M$, and we have an exact sequence:

$$0 \to R/M \to M^{-1}/M \to M^{-1}/R \to 0$$

over the field R/M. Therefore, the sequence splits, and we have $M^{-1}/M \cong R/M \oplus M^{-1}/R$. Now M^{-1}/M is a vector space of dimension two over R/M, and so M^{-1}/R is one dimensional over R/M. Thus $M^{-1}/R \cong R/M$.

Now assume that K is injective. Let S be a proper, non-zero R-submodule of Q. We can assume that $R \subset S$. Then $S/R \subset K$, and $S/R \neq K$. By Proposition 2, S/R is not a faithful submodule of K. Hence there exists $a \neq 0 \in R$ such that $a(S/R) = 0$. Therefore, $aS \subset R$, and so S is a finitely generated R-module.

COROLLARY. *Let R be a Noetherian, local domain such that* $\dim R = 1$ *and $\bar{R}$ is an integral domain. If M is generated by at most two elements, then K is injective.*

Proof. By Theorem 6 it will be sufficient to prove that M^{-1} is generated by at most two elements. Let $u_1, u_2 \in M$ be the generators of M. We can assume that $u_2 \notin u_1R$. There exists an integer $n > 0$ such that $u_2^n \notin u_1R$, but $u_2^{n+1} \in u_1R$. Let B be the R-module generated by 1, u_2^n/u_1. Then $B \subset M^{-1}$. Conversely, let $x \in M^{-1}$. Then $x = b/u_1$, where $b \in M$. If $b \in u_1R$, then $x \in B$; hence assume that $b \notin u_1R$. Since $Ru_1 + Ru_2^k = Ru_1 + M^k$, and since

$$\bigcap_k (Ru_1 + M^k) = Ru_1,$$

there exists an integer $k > 0$ such that $b \in Ru_1 + Ru_2^k$, but $b \notin Ru_1 + Ru_2^{k+1}$. Thus $b = ru_1 + su_2^k$, where $r \in R$ and $s \in R - M$. Hence $x = b/u_1 = r + s(u_2^k/u_1)$; and since s is a unit, $(u_2^k/u_1) \in M^{-1}$. Hence $u_2^{k+1} \in u_1R$, which means that $k \geqslant n$, and so $x \in B$. Thus $M^{-1} = B$, and so M^{-1} is generated by at most two elements.

PROPOSITION 7. *Let R be an integral domain and $S \neq 0$ a torsion-free R-module such that S, and every factor module of S by a cyclic submodule, is indecomposable. Let A be an extension of R by S. Then, if A is not the split extension, A is an indecomposable, torsion-free R-module and rank A = rank $S + 1$.*

Proof. Suppose that A decomposes into a direct sum: $A = A_1 \oplus A_2$, where A_1, A_2 are non-zero submodules of A. Now there exist $x_1 \in A_1$, $x_2 \in A_2$ such that, if $x = x_1 + x_2$, then $A/Rx \cong S$. We must prove that Rx is a direct summand of A. Let $B = Rx_1 \oplus Rx_2$; then $B = Rx + Rx_2$, and so B/Rx is cyclic. Since $B/Rx \subset A/Rx$, B/Rx is torsion-free. Thus Rx is a direct summand of B, and $B = Rx \oplus T$, where T is a submodule of B such that either $T = 0$ or $T \cong R$.

Now we have $A_1/Rx_1 \oplus A_2/Rx_2 \cong A/B \cong (A/Rx)/(B/Rx) \cong S/T'$, where T' is a submodule of S such that $T' \cong T$. Since S/T' is indecomposable by

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assumption, we can assume that $A_2/Rx_2 = 0$; that is, $A_2 = Rx_2$. If $y \in A_2$, then $y = rx_2 = rx - rx_1$, where $r \in R$. Thus $A_2 \subset Rx + A_1$, and so $A = Rx + A_1$. Suppose $z \neq 0 \in Rx \cap A_1$. Then $z = rx$, $r \in R$, and $z = y_1 \in A_1$. Hence $rx_2 = 0$. Since $z \neq 0$, we have $r \neq 0$, and thus $x_2 = 0$. But then $A_2 = 0$, which is a contradiction to assumption. Hence $Rx \cap A_1 = 0$, and so $A = Rx \oplus A_1$. Thus Rx is a direct summand of A which makes A the split extension.

We now list four examples of integral domains that have indecomposable, torsion-free modules of rank two.

(1) *R is an integral domain such that K is indecomposable and* $\mathrm{Hom}_R(K, K) \neq R$.

Proof. By Theorem A5 we have $\mathrm{Ext}_R{}^1(Q, R) \neq 0$. The conclusion follows from Proposition 7 and the fact that there is an extension of R by Q that is not the split extension.

(2) *R is a Noetherian, local domain such that K is injective, but R is not complete.*

Proof. By Theorem 5 we have $K \cong E$, and so K is indecomposable. Since R is not complete, $\mathrm{Hom}_R(K, K) \neq R$ by Theorem A1. The conclusion now follows from example (1).

(3) *R is a Noetherian domain such that every R-homomorphic image of Q is indecomposable, but R is not complete.*

Proof. Since R is not complete, we have by Theorem 4 that $\mathrm{Hom}_R(K, K) \neq R$. The conclusion now follows from example (1).

(4) *R is an integral domain such that every principal ideal of R is irreducible, but K is not injective.*

Proof. Since K is not injective, there exists a non-zero ideal of R such that $\mathrm{Ext}_R{}^2(R/I, R) \neq 0$. Hence $\mathrm{Ext}_R{}^1(I, R) \neq 0$, and so there exist extensions of R by I that are not split. Since every principal ideal of R is irreducible, every factor module of I by a cyclic submodule is indecomposable. The conclusion now follows from Proposition 7.

COROLLARY. *Let R be a valuation ring. Then R is a maximal valuation ring if and only if every torsion-free R-module of rank two decomposes into a direct sum of two R-modules of rank one.*

Proof. If R is maximal, see (**5**, Th. 12) or Theorem A4. Conversely, assume that every torsion-free R-module of rank two is decomposable. By the preceding example (4) we have that K is injective. Hence by Theorem A3, R is almost maximal. By the preceding example (1) we have that $\mathrm{Hom}_R(K, K) \cong R$. Thus R is a maximal valuation ring by Theorem A4.

Appendix

THEOREM A1. *Let R be a commutative, Noetherian ring. Then:*

(1) *Every injective R-module is a direct sum of indecomposable injective R-modules.*

(2) *There is a one-to-one correspondence between the prime ideals P of R and the indecomposable injective R-modules given by $P \leftrightarrow E(R/P)$.*

(3) *If P is a prime ideal of R, then I is an irreducible, P-primary ideal of R if and only if there exists an element $x \neq 0 \in E(R/P)$ such that $I = 0(x)$.*

(4) $\mathrm{Hom}_R(E(R/P), E(R/P)) \cong \bar{R}_P$, *where $\bar{R}_P$ is the completion of R_P.*

(5) *If I is any ideal of R contained in the prime ideal P, then* $\mathrm{Hom}_R(R/I, E(R/P))$ *is the injective envelope over R/I of $(R/I)/(P/I)$. Thus, in particular,* $\mathrm{Hom}_R(R/P, E(R/P))$ *is isomorphic over R/P to the quotient field of R/P.*

Proof. (**7**, Th. 2.5), (**7**, Prop. 3.1), (**7**, Lemma 3.2), (**7**, Th. 3.4), (**7**, Th. 3.7).

THEOREM A2. *Let R be a complete, Noetherian, local ring with maximal ideal M and $E = E(R/M)$. Then there is a one-to-one, lattice-order inverting correspondence between the ideals I of R and the submodules A of E given by:*

$$I \leftrightarrow \mathrm{Ann}_E(I) \qquad \text{and} \qquad A \leftrightarrow 0(A),$$

such that $I = 0(\mathrm{Ann}_E(I))$ and $A = \mathrm{Ann}_E(0(A))$. If B is a finitely generated R-module (resp. has D.C.C.*), then* $\mathrm{Hom}_R(B, E)$ *has* D.C.C. *(resp. is finitely generated); and we have* $\mathrm{Hom}_R(\mathrm{Hom}_R(B, E), E) \cong B$.

Proof. (**7**, Th. 4.2), (**7**, Cor. 4.3).

THEOREM A3. *Let V be a valuation ring with maximal ideal N and quotient field Q. Then V is almost maximal if and only if Q/V is injective. If V is almost maximal, then V is maximal if and only if* $\mathrm{Hom}_V(Q/N, Q/N) \cong V$. *And if V is almost maximal, Q/N is the injective envelope of V/N.*

Proof. (**8**, Th. 4), (**8**, Lemma 7), (**8**, Th. 9).

THEOREM A4. *Let R be an integral domain. Then the following are equivalent:*

(1) *R is a maximal valuation ring.*

(2) *R is an almost maximal valuation ring and* $\mathrm{Hom}_R(K, K) \cong R$.

(3) $\mathrm{Ext}_R{}^1(A, S) = 0$ *for every torsion-free R-module A, and every torsion-free R-module S of rank one.*

Proof. (**8**, Th. 9).

THEOREM A5. *Let R be an integral domain. Then* $\mathrm{Ext}_R{}^1(Q, R) = 0$ *if and only if* $\mathrm{Hom}_R(K, K) \cong R$*; and in this case* $\mathrm{Ext}_R{}^1(Q, S) = 0$ *for every finitely generated R-submodule S of Q. If R is a complete, Noetherian, local domain, then the above conditions are satisfied.*

Proof. (**8**, Lemma 6), (**7**, Th. 4.2), (**2**, Prop. 6.5.1).

THEOREM A6. *Let R be a Noetherian integral domain of dimension one. Then*

(1) $\mathrm{Hd}_R Q = 1$.

(2) *Every divisible R-module is a homomorphic image of an injectiev R-module.*

Proof. **(9**, Lemma 3.2**)**, **(9**, Th. 3.3**)**.

THEOREM A7. *Let R be a commutative, Noetherian ring, and A an R-module such that if $x \neq 0 \in A$, then every prime ideal that belongs to $0(x)$ is maximal. Then:*

(1) $A = \sum \oplus A_{M_\alpha}$, *where M_α ranges over the maximal ideals of R.*

(2) $A_{M_\alpha} = A \otimes_R \bar{R}_{M_\alpha}$, *where $\bar{R}_{M_\alpha}$ is the completion of R_{M_α}.*

(3) A_{M_α} *is finitely generated (resp. has* D.C.C.) *over R_{M_α} if and only if the same is true over R.*

Proof. **(9**, Lemma 3.1**)** and **(10)**.

THEOREM A8. *Let A be a module over a Noetherian ring R. Then A has* D.C.C. *if and only if A is contained in a finite direct sum of modules of the form $E(R/M_\alpha)$, where M_α is a maximal ideal of R.*

Proof. **(7**, Cor. 4.3**)** and **(10)**.

THEOREM A9. *Let A be a module with* D.C.C. *over a commutative, Noetherian ring R. Let I be an ideal of R. Then $IA = A$ if and only if there exists an element $r \in I$ such that $rA = A$.*

Proof. **(10)**.

THEOREM A10. *Let R be a commutative, Noetherian ring such that* $\dim R \geqslant 2$. *Then R has an infinite number of prime ideals of rank one.*

Proof. **(1**, Prop. 2.6**)**.

THEOREM A11. *Let R, S be rings and consider the situation described by the symbol $(A_R, {}_RB_S, C_S)$. Then there is a natural isomorphism:*

$$\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(A \otimes_R B, C).$$

If C is S-injective, then we have an isomorphism:

$$\mathrm{Ext}_R(A, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(\mathrm{Tor}^R(A, B), C).$$

Proof. **(2**, Prop. 2.5.2′**)**, **(2**, Prop. 6.5.1**)**.

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Northwestern University