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Obstructions of Connectivity Two for Embedding Graphs into the Torus

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Abstract. The complete set of minimal obstructions for embedding graphs into the torus is still not determined. In this paper, we present all obstructions for the torus of connectivity 2. Furthermore, we describe the building blocks of obstructions of connectivity 2 for any orientable surface.

1 Introduction

The problem of determining the graphs that can be embedded in a given surface is a fundamental question in topological graph theory. Robertson and Seymour [9] proved that for each surface $\mathbb{S}$ the class of graphs that embeds into $\mathbb{S}$ can be characterized by a finite list, Forb($\mathbb{S}$), of minimal forbidden minors (or *obstructions*). For the 2-sphere $\mathbb{S}_0$, Forb($\mathbb{S}_0$) consists of the *Kuratowski graphs*, K_5 and $K_{3,3}$. The list of obstructions Forb($\mathbb{N}_1$) for the projective plane $\mathbb{N}_1$ contains 35 graphs, and $\mathbb{N}_1$ is the only other surface for which the complete list of forbidden minors is known. The number of obstructions for both orientable and non-orientable surfaces seems to grow fast with the genus and that can be one of the reasons why even for the torus $\mathbb{S}_1$ the complete list of obstructions is still not known, although thousands of obstructions have been generated by the computer (see [2, 5, 8, 11]).

In this paper, we study the obstructions for orientable surfaces of low connectivity. A graph G is a *k-sum* of graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ if G can be written as $G = (V_1 \cup V_2, E_1 \cup E_2)$ such that $|V_1 \cap V_2| = k$. It is easy to show that obstructions that are not 2-connected can be obtained as disjoint unions and 1-sums of obstructions for surfaces of smaller genus (see [1]). Stahl [10] and Decker et al. [3] showed that the genus of a 2-sum differs by at most 1 from the sum of genera of its parts. Decker et al. [4] provided a simple formula for the genus of a 2-sum that will be used in this paper. We shall prove that obstructions for an orientable surface of connectivity 2 can be obtained as a 2-sum of building blocks that fall (roughly) into two families of graphs. One family consists of obstructions for embeddings into surfaces of smaller genus. The graphs in the second family are critical with respect to the graph parameter g_a defined in Section 3. We use this characterization in Section 8 to construct all obstructions for the torus of connectivity 2.

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2 Notation

Let G be a connected multigraph. An (orientable) *embedding* Π of G is a mapping that assigns to each vertex $v \in V(G)$ a cyclic permutation π_v of the edges incident with v, called the *local rotation* around v. Thus $\pi_v(e)$ for an edge e incident with v is the edge that follows e in clockwise order around v in the embedding Π. If Π is an embedding of G, then we also say that G is Π-*embedded*. Given a Π-embedded graph G, a Π-*face* (or Π-*facial walk*) is a cyclic sequence $(v_1, e_1, \dots, v_k, e_k)$ such that $e_i = v_i v_{i+1}$, $e_i = \pi_{v_i}(e_{i-1})$ for each $i = 1, \dots, k$ (where $v_{k+1} = v_1$ and $e_0 = e_k$), and all pairs (v_i, e_i) are distinct. The linear subsequence e_{i-1}, v_i, e_i of a Π-face W is called a Π-*angle* at v_i. Note that edges of Π-angles are formed precisely by the pairs of edges that are consecutive in the local rotation around a vertex. The linear subsequence of W that is obtained from W by removing v_i is said to be obtained by *opening* W at v_i.

The above combinatorial definition of an embedding only uses local rotations and the notion of facial walks. This is equivalent to the notion of topological 2-cell embeddings of graphs in surfaces, and we refer the reader to [6] for detailed treatment of this relationship.

Each edge e of a Π-embedded graph appears twice in the Π-faces. If there exists a single Π-face where e appears twice, we say that e is *singular*. Otherwise, e is *nonsingular*.

The *genus* of an orientable embedding Π of a graph G is given by the Euler formula,

$$g(\Pi) = \frac{1}{2}(2 - n + m - f), \tag{2.1}$$

where n is the number of vertices, m the number of edges, and f the number of Π-faces of G. The *genus* $g(G)$ of a connected multigraph G is the minimum genus of an orientable embedding of G.

In this paper, we will deal mainly with the class $\mathcal{G}$ of simple graphs. Let $G \in \mathcal{G}$ be a simple graph and let e be an edge of G. Then $G - e$ denotes the graph obtained from G by *deleting* e and G/e denotes the graph[1] obtained from G by *contracting* e. It is convenient for us to formalize these graph operations. The set $\mathcal{M}(G) = E(G) \times \{-, /\}$ is the *set of minor-operations* available for G. An element $\mu \in \mathcal{M}(G)$ is called a *minor-operation* and μG denotes the graph obtained from G by applying μ. For example, if $\mu = (e, -)$, then $\mu G = G - e$. A graph H is a *minor* of G if H can be obtained from a subgraph of G by contracting some edges. If G is connected, then H can be obtained from G by a sequence of minor-operations.

Let H be a subgraph of G. We say that H is *minor-tight* in G (for the genus parameter g) if $g(\mu G) < g(G)$ for every minor-operation $\mu \in \mathcal{M}(H)$. The following observation asserts that being an obstruction for a surface is equivalent to having all subgraphs minor-tight.

Lemma 2.1 *Let $H_1, \dots, H_s$ be subgraphs of a graph G with $g(G) = k+1$. If $E(H_1) \cup \dots \cup E(H_s) = E(G)$, then G is an obstruction for $\mathbb{S}_k$ if and only if $H_1, \dots, H_s$ are minor-tight in G.*

[1] When contracting an edge, one may obtain multiple edges. We shall replace any multiple edges by single edges as such a simplification has no effect on the genus.

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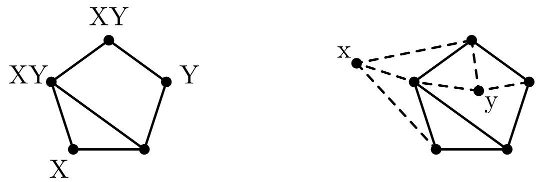

Figure 1: An example of an XY-labelled graph and its corresponding graph in $\mathcal{G}_{xy}^{\circ}$.

It is well known that each closed orientable surface is homeomorphic, for some $k \geq 0$, to the surface $\mathbb{S}_k$, which is the surface obtained from the sphere by adding k handles. A graph with an embedding of genus k can be viewed as embedded onto $\mathbb{S}_k$ (see [6]).

A graph has *connectivity* k when it is k-connected but not $(k+1)$-connected.[2] An edge whose deletion disconnects the graph is a *cut-edge*. The structure of obstructions for orientable surfaces that have connectivity at most 1 is very simple. They are disjoint unions and 1-sums of obstructions for surfaces of smaller genus. This can be easily seen as an application of the following theorem that states that the genus of graphs is additive with respect to their 2-connected components (or *blocks*).

Theorem 2.2 (Battle et al. [1]) *The genus of a graph is the sum of the genera of its blocks.*

3 Graphs with Terminals

In this paper, we study obstructions for embedding graphs into orientable surfaces that have connectivity 2. Given graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \{x, y\}$, we say that the graph $G = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ is the *xy-sum* of G_1 and G_2. The graphs G_1 and G_2 are the *parts* of the xy-sum.

We wish to study the parts of a 2-sum separately, and in order to do so, we mark the vertices of the separation as *terminals*. This prompts us to study the class of graphs $\mathcal{G}_{xy}$ with two terminals, x and y. The letters x and y will be consistently used for the two distinguished terminals. Most notions that are used for graphs can be used in the same way for graphs with terminals. Some notions differ though, and to distinguish between graphs with and without terminals, let $\widehat{G}$ be the underlying graph of G without terminals (for $G \in \mathcal{G}_{xy}$). Two graphs, G_1 and G_2, in $\mathcal{G}_{xy}$ are *isomorphic* if there is an isomorphism of the graphs $\widehat{G_1}$ and $\widehat{G_2}$ that maps terminals of G_1 onto terminals of G_2 (and non-terminals onto non-terminals) possibly exchanging x and y. Note that it is possible that $G_1, G_2 \in \mathcal{G}_{xy}$ are non-isomorphic, but $\widehat{G_1}, \widehat{G_2}$ are isomorphic. We define minor-operations on graphs in $\mathcal{G}_{xy}$ in the way that $\mathcal{G}_{xy}$ is a minor-closed family. When performing edge contractions on $G \in \mathcal{G}_{xy}$, we do not allow contraction of the edge xy (if $xy \in E(G)$) and contracting an edge incident with x results

[2]Here and thereafter we only discuss vertex-connectivity. Recall that a graph is k-connected if it has at least $k+1$ vertices and it remains connected after removal of any set of at most $k-1$ vertices.

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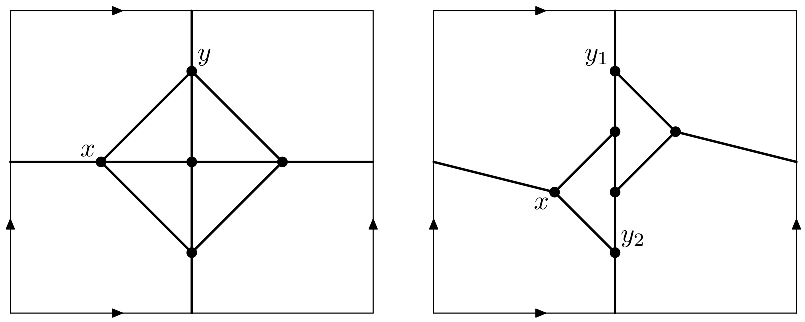

Figure 2: xy-alternating embeddings of K_5 and $K_{3,3}$ in the torus.

in a vertex labelled x and similarly for y. We use $\mathcal{M}(G)$ to denote the set of available minor-operations for G. Since $(xy, /) \notin \mathcal{M}(G)$ for $G \in \mathcal{G}_{xy}$, we shall use G/xy to denote the underlying simple graph in $\mathcal{G}$ obtained from G by identification of x and y; for this operation, we do not require the edge xy to be present in G.

For convenience, we use $\mathcal{G}^\circ_{xy}$ for the subclass of $\mathcal{G}_{xy}$ of graphs without the edge xy. We shall sometimes depict the graphs in $\mathcal{G}^\circ_{xy}$ as *XY-labelled* graphs, which are defined as follows. Given a graph $G \in \mathcal{G}^\circ_{xy}$, let H be the graph $G - x - y$, where a vertex of H is labelled X if it is adjacent to x in G and is labelled Y if it is adjacent to y in G (see Figure 1). We say that H is the *XY-labelled* graph *corresponding* to G.

A *graph parameter* is a function $\mathcal{G} \to \mathbb{R}$ that is constant on each isomorphism class of $\mathcal{G}$. Similarly, we call a function $\mathcal{G}_{xy} \to \mathbb{R}$ a *graph parameter* if it is constant on each isomorphism class of $\mathcal{G}_{xy}$. A graph parameter $\mathcal{P}$ is *minor-monotone* if $\mathcal{P}(H) \le \mathcal{P}(G)$ for each graph $G \in \mathcal{G}_{xy}$ and each minor H of G. The graph genus is an example of a minor-monotone graph parameter.

Several other graph parameters will be used in this paper. We use G^+ for the graph G plus the edge xy if it is not already present. The genus of G^+ can also be viewed as a graph parameter g^+ defined as $g^+(G) = g(G^+)$. The graph parameter $\theta = g^+ - g$ captures the difference between the genera of G^+ and G; that is, $\theta(G) = g^+(G) - g(G)$. Note that $\theta(G) \in \{0, 1\}$.

In order to compute the genus of an xy-sum of graphs, it is necessary to know whether G has a minimum genus embedding Π with x and y appearing at least twice in an alternating order on a Π-face. More precisely, we say that an embedding Π is *xy-alternating* if there is a Π-face W such that (x, y, x, y) is a cyclic subsequence of W. A graph $G \in \mathcal{G}_{xy}$ is *xy-alternating* if it has a minimum genus embedding that is xy-alternating. Figure 2 shows two examples of xy-alternating embeddings in the torus. Note that, up to isomorphism, there are precisely two graphs in $\mathcal{G}_{xy}$, whose underlying simple graph is $K_{3,3}$, and Figure 2 gives xy-alternating embeddings for both of them (one for terminals x, y_1 and second for terminals x, y_2). We associate a graph parameter with this property. Let $\epsilon(G) = 1$ if G is xy-alternating and $\epsilon(G) = 0$ otherwise.[3] We shall also use the graph parameter ϵ^+ defined as $\epsilon^+(G) = \epsilon(G^+)$.

In order to describe minimum genus embeddings of an xy-sum G of graphs G_1 and G_2, it is sufficient to consider two types of embeddings. To construct them, we take particular minimum genus embeddings Π_1 and Π_2 of G_1 and G_2 (respectively) and combine them into an embedding Π of G. For a non-terminal vertex v, let the local rotation around v in Π be the same as the local rotation around v in Π_i (if $v \in V(G_i)$ for $i \in \{1,2\}$). Consider Π_1-faces W_1 and W_2 incident with x and y, respectively, and Π_2-faces W_3 and W_4 incident with x and y, respectively. Note that the faces W_1 and W_2 (and also W_3 and W_4) need not be distinct. We distinguish three cases.

Case 1: W_1, W_2, W_3, W_4 are distinct faces.

Write the face W_1 as (x, e_1, U_1, e_2), W_2 as (y, f_1, U_2, f_2), W_3 as (x, e_3, U_3, e_4), and W_4 as (y, f_3, U_4, f_4).[4] Note that e_1 follows the edge e_2 in the clockwise rotation around x in the embedding Π_1. Let e_1, S_1, e_2 be the linear sequence obtained from $\Pi_1(x)$ by opening it at e_1, e_2. Similarly, let e_3, S_2, e_4 be the linear sequence obtained from $\Pi_2(x)$ by opening it at e_3, e_4. We let $\Pi(x)$ be the cyclic sequence $(e_1, S_1, e_2, e_3, S_2, e_4)$. Similarly, we define $\Pi(y)$ as the concatenation of the two linear sequences obtained from $\Pi_1(y)$ and $\Pi_2(y)$ by opening each of them at f_1, f_2 and f_3, f_4, respectively. Each Π_1-face and Π_2-face different from W_1, W_2, W_3, and W_4 is also a Π-face. The faces W_1 and W_3 combine into the Π-face $(x, e_1, U_1, e_2, x, e_3, U_3, e_4)$, and the faces W_2 and W_4 combine into the Π-face $(y, f_1, U_2, f_2, y, f_3, U_4, f_4)$. Thus, the total number of faces decreases by two, and (2.1) gives the following value $h_0(G)$ of $g(\Pi)$:

$$g(\Pi) = g(G_1) + g(G_2) + 1 = h_0(G). \tag{3.1}$$

Case 2: W_1, W_2, W_3, W_4 consist of three distinct faces.

We may assume that $W_3 = W_4 = (x, e_3, U_3, f_4, y, f_3, U_4, e_4)$. The same construction as in the previous case (with W_1 and W_2 expressed as above) combines W_1, W_2, and W_3 into a single Π-face $(x, e_1, U_1, e_2, e_3, U_3, f_4, y, f_1, U_2, f_2, y, f_3, U_4, e_4, x)$. As before, the total number of faces decreases by two and the genus of Π is given by (3.1).

Case 3: $W_1 = W_2$ and $W_3 = W_4$.

Observe that since $W_1 = W_2$, we have that $\theta(G_1) = 0$, and similarly, we have $\theta(G_2) = 0$. Write $W_1 = W_2 = (x, e_1, U_1, f_2, y, f_1, U_2, e_2)$ and $W_3 = W_4 = (x, e_3, U_3, f_4, y, f_3, U_4, e_4)$. The above construction combines W_1 and W_3 into the Π-faces $(x, e_1, U_1, f_2, y, f_3, U_4, e_4)$ and $(y, f_1, U_2, e_2, x, e_3, U_3, f_4)$. Thus, the total number of faces does not change, and (2.1) gives the following value of $g(\Pi)$,

$$g(\Pi) = g(G_1) + g(G_2). \tag{3.2}$$

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[3]Parameters θ, ϵ and ϵ^+ encode basic properties of parts of xy-sums. They are used throughout the paper. An easy way to remember their meaning is that θ describes whether adding the edge xy increases the genus, while ϵ and ϵ^+ describe whether G and $G + xy$ are xy-alternating, respectively.

[4]In this notation, U_1, U_2, U_3, U_4 denote the corresponding subsequences of vertices and edges in these facial walks.

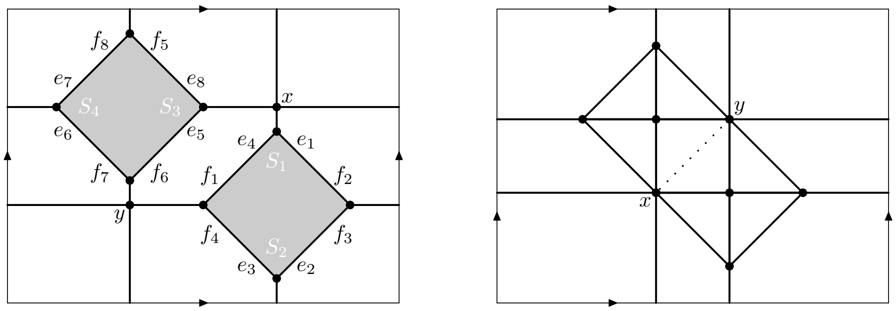

Figure 3: (a) An illustration of an embedding of an xy-sum of two xy-alternating graphs on the torus. For better clarity, the vertices x and y were split into 5 vertices each. Contract all the edges incident with x and y to get the xy-sum. (b) The 2-sum of two copies of K_5 embedded into the torus.

Suppose that Π_1 and Π_2 are minimum genus embeddings of G_1 and G_2 (respectively) that are both xy-alternating. In this case, we construct an embedding of G whose genus is smaller than given above. Let W_1 and W_2 be the xy-alternating faces of Π_1 and Π_2, respectively, and write W_1 as $(x, e_1, U_1, f_2, y, f_1, U_2, e_4, x, e_3, U_3, f_4, y, f_3, U_4, e_2)$ and W_2 as $(x, e_5, U_5, f_6, y, f_5, U_6, e_8, x, e_7, U_7, f_8, y, f_7, U_8, e_6)$. Again, the local rotation $\Pi(v)$ of a non-terminal vertex $v \in V(G_i)$ is set to $\Pi_i(v)$, $i = 1, 2$. To construct $\Pi(x)$, open $\Pi_1(x)$ at e_1, e_2 and e_3, e_4 to obtain two linear sequences e_1, S_1, e_4 and e_3, S_2, e_2; open $\Pi_2(x)$ at e_5, e_6 and e_7, e_8 to obtain e_5, S_3, e_8 and e_7, S_4, e_6. Let $\Pi(x)$ be the cyclic sequence $(e_1, S_1, e_4, e_5, S_3, e_8, e_3, S_2, e_2, e_7, S_4, e_6)$. We construct $\Pi(y)$ similarly. Figure 3 illustrates this process and gives an example of a 2-sum of two K_5's. The faces W_1 and W_2 are combined into Π-faces $(x, e_1, U_1, f_2, y, f_7, U_8, e_6)$, $(y, f_1, U_2, e_4, x, e_5, U_5, f_6)$, $(x, e_3, U_3, f_4, y, f_5, U_6, e_8)$, and $(y, f_3, U_4, e_2, x, e_7, U_7, f_8)$. As the total number of faces increases by two, (2.1) gives the following value of $g(\Pi)$:

$$g(\Pi) = g(G_1) + g(G_2) - 1. \tag{3.3}$$

Usually, there is a minimum genus embedding of G constructed from the minimum genus embeddings of G_1 and G_2. Suppose now that $\theta(G_1) = 1$, $\epsilon^+(G_1) = 1$, and $\epsilon(G_2) = 1$. Since $\theta(G_1) = 1$, the embedding constructed from minimum genus embeddings of G_1 and G_2 as described above has genus $g(G_1) + g(G_2) + 1$. On the other hand, $g(G_1^+) = g(G_1) + 1$ and both G_1^+ and G_2 are xy-alternating. Thus we obtain an embedding of G of genus

$$g(G_1^+) + g(G_2) - 1 = g(G_1) + g(G_2) < g(G_1) + g(G_2) + 1.$$

Hence it is necessary also to consider the embeddings of G_1^+ and G_2^+. The minima of the genera given by equations (3.2) and (3.3) can be combined into a single value, denoted $h_1(G)$:

$$h_1(G) = g^+(G_1) + g^+(G_2) - \epsilon^+(G_1)\epsilon^+(G_2). \tag{3.4}$$

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Using the parameters defined above, we can write

$$h_1(G) = g(G_1) + g(G_2) + \theta(G_1) + \theta(G_2) - \epsilon^+(G_1)\epsilon^+(G_2).$$

The similarity of this equation to (3.1) leads us to define the graph parameter $\eta(G_1, G_2) = \theta(G_1) + \theta(G_2) - \epsilon^+(G_1)\epsilon^+(G_2)$. Note that $\eta(G_1, G_2) \in \{-1, 0, 1, 2\}$. This gives another expression for h_1:

$$h_1(G) = g(G_1) + g(G_2) + \eta(G_1, G_2). \tag{3.5}$$

Decker et al. [4] proved the following formula for the genus of a 2-sum of graphs.

Theorem 3.1 (Decker, Glover, and Huneke [4]) *Let G be an xy-sum of connected graphs $G_1, G_2 \in \mathcal{G}^{\circ}_{xy}$. Then*

(i) $g(G) = \min(h_0(G), h_1(G))$,
(ii) $g^+(G) = h_1(G)$,
(iii) $\epsilon^+(G) = 1$ *if and only if* $\epsilon^+(G_1) \neq \epsilon^+(G_2)$, *and*
(iv) $\theta(G) = 1$ *if and only if* $\eta(G_1, G_2) = 2$.

Often, we consider minor-operations in the graph G_1 while the graph G_2 is fixed. When $\epsilon^+(G_2) = 1$, the genus of G depends on the graph parameter $g_a = g - \epsilon$, called the *alternating genus* of G. Let $g_a^+ = g^+ - \epsilon^+$ be the graph parameter defined as $g_a^+(G) = g_a(G^+) = g^+(G) - \epsilon^+(G)$. If we know the value of the parameter $\epsilon^+(G_2)$, then we can express $h_1(G)$ as follows. If $\epsilon^+(G_2) = 1$, then (3.4) can be rewritten as

$$h_1(G) = g_a^+(G_1) + g^+(G_2). \tag{3.6}$$

Otherwise, (3.4) is equivalent to

$$h_1(G) = g^+(G_1) + g^+(G_2). \tag{3.7}$$

The next lemma shows that alternating genus is a minor-monotone graph parameter.

Lemma 3.2 *Let $G \in \mathcal{G}_{xy}$. If H is a minor of G, then $g_a(H) \leq g_a(G)$.*

Proof If $g(H) < g(G)$ or $\epsilon(H) \geq \epsilon(G)$, then the result trivially holds. Hence if the claimed inequality is violated, then $g(H) = g(G)$, $\epsilon(H) = 0$, and $\epsilon(G) = 1$. Thus, there is an xy-alternating minimum genus embedding Π of G. Let W_a be an xy-alternating Π-face.

We may assume without loss of generality that H is obtained from G by a single minor-operation. Suppose first that $H = G - e$ for some edge $e \in E(G)$. Let Π' be the embedding of H induced by Π. If e is a singular edge that appears in a Π-face W, then W is split into two Π'-faces in Π'. Thus $g(H) \leq g(\Pi') = g(\Pi) - 1 = g(G) - 1$, which contradicts the assumption that $g(H) = g(G)$. Hence e appears in two different Π-faces W_1 and W_2. The faces W_1 and W_2 combine to form a single Π'-face W' in Π'. Thus $g(\Pi') = g(\Pi)$. As either W_a is a Π'-face or $W_a - e$ is a subsequence of W', we conclude that Π' is also xy-alternating. This contradicts the assumption that $g(H) = g(G)$ and $\epsilon(H) = 0$.

Suppose now that $H = G/e$ for some edge $e \in E(G)$. Let Π' be the induced embedding of H obtained from Π by contracting e. That is, the local rotation $\Pi'(v_e)$

around the vertex v_e obtained by contraction of $e = uv$ contains as a subsequence both linear sequences obtained from $\Pi(u)$ and $\Pi(v)$ by deleting the subsequence u, e, v. If e does not appear in W_a, then W_a is also a Π'-face. Otherwise, as $e \neq xy$, Π' contains a facial walk W'_a that can be obtained from W_a by replacing the occurrence of u, e, v by v_e. It is immediate that W'_a is an xy-alternating Π'-face. This again contradicts the choice of H. ■

The following lemma shows how the property of being xy-alternating can be expressed in terms of $\theta(G)$ and $\epsilon(G^+)$.

Lemma 3.3 *Let $G \in \mathcal{G}^\circ_{xy}$. The graph G is xy-alternating if and only if $\theta(G) = 0$ and G^+ is xy-alternating. In symbols, $\epsilon(G) = 1$ if and only if $\theta(G) = 0$ and $\epsilon^+(G) = 1$.*

Proof Assume that G is xy-alternating and let Π be an xy-alternating embedding of G of genus $g(G)$. By embedding the edge xy into the xy-alternating Π-face, we obtain an embedding of G^+ into the same surface that is also xy-alternating. This shows that $\theta(G) = 0$ and $\epsilon^+(G) = 1$.

For the converse, assume that $\theta(G) = 0$ and that G^+ is xy-alternating. Let Π be an xy-alternating embedding of G^+ with an xy-alternating Π-face W. Since $\theta(G) = 0$, the edge xy is not a singular edge. Thus by deleting xy from Π, we obtain an embedding Π' of G in the same surface where either W is a Π'-face or $W - xy$ is a subsequence of a Π'-face. Hence Π' is an xy-alternating embedding of G. Since $g(\Pi') = g(G)$, the graph G is xy-alternating. ■

Figure 4 shows the relationship between the parameters g, g^+, g_a, and g_a^+. In addition to the constraints given in the figure, there is one more interrelationship that is described by the following lemma.

Lemma 3.4 *For a graph $G \in \mathcal{G}_{xy}$, we have either $g_a(G) = g(G)$ or $g_a(G) = g_a^+(G)$.*

Proof If $\epsilon(G) = 0$, then $g_a(G) = g(G)$, and we are done. Otherwise, $\epsilon(G) = 1$ and Lemma 3.3 gives that $\epsilon^+(G) = 1$ and $\theta(G) = 0$. Therefore, $g_a^+(G) = g_a(G) + \epsilon(G) + \theta(G) - \epsilon^+(G) = g_a(G)$. ■

For a graph parameter $\mathcal{P}$, we say that a minor-operation $\mu \in \mathcal{M}(G)$ *decreases* $\mathcal{P}$ by at least k if $\mathcal{P}(\mu G) \leq \mathcal{P}(G) - k$. The subset of $\mathcal{M}(G)$ that decreases $\mathcal{P}$ by at least k is denoted by $\Delta_k(\mathcal{P}, G)$. We write just $\Delta_k(\mathcal{P})$ when the graph is clear from the context.

We shall show that each minor-operation in a 2-connected minor-tight part of an xy-sum decreases at least one of the graph parameters g, g^+, and g_a^+ by at least 1. Note that several parameters can be decreased by a single minor-operation and it depends on the relations between the parameters. For example, if G is $K_{3,3}$ with the terminals that are non-adjacent and we consider an edge e of G, then the contraction $(e, /)$ belongs both to $\Delta_1(g)$ and $\Delta_1(g^+)$ as $g(G/e) = g^+(G/e) = 0$. But G is xy-alternating (see Figure 2), so $g_a(G) = g_a^+(G) = 0$ and $(e, /)$ belongs neither to $\Delta_1(g_a)$ nor to $\Delta_1(g_a^+)$.

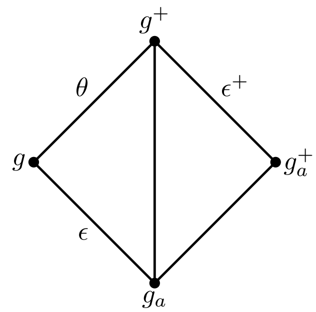

Figure 4: Hasse diagram showing relations of several graph parameters. An edge indicates that the values of parameters differ by at most one and the parameter below is bounded from above by the parameter above.

Lemma 3.5 *Let $\mathcal{P}$ and $\mathcal{Q}$ be graph parameters such that $\mathcal{P}(G) \le \mathcal{Q}(G) \le \mathcal{P}(G) + 1$ for every graph $G \in \mathcal{G}_{xy}$. Then the following holds for each k and each $G \in \mathcal{G}_{xy}$:*

(S1) *If $\mathcal{Q}(G) - \mathcal{P}(G) = 1$, then $\Delta_k(\mathcal{P}) \subseteq \Delta_k(\mathcal{Q})$;*
(S2) *if $\mathcal{Q}(G) - \mathcal{P}(G) = 0$, then $\Delta_k(\mathcal{Q}) \subseteq \Delta_k(\mathcal{P})$;*
(S3) *$\Delta_{k+1}(\mathcal{P}) \subseteq \Delta_k(\mathcal{Q})$ and $\Delta_{k+1}(\mathcal{Q}) \subseteq \Delta_k(\mathcal{P})$.*

The proof of the lemma is easy and is omitted. The lemma will be frequently used for the pairs $\mathcal{P}$ and $\mathcal{Q}$ of graph parameters indicated by connecting lines in Figure 4: g and g^+, g_a and g, g_a and g_a^+, g_a^+ and g^+. The following result shows that Lemma 3.5 also applies to the remaining pair g_a and g^+ from Figure 4.

Lemma 3.6 *For any graph $G \in \mathcal{G}_{xy}$, we have that $g_a(G) \le g^+(G) \le g_a(G) + 1$.*

Proof The first inequality is obvious. To prove the second one, observe that by Lemma 3.4, either $g_a(G) = g(G)$ or $g_a(G) = g_a^+(G)$. In the former case, $g^+(G) = g(G) + \theta(G) \le g_a(G) + 1$. In the latter case, $g^+(G) = g_a^+(G) + \epsilon^+(G) \le g_a(G) + 1$. ■

Using the new notation we can state the following corollary of Lemma 3.4.

Corollary 3.7 *For each $G \in \mathcal{G}_{xy}$, $\Delta_1(g_a) \subseteq \Delta_1(g) \cup \Delta_1(g_a^+)$.*

Proof Let $\mu \in \Delta_1(g_a)$. If $\mu \notin \Delta_1(g) \cup \Delta_1(g_a^+)$, then $g(\mu G) = g(G) > g_a(\mu G)$ and $g_a^+(\mu G) = g_a^+(G) > g_a(\mu G)$, which contradicts Lemma 3.4 (for the graph μG). ■

The next lemma describes necessary and sufficient conditions for a single part of a 2-sum of graphs to be minor-tight. This is a key lemma, and its outcome, summarized in Table 1, will be used heavily throughout this paper.

$xy \in E(G)$	$\epsilon^+(G_2)$	$\eta(G_1, G_2)$	μ
yes	0	—	$\Delta_1(g^+)$
	1		$\Delta_1(g_a^+)$
no	0	0	$\Delta_1(g^+)$
		1	$\Delta_1(g)$ or $\Delta_1(g^+)$
		2	$\Delta_1(g)$
	1	-1	$\Delta_1(g_a^+)$
		0	$\Delta_2(g)$ or $\Delta_1(g_a^+)$
		1	$\Delta_1(g)$ or $\Delta_1(g_a^+)$
		2	$\Delta_1(g)$ or $\Delta_2(g_a^+)$

Table 1: Possible results for a minor-operation in a minor-tight part of a 2-sum of graphs.

Lemma 3.8 *Let G be an xy-sum of connected graphs $G_1, G_2 \in \mathcal{G}_{xy}^{\circ}$ and $\mu \in \mathcal{M}(G_1)$ such that μG_1 is connected. Then $g(\mu G) < g(G)$ if and only if the following are true (where $\Delta_k(\cdot)$ always refer to the decrease of the parameter in G_1):*

(i) *If $xy \in E(G)$, then $\mu \in \Delta_1(g^+)$ if $\epsilon^+(G_2) = 0$ and $\mu \in \Delta_1(g_a^+)$ if $\epsilon^+(G_2) = 1$.*
(ii) *If $xy \notin E(G)$ and $\eta(G_1, G_2) = -1$, then $\mu \in \Delta_1(g_a^+)$.*
(iii) *If $xy \notin E(G)$ and $\eta(G_1, G_2) = 0$, then $\mu \in \Delta_1(g^+)$ when $\epsilon^+(G_2) = 0$ and $\mu \in \Delta_2(g) \cup \Delta_1(g_a^+)$ when $\epsilon^+(G_2) = 1$.*
(iv) *If $xy \notin E(G)$ and $\eta(G_1, G_2) = 1$, then $\mu \in \Delta_1(g) \cup \Delta_1(g^+)$ when $\epsilon^+(G_2) = 0$ and $\mu \in \Delta_1(g) \cup \Delta_1(g_a^+)$ when $\epsilon^+(G_2) = 1$.*
(v) *If $xy \notin E(G)$ and $\eta(G_1, G_2) = 2$, then $\mu \in \Delta_1(g)$ when $\epsilon^+(G_2) = 0$ and $\mu \in \Delta_1(g) \cup \Delta_2(g_a^+)$ when $\epsilon^+(G_2) = 1$.*

Proof Let us start with the "only if" part. Since μG_1 is connected, Theorem 3.1 can be used to determine $g(\mu G)$. In order to show (i), suppose that $xy \in E(G)$. By Theorem 3.1, $g(G)$ and $g(\mu G)$ are equal to $h_1(G)$ and $h_1(\mu G)$, respectively. If $\epsilon^+(G_2) = 0$, then by (3.7),

$$g^+(\mu G_1) + g^+(G_2) = g(\mu G) < g(G) = g^+(G_1) + g^+(G_2).$$

Thus $g^+(\mu G_1) < g^+(G_1)$, yielding that $\mu \in \Delta_1(g^+)$. If $\epsilon^+(G_2) = 1$, then by (3.6),

$$g_a^+(\mu G_1) + g^+(G_2) = g(\mu G) < g(G) = g_a^+(G_1) + g^+(G_2).$$

Thus $g_a^+(\mu G_1) < g_a^+(G_1)$, yielding that $\mu \in \Delta_1(g_a^+)$.

Assume now that $xy \notin E(G)$. We will prove the cases (ii), (iii), and (iv) together. Assume that $\eta(G_1, G_2) \le 1$. If $\epsilon^+(G_2) = 0$, let us assume that $\mu \notin \Delta_1(g^+)$ and if $\epsilon^+(G_2) = 1$, let us assume that $\mu \notin \Delta_1(g_a^+)$. By (3.7) and (3.6), $h_1(\mu G) = h_1(G)$. By Theorem 3.1, $g(\mu G) = h_0(G) < g(G)$. By using the definition of $h_0(G)$ in (3.1), we obtain

$$h_0(G) = g(\mu G_1) + g(G_2) + 1 = g(\mu G) < g(G) = g(G_1) + g(G_2) + \eta(G_1, G_2).$$

Thus $g(\mu G_1) \le g(G_1) + \eta(G_1, G_2) - 2$. If $\eta(G_1, G_2) = -1$, then $\mu \in \Delta_3(g)$, which implies that $\mu \in \Delta_2(g^+)$ by Lemma 3.5(S3) (applied to g and g^+). By another application of (S3), we obtain that $\mu \in \Delta_1(g_a^+)$, yielding (ii). If $\eta(G_1, G_2) = 0$, then $\mu \in \Delta_2(g)$. This proves (iii) when $\epsilon^+(G_2) = 1$. If $\epsilon^+(G_2) = 0$, then also $\mu \in \Delta_1(g^+)$ by (S3). This yields (iii). If $\eta(G_1, G_2) = 1$, then $\mu \in \Delta_1(g)$ which yields (iv).

Suppose now that $\eta(G_1, G_2) = 2$ and that $\mu \notin \Delta_1(g)$. Then $h_0(G) = h_0(\mu G)$. By Theorem 3.1 and (3.5), $g(G) = h_0(G)$. Since $g(\mu G) < g(G)$, we conclude that $g(\mu G) = h_1(\mu G) < g(G)$. As $\eta(G_1, G_2) = 2$, we know that $\theta(G_1) = \theta(G_2) = 1$ and $\epsilon^+(G_1)\epsilon^+(G_2) = 0$. Thus we can write

$$g(G) = h_0(G) = g(G_1) + g(G_2) + 1 = g^+(G_1) + g^+(G_2) - 1.$$

If $\epsilon^+(G_2) = 0$, then we obtain using (3.7) that

$$g^+(\mu G_1) + g^+(G_2) = g(\mu G) < g(G) = g^+(G_1) + g^+(G_2) - 1.$$

Hence $\mu \in \Delta_2(g^+)$, which implies by (S3) that also $\mu \in \Delta_1(g)$, a contradiction.

If $\epsilon^+(G_2) = 1$, then $\epsilon^+(G_1) = 0$ and $g_a^+(G_1) = g^+(G_1)$. We use (3.6) to obtain that

$$g_a^+(\mu G_1) + g^+(G_2) = g(\mu G) < g(G) = g_a^+(G_1) + g^+(G_2) - 1.$$

Hence $\mu \in \Delta_2(g_a^+)$. This finishes the "only if" part.

To prove the "if" part, we assume that (i)–(v) hold and show that $g(\mu G) < g(G)$. We start by proving that if $\mu \in \Delta_1(g)$, $xy \notin E(G)$, and $\eta(G_1, G_2) \ge 1$, then $g(\mu G) < g(G)$. By Theorem 3.1, $g(G) = h_0(G)$. Since $g(\mu G) \le h_0(\mu G)$, we obtain that

$$g(\mu G) \le g(\mu G_1) + g(G_2) + 1 < g(G_1) + g(G_2) + 1 = g(G).$$

If $\mu \in \Delta_2(g)$, $xy \notin E(G)$, and $\eta(G_2, G_2) = 0$, we have a similar inequality:

$$g(\mu G) \le g(\mu G_1) + g(G_2) + 1 < g(G_1) + g(G_2) = g(G).$$

Similarly, we handle the cases when $\mu \in \Delta_1(g^+)$ and when $\mu \in \Delta_1(g_a^+)$. Suppose that $\mu \in \Delta_1(g^+)$, $\epsilon^+(G_2) = 0$, and $xy \in E(G)$ or $\eta(G_1, G_2) \le 1$. By Theorem 3.1, $g(G) = h_1(G)$. We obtain from Theorem 3.1 and (3.7) that

$$g(\mu G) \le g^+(\mu G_1) + g^+(G_2) < g^+(G_1) + g^+(G_2) = g(G).$$

Suppose now that $\mu \in \Delta_1(g_a^+)$, $\epsilon^+(G_2) = 1$, and $xy \in E(G)$ or $\eta(G_1, G_2) \le 1$. We obtain from Theorem 3.1 and (3.6) that

$$g(\mu G) \le g_a^+(\mu G_1) + g^+(G_2) < g_a^+(G_1) + g^+(G_2) = g(G).$$

In the remaining case, when $xy \notin E(G)$, $\eta(G_2, G_2) = 2$, $\epsilon^+(G_2) = 1$, and $\mu \in \Delta_2(g_a^+)$, we have a similar inequality:

$$g(\mu G) \le g_a^+(\mu G_1) + g^+(G_2) < g_a^+(G_1) + g^+(G_2) - 1 \le g(G_1) + g(G_2) + 1 = g(G).$$

This finishes the proof of the lemma. ■

Since for each graph precisely one hypothesis in the cases (i)–(v) of Lemma 3.8 holds, we obtain the following corollary.

Corollary 3.9 *Let G be an xy-sum of connected graphs $G_1, G_2 \in \mathcal{G}_{xy}^{\circ}$ and $\mu \in \mathcal{M}(G_1)$ such that μG_1 is connected and $g(\mu G) < g(G)$. Then $\mu \in \Delta_1(g) \cup \Delta_1(g^+) \cup \Delta_1(g_a^+)$. Furthermore, if $\epsilon^+(G_2) = 0$, then $\mu \in \Delta_1(g) \cup \Delta_1(g^+)$.*

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Lemma 3.8 characterizes when a graph with two terminals is a part of an obstruction for an orientable surface. The next lemma describes when the edge xy is minor-tight in an xy-sum of graphs.

Lemma 3.10 *Let G be an xy-sum of connected graphs $G_1, G_2 \in \mathcal{G}^{\circ}_{xy}$. If $xy \in E(G)$, then the subgraph of G induced by the edge xy is minor-tight if and only if $\eta(G_1, G_2) = 2$ and either $g(G_1/xy) < g^+(G_1)$ or $g(G_2/xy) < g^+(G_2)$.*

Proof By Theorem 3.1(ii), $\theta(G - xy) = 1$ if and only if $\eta(G_1, G_2) = 2$. Thus, $g(G - xy) < g(G)$ if and only if $\eta(G_1, G_2) = 2$. We can therefore assume that $\eta(G_1, G_2) = 2$.

Theorem 2.2 implies that $g(G/xy) = g(G_1/xy) + g(G_2/xy)$. Since $\epsilon^+(G_1)\epsilon^+(G_2) = 0$, Theorem 3.1 and (3.4) gives

$$g(G) = h_1(G) = g^+(G_1) + g^+(G_2).$$

Therefore, $g(G/xy) < g(G)$ if and only if $g(G_1/xy) + g(G_2/xy) < g^+(G_1) + g^+(G_2)$. Since $g(G_1/xy) \le g^+(G_1)$ and $g(G_2/xy) \le g^+(G_2)$, we obtain that $g(G/xy) < g(G)$ if and only if $g(G_1/xy) < g^+(G_1)$ or $g(G_2/xy) < g^+(G_2)$. ■

4 Critical Classes for Graph Parameters

Lemma 3.8 provides necessary and sufficient conditions on the parts of an xy-sum for being minor-tight. In this section, we shall study and categorize graphs that satisfy these conditions.

For a graph parameter $\mathcal{P}$, let $\mathcal{C}(\mathcal{P})$ denote the family of graphs $G \in \mathcal{G}_{xy}$ such that each minor-operation in G decreases $\mathcal{P}$ by at least 1, i.e., $\mathcal{M}(G) = \Delta_1(\mathcal{P})$. We call $\mathcal{C}(\mathcal{P})$ the *critical class* for $\mathcal{P}$. Let $\mathcal{C}^{\circ}(\mathcal{P})$ be the subfamily of $\mathcal{C}(\mathcal{P})$ of graphs without the edge xy. We refine the class $\mathcal{C}(\mathcal{P})$ according to the value of $\mathcal{P}$. Let $\mathcal{C}_k(\mathcal{P})$ denote the subfamily of $\mathcal{C}(\mathcal{P})$ that contains precisely the graphs G for which $\mathcal{P}(G) = k + 1$. The classes $\mathcal{C}^{\circ}_k(\mathcal{P})$ are defined similarly as subfamilies of $\mathcal{C}^{\circ}(\mathcal{P})$.

In this section, we shall study the classes $\mathcal{C}^{\circ}(g)$, $\mathcal{C}^{\circ}(g^+)$, $\mathcal{C}^{\circ}(g_a)$, and $\mathcal{C}^{\circ}(g_a^+)$. It is easy to see that, for each graph $G \in \mathcal{C}^{\circ}_k(g)$, the graph $\widehat{G}$ is an obstruction for $\mathbb{S}_k$. On the other hand, for each graph $G \in \mathrm{Forb}(\mathbb{S}_k)$ and two non-adjacent vertices x and y of G, the graph in $\mathcal{G}_{xy}$ obtained from G by making x and y terminals belongs to $\mathcal{C}^{\circ}_k(g)$. Similarly to $\mathcal{C}^{\circ}_k(g)$, the family $\mathcal{C}^{\circ}_k(g^+)$ can be constructed from the graphs in $\mathrm{Forb}(\mathbb{S}_k)$.

We shall denote by $\mathrm{Forb}^*(\mathbb{S})$ the class of graphs of minimum degree at least 3 that are not embeddable in the surface $\mathbb{S}$, but every proper subgraph is embeddable. These are minimally non-embeddable graphs with respect to deletion of edges and are sometimes called *minimal forbidden topological minors* for the surface $\mathbb{S}$.

Lemma 4.1 *Let $G \in \mathcal{C}^{\circ}_k(g^+)$. If $\theta(G) = 0$, then $\widehat{G} \in \mathrm{Forb}(\mathbb{S}_k)$. If $\theta(G) = 1$, then either $\widehat{G^+} \in \mathrm{Forb}(\mathbb{S}_k)$, or $\widehat{G^+} \in \mathrm{Forb}^*(\mathbb{S}_k)$ and $\widehat{G/xy} \in \mathrm{Forb}(\mathbb{S}_k)$.*

Proof If $\theta(G) = 0$, then $\mathcal{M}(G) = \Delta_1(g)$ by claim (S2) in Lemma 3.5 (applied to g and g^+). Thus $G \in \mathcal{C}^{\circ}_k(g)$. Therefore, $\widehat{G} \in \mathrm{Forb}(\mathbb{S}_k)$ as explained above. Suppose now that $\theta(G) = 1$. Since $G \in \mathcal{C}^\circ(g^+)$, $\mathcal{M}(G) \subseteq \Delta_1(g, G^+)$. As $g(G^+ - xy) < g(G^+)$ (and all other minor-operations in $\widehat{G^+}$ except contracting the edge xy decrease the genus of G^+), we have that $\widehat{G^+} \in \mathrm{Forb}^*(\mathbb{S}_k)$. If $g(G/xy) < g^+(G)$, then $\widehat{G^+} \in \mathrm{Forb}(\mathbb{S}_k)$ since both deletion and contraction of xy decreases the genus of G^+. On the other hand, if $g(G/xy) = g^+(G)$, take any minor-operation $\mu \in \mathcal{M}(G/xy)$. Since μ is also a minor-operation in G, we obtain that $g(\mu(G/xy)) \leq g^+(\mu G) < g^+(G) = g(G/xy)$ as $\mu(G/xy)$ is a minor of $\widehat{\mu G^+}$. Since μ was chosen arbitrarily, $G/xy \in \mathrm{Forb}(\mathbb{S}_k)$. ■

Next, we prove that graphs, whose minor-operations decrease either g or g^+ by at least 1 belong to $\mathcal{C}^\circ(g) \cup \mathcal{C}^\circ(g^+)$.

Lemma 4.2 *Let $G \in \mathcal{G}^\circ_{xy}$. If $\mathcal{M}(G) = \Delta_1(g) \cup \Delta_1(g^+)$, then G belongs to either $\mathcal{C}^\circ(g)$ or $\mathcal{C}^\circ(g^+)$.*

Proof If $\theta(G) = 0$, then $\Delta_1(g^+) \subseteq \Delta_1(g)$ by (S2) in Lemma 3.5 applied to g and g^+. Thus $\mathcal{M}(G) = \Delta_1(g)$ and $G \in \mathcal{C}^\circ(g)$. Similarly, if $\theta(G) = 1$, then $\Delta_1(g) \subseteq \Delta_1(g^+)$ by (S1). We conclude that $\mathcal{M}(G) = \Delta_1(g^+)$ and $G \in \mathcal{C}^\circ(g^+)$. ■

The classes $\mathcal{C}^\circ(g_a)$ and $\mathcal{C}^\circ(g_a^+)$ are related to the class $\mathcal{C}(g_a)$, which was introduced in Mohar and Škoda [7] where it was proved that the classes $\mathcal{C}_k(g_a)$ are finite (for each $k \geq 1$). By the following lemma, this implies that both $\mathcal{C}^\circ_k(g_a)$ and $\mathcal{C}^\circ_k(g_a^+)$ are finite. Observe that a graph $G \in \mathcal{G}^\circ_{xy}$ belongs to $\mathcal{C}(g_a)$ if and only if it belongs to $\mathcal{C}^\circ(g_a)$. The graphs in $\mathcal{C}(g_a) \setminus \mathcal{C}^\circ(g_a)$ can be characterized as follows.

Lemma 4.3 *For a graph $G \in \mathcal{G}^\circ_{xy}$ and $k \geq 0$, we have that $G^+ \in \mathcal{C}_k(g_a)$ if and only if $G \in \mathcal{C}^\circ_k(g_a^+) \setminus \mathcal{C}^\circ_k(g_a)$.*

Proof Suppose that $G^+ \in \mathcal{C}_k(g_a)$. It is immediate that $G \in \mathcal{C}^\circ_k(g_a^+)$. Since $g_a(G) = g_a(G^+ - xy) < g_a(G^+) = k + 1$, the graph G does not belong to $\mathcal{C}^\circ_k(g_a)$.

Suppose now that $G \in \mathcal{C}^\circ_k(g_a^+) \setminus \mathcal{C}^\circ_k(g_a)$. If $g_a(G) = g_a^+(G)$, then $\mathcal{M}(G) = \Delta_1(g_a)$ by (S2) in Lemma 3.5 applied to g_a and g_a^+. It follows that $G \in \mathcal{C}^\circ_k(g_a)$. Thus $g_a(G) < g_a^+(G)$. Hence $g_a(G^+) > g_a(G) = g_a(G^+ - xy)$ and $(xy, -) \in \Delta_1(g_a, G^+)$. We conclude that $G^+ \in \mathcal{C}_k(g_a)$ as $g_a(G^+) = g_a^+(G) = k + 1$. ■

Also, the graphs that do not belong to $\mathcal{C}^\circ(g_a^+)$ can be characterized.

Lemma 4.4 *If $G \in \mathcal{C}^\circ(g_a)$, then $G \notin \mathcal{C}^\circ(g_a^+)$ if and only if there exists $\mu \in \mathcal{M}(G)$ such that $\mu \in \Delta_1(g) \setminus \Delta_1(g_a^+)$.*

Proof The "if" part follows from the fact that $\mathcal{M}(G) \neq \Delta_1(g_a^+)$. The "only if" part follows from Corollary 3.7, as there is $\mu \in \mathcal{M}(G)$ such that $\mu \notin \Delta_1(g_a^+)$. ■

Corollary 3.7 says that each minor-operation that decreases alternating genus also decreases g or g_a^+. We have the following weakly converse statement.

Lemma 4.5 *Let $G \in \mathcal{G}^\circ_{xy}$. If $\mathcal{M}(G) = \Delta_1(g) \cup \Delta_1(g_a^+)$, then G belongs to at least one of $\mathcal{C}^\circ(g)$, $\mathcal{C}^\circ(g_a)$, or $\mathcal{C}^\circ(g_a^+)$.*

Proof By Lemma 3.4, either $g_a(G) = g(G)$ or $g_a(G) = g_a^+(G)$. If $g(G) = g_a^+(G) = g_a(G)$, then $\Delta_1(g) \subseteq \Delta_1(g_a)$ and $\Delta_1(g_a^+) \subseteq \Delta_1(g_a)$ by property (S2) of Lemma 3.5. Thus $G \in \mathcal{C}^\circ(g_a)$.

If $g(G) > g_a(G)$, then $\Delta_1(g_a^+) \subseteq \Delta_1(g_a)$ by (S2). By (S1), $\Delta_1(g_a) \subseteq \Delta_1(g)$. We conclude that $G \in \mathcal{C}^\circ(g)$. Similarly, if $g_a^+(G) > g_a(G)$, then $\Delta_1(g) \subseteq \Delta_1(g_a)$ by (S2). By (S1), $\Delta_1(g_a) \subseteq \Delta_1(g_a^+)$. We conclude that $G \in \mathcal{C}^\circ(g_a^+)$. ∎

5 Hoppers

In this section, we describe three subfamilies of $\mathcal{C}^\circ(g^+)$ all of which we call *hoppers*. Two kinds of hoppers (hoppers of level 0 and 1 as defined below) appear as parts of obstructions of connectivity 2. A graph $G \in \mathcal{G}_{xy}^\circ$ is a *hopper of level 0* if $\mathcal{M}(G) = \Delta_1(g) \cup \Delta_2(g_a^+)$ and $G \notin \mathcal{C}^\circ(g)$. A graph $G \in \mathcal{G}_{xy}^\circ$ is a *hopper of level 1* if $\mathcal{M}(G) = \Delta_1(g_a^+) \cup \Delta_2(g)$ and $G \notin \mathcal{C}^\circ(g_a^+)$. If G is a graph in $\mathcal{C}^\circ(g_a^+)$ such that $\epsilon^+(G) = 1$, then we call G a *hopper of level 2*. It is immediate from (S1) in Lemma 3.5 that $G \in \mathcal{C}^\circ(g^+)$. The level of the hopper vaguely corresponds to the difficulty to construct such a graph.

Let $\mathcal{H}^l$, $0 \le l \le 2$, denote the family of hoppers of level l. Let $\mathcal{H}_k^l$ denote the subfamily of $\mathcal{H}^l$ containing graphs G with $g^+(G) = k$.

Lemma 5.1 *If $G \in \mathcal{H}^0$, then $G \in \mathcal{C}^\circ(g^+)$, $\epsilon^+(G) = 0$, and $\theta(G) = 1$.*

Proof By (S3) in Lemma 3.5, $\Delta_2(g_a^+) \subseteq \Delta_1(g^+)$. If $\theta(G) = 0$, then $\Delta_1(g^+) \subseteq \Delta_1(g)$ by (S2), a contradiction with $G \notin \mathcal{C}^\circ(g)$. Hence $\theta(G) = 1$. By (S1), $\Delta_1(g) \subseteq \Delta_1(g^+)$ and we conclude that $G \in \mathcal{C}^\circ(g^+)$.

If $\epsilon^+(G) = 1$, then $\Delta_2(g_a^+) \subseteq \Delta_2(g^+)$ by (S1) and, since $\Delta_2(g^+) \subseteq \Delta_1(g)$ by (S3), we have that $\Delta_2(g_a^+) \subseteq \Delta_1(g)$, a contradiction. Thus $\epsilon^+(G) = 0$. ∎

Note that the proof of the next lemma is analogous to the proof of Lemma 5.1.

Lemma 5.2 *If $G \in \mathcal{H}^1$, then $G \in \mathcal{C}^\circ(g^+)$, $\epsilon^+(G) = 1$, and $\theta(G) = 0$.*

Proof By (S3) in Lemma 3.5, $\Delta_2(g) \subseteq \Delta_1(g^+)$. If $\epsilon^+(G) = 0$, then $\Delta_1(g^+) \subseteq \Delta_1(g_a^+)$ by (S2), a contradiction with $G \notin \mathcal{C}^\circ(g_a^+)$. Hence $\epsilon^+(G) = 1$. By (S1), $\Delta_1(g_a^+) \subseteq \Delta_1(g^+)$ and we conclude that $G \in \mathcal{C}^\circ(g^+)$.

If $\theta(G) = 1$, then $\Delta_2(g) \subseteq \Delta_2(g^+)$ by (S1), and since $\Delta_2(g^+) \subseteq \Delta_1(g_a^+)$ by (S3), we have that $\Delta_2(g) \subseteq \Delta_1(g_a^+)$, a contradiction. Thus $\theta(G) = 0$. ∎

Similarly to the genus, alternating genus decreases by at most 1 when an edge is deleted.

Lemma 5.3 *Let $G \in \mathcal{G}_{xy}$. For each $e \in E(G)$, $g_a(G - e) \ge g_a(G) - 1$.*

Proof Suppose that $g_a(G - e) < g_a(G) - 1$. Since $g(G - e) \ge g(G) - 1$, we have that $\epsilon(G) = 0$, $\epsilon(G - e) = 1$, and $g(G - e) = g(G) - 1$. Let Π be an xy-alternating embedding of $G - e$ in $\mathbb{S}_k$, $k = g(G - e)$ and let W be an xy-alternating Π-face. If the endvertices u and v of e are Π-cofacial, then Π can be extended to an embedding of

G in $\mathbb{S}_k$, a contradiction. Otherwise, let Π' be the embedding of G on $\mathbb{S}_{k+1}$ obtained from Π by embedding e into a new handle connecting faces incident with u and v. Since W is a subwalk of a Π'-face, Π' is xy-alternating. Since $g(\Pi') = g(G-e)+1 = g(G)$, we have that $\epsilon(G) = 1$, which is a contradiction. ∎

Lemma 5.3 has the following corollary that shows the motivation for introducing the notion of hoppers of level 2.

Corollary 5.4 *A graph $G \in \mathcal{C}_k^\circ(g_a^+)$ does not embed into $\mathbb{S}_{k+1}$ if and only if $\epsilon^+(G) = 1$.*

Mohar and Škoda conjectured that all graphs in $\mathcal{C}_k(g_a)$ embed into $\mathbb{S}_{k+1}$.

Conjecture 5.5 (Mohar and Škoda [7]) *Each $G \in \mathcal{C}_k(g_a)$ embeds into $\mathbb{S}_{k+1}$.*

We suspect that there are no hoppers of level 1 and 2.

Conjecture 5.6 *There are no hoppers of level 1 and 2.*

Thus Conjecture 5.6 is a stronger version of Conjecture 5.5. The following lemma shows that Conjecture 5.5 is true if $xy \in E(G)$.

Lemma 5.7 *Let $G \in \mathcal{G}_{xy}^\circ$. Then $g_a(G) < g_a^+(G)$ if and only if $\epsilon^+(G) = 0$ and $\theta(G) = 1$.*

Proof If $g_a(G) < g_a^+(G)$, then $g_a(G) = g(G)$ by Lemma 3.4. From Figure 4, we conclude that $g_a^+(G) - g_a(G) = 1 = \epsilon(G) + \theta(G) - \epsilon^+(G)$. Since $\epsilon(G) = 0$, we obtain that $\theta(G) = 1$ and $\epsilon^+(G) = 0$, as required.

If $\epsilon^+(G) = 0$ and $\theta(G) = 1$, then $\epsilon(G) = 0$ by Lemma 3.3. Thus $g_a(G) < g_a(G) + \epsilon(G) + \theta(G) - \epsilon^+(G) = g_a^+(G)$. ∎

Lemmas 4.3 and 5.7 assert that a hopper of level 2 belongs to the class $\mathcal{C}_k^\circ(g_a)$.

6 Dumbbells

Lemma 3.8 provides useful information about minor-operations in a part G_1 of an xy-sum. However, the lemma cannot be used for deletion of cut-edges of G_1. Since the xy-sum is 2-connected, deletion of a cut-edge of G_1 separates x and y. In this section, we determine how minor-tight parts of an xy-sum with such a cut-edge look like.

If $G_1 \in \mathcal{G}_{xy}^\circ$ and $b \in E(G_1)$ is a cut-edge of G_1 whose deletion separates x and y, we say that G_1 is a *dumbbell* with *bar* b.

Lemma 6.1 *If G_1 is a dumbbell with bar b, then $\epsilon^+(G_1) = 0$ and $(b, /) \notin \Delta_1(g) \cup \Delta_1(g^+)$.*

Proof Suppose for a contradiction that $\epsilon^+(G_1) = 1$; then there exists an xy-alternating minimum-genus embedding Π of G_1^+. Let W be an xy-alternating Π-facial walk. The walk W can be split into 4 subwalks containing x and y. Each of the edges xy and b appears precisely twice in the Π-facial walks (either once in two different Π-facial walks or twice in a single Π-facial walk). Since each walk from x to y has to use either xy or b, both xy and b are singular edges that appear twice in W. Since Π is an orientable embedding, the edge xy appears in W once in the direction from x to y and once from y to x. Hence, there is another appearance of one of the terminals, say x, in W that is not incident with the edge xy. We can write W as $W = (x, xy, y, W_1, e_1, x, e_2, W_2, y, xy, x, e_3, W_3, e_4, x)$. The local rotation around x can be written as $(xy, e_4, S_1, e_2, e_1, S_2, e_3)$. Let Π' be the embedding obtained from Π by letting $\Pi'(v) = \Pi(v)$ for $v \in V(G_1) \setminus \{x\}$ and $\Pi(x) = (e_4, S_1, e_2, xy, e_1, S_2, e_3)$. All Π-facial walks except W are also Π'-facial walks, as all Π-angles not incident with W are also Π'-angles. The Π-facial walk W is split into three Π'-facial walks: (x, xy, y, W_1, e_1, x), (x, e_3, W_3, e_4, x), and (x, e_2, W_2, y, xy, x). Thus $g(\Pi') < g(\Pi)$, a contradiction with Π being a minimum-genus embedding of G_1^+. We conclude that $\epsilon^+(G_1) = 0$.

Let $\mu = (b, /)$ be the contraction operation of b in G_1. We shall show that $\mu \notin \Delta_1(g) \cup \Delta_1(g^+)$. Let H_1 and H_2 be the components of $G_1 - b$. By Theorem 2.2, $g(G_1) = g(H_1) + g(H_2) = g(\mu G_1)$. If b is incident with a terminal, say $b = zy$, $z \in V(H_1)$, then G_1^+ is the 1-sum of $H_1 + b + xy$ and H_2. By Theorem 2.2,

$$g(G_1^+) = g(H_1 + b + xy) + g(H_2) = g(H_1 + xz) + g(H_2) = g(\mu G_1^+).$$

Thus $g^+(G_1) = g^+(\mu G_1)$.

Suppose that b is not incident with a terminal and let $z \in V(H_1)$ be an endpoint of b. Consider the graphs $H_1' = H_1 + xy$ and $H_2' = H_2 + b$ as members of the class $\mathcal{G}_{yz}^{\circ}$. Observe that H_1' and H_2' are dumbbells (in $\mathcal{G}_{yz}^{\circ}$). We have already shown that $\epsilon^+(H_1') = \epsilon^+(H_2') = 0$ and $g(\mu H_2') = g(H_2')$, and since the bar of H_2' is incident with a terminal, $g^+(\mu H_2') = g^+(H_2')$. By Theorem 3.1 (when G_1^+ is viewed as a yz-sum of H_1' and H_2'),

$$g(G_1^+) = \min\{g(H_1') + g(H_2') + 1, g^+(H_1') + g^+(H_2')\},$$
$$g(\mu G_1^+) = \min\{g(H_1') + g(\mu H_2') + 1, g^+(H_1') + g^+(\mu H_2')\}.$$

Since $g(\mu H_2') = g(H_2')$ and $g^+(\mu H_2') = g^+(H_2')$, we conclude that $g(\mu G_1^+) = g(G_1^+)$. Thus $g^+(\mu G_1) = g^+(G_1)$. This shows that $\mu \notin \Delta_1(g) \cup \Delta_1(g^+)$. ■

Lemma 6.2 *Let G be an xy-sum of connected graphs $G_1, G_2 \in \mathcal{G}_{xy}^{\circ}$. If G_1 is a dumbbell with bar b and G_1 is minor-tight in G, then $\epsilon^+(G_1/b) = 1$ and b is unique, that is, G_1 has a single cut-edge separating x and y.*

Proof By Lemma 6.1 and Corollary 3.9, $(b, /) \in \Delta_1(g_a^+) \setminus \Delta_1(g^+)$. It is immediate that $\epsilon^+(G_1/b) = 1$.

For the second part, suppose that there is another bar $e \neq b$ in G_1. By Lemma 6.1, $\epsilon^+(G_1/b) = 0$ as G_1/b is a dumbbell with bar e, a contradiction. We conclude that b is unique. ■

Let $\mathcal{D}$ be the class of dumbbells G_1 with bar b such that $\theta(G_1) = 0$, $\mu \in \Delta_1(g)$ for each $\mu \in \mathcal{M}(G_1) \setminus \{(b, -), (b, /)\}$, and $\epsilon^+(G_1/b) = 1$.

Lemma 6.3 *Let G be an xy-sum of connected graphs $G_1, G_2 \in \mathcal{G}^{\circ}_{xy}$ such that G_1 is a dumbbell. Then G_1 is minor-tight in G if and only if $\epsilon^+(G_2) = 1$ and one of the following holds:*

(i) $G_1 \in \mathcal{C}^{\circ}(g_a^+) \setminus \mathcal{C}^{\circ}(g_a)$, $\theta(G_1) = 1$, *and either* $xy \in E(G)$ *or* $\eta(G_1, G_2) = 1$.
(ii) $G_1 \in \mathcal{D}$, $xy \notin E(G)$, *and* $\eta(G_1, G_2) = 1$.

Proof Assume that G_1 is minor-tight in G. By Lemmas 6.1 and 6.2, $\epsilon^+(G_1) = 0$ and G_1 has a unique bar b for which it holds that $(b, /) \notin \Delta_1(g) \cup \Delta_1(g^+)$ and $\epsilon^+(G_1/b) = 1$. Hence $g_a^+(G_1/b) = g_a^+(G_1) - 1$, and we have that $(b, /) \in \Delta_1(g_a^+) \setminus \Delta_2(g_a^+)$. By Corollary 3.9, $\epsilon^+(G_2) = 1$.

Assume first that $\theta(G_1) = 1$. We shall show that (i) holds. If $xy \notin E(G)$ and $\eta(G_1, G_2) = 2$, then $(b, /)$ violates Lemma 3.8 as $(b, /) \notin \Delta_1(g) \cup \Delta_2(g_a^+)$. Thus either $xy \in E(G)$ or $\eta(G_1, G_2) \le 1$. Since $\epsilon^+(G_1) = 0$ and $\theta(G_1) = 1$, we conclude that either $xy \in E(G)$ or $\eta(G_1, G_2) = 1$.

Since $g(G_1 - b) = g(G_1)$ and $\epsilon(G_1 - b) = 0$ (as the terminals of $G_1 - b$ are not connected), $(b, -) \notin \Delta_1(g_a)$. Hence $G_1 \notin \mathcal{C}^{\circ}(g_a)$. It remains to show that $G_1 \in \mathcal{C}^{\circ}(g_a^+)$.

Since $\theta(G_1) = 1$, we have that $g^+(G_1 - b) = g(G_1^+ - b) = g(G_1) < g^+(G_1)$ and thus $(b, -) \in \Delta_1(g^+)$. By Lemma 6.1, $\epsilon^+(G_1) = 0$. By (S2) (Lemma 3.5), $(b, -) \in \Delta_1(g_a^+)$.

Let $\mu \in \mathcal{M}(G_1) \setminus \{(b, -), (b, /)\}$. Since μG_1 is connected, Lemma 3.8 gives that $\mu \in \Delta_1(g_a^+)$ if $xy \in E(G)$ and $\mu \in \Delta_1(g) \cup \Delta_1(g_a^+)$ if $xy \notin E(G)$ and $\eta(G_1, G_2) = 1$. By (S1), $\Delta_1(g) \subseteq \Delta_1(g^+)$. By (S2), $\Delta_1(g^+) \subseteq \Delta_1(g_a^+)$. We conclude that $\mu \in \Delta_1(g_a^+)$. Since μ was arbitrary and $(b, -), (b, /) \in \Delta_1(g_a^+)$, we have that $\mathcal{M}(G_1) = \Delta_1(g_a^+)$ and $G_1 \in \mathcal{C}^{\circ}(g_a^+)$. Therefore, (i) holds.

Assume now that $\theta(G_1) = 0$. We shall show that (ii) holds. In $G - b$, the two components of $G_1 - b$ are joined to G_2 by single vertices. If $xy \in E(G)$, Theorems 2.2 and 3.1 imply (using $\epsilon^+(G_1) = 0$ and $\theta(G_1) = 0$) that

$$g(G-b) = g(\widehat{G_1 - b}) + g(\widehat{G_2^+}) = g(G_1) + g^+(G_2) = g^+(G_1) + g^+(G_2) = h_1(G) = g(G).$$

This contradicts the assumption that G_1 is minor-tight. We conclude that $xy \notin E(G)$. If $\eta(G_2) = 0$, we obtain a similar contradiction:

$$g(G-b) = g(\widehat{G_1 - b}) + g(\widehat{G_2}) = g(G_1) + g(G_2) = g^+(G_1) + g^+(G_2) = h_1(G) = g(G).$$

Thus $\eta(G_1, G_2) \ge 1$. Since $\theta(G_1) = 0$, we conclude that $\theta(G_2) = 1$ and $\eta(G_1, G_2) = 1$.

It remains to show that $G_1 \in \mathcal{D}$, namely that $\mu \in \Delta_1(g)$ for each $\mu \in \mathcal{M}(G) \setminus \{(b, -), (b, /)\}$. Let $\mu \in \mathcal{M}(G) \setminus \{(b, -), (b, /)\}$. Since μG_1 is connected, $\mu \in \Delta_1(g) \cup \Delta_1(g_a^+)$ by Lemma 3.8. Since μG is still a dumbbell, $\epsilon^+(\mu G) = 0$ by Lemma 6.1. Hence $g^+(\mu G) = g_a^+(\mu G)$ and $\Delta_1(g_a^+) \subseteq \Delta_1(g^+)$. By (S2), $\Delta_1(g^+) \subseteq \Delta_1(g)$. Therefore, $\mu \in \Delta_1(g)$. We conclude that $G_1 \in \mathcal{D}$. Thus (ii) holds.

Let us prove the "if" part of the theorem. Assume that $\epsilon^+(G_2) = 1$ and that (i) holds. Let $\mu \in \mathcal{M}(G_1)$. We have that $\mu \in \Delta_1(g_a^+)$. If μG_1 is connected, $g(\mu G) < g(G)$ by Lemma 3.8, since $\epsilon^+(G_2) = 1$. Otherwise, $\mu = (b, -)$. If $xy \in E(G)$, then by

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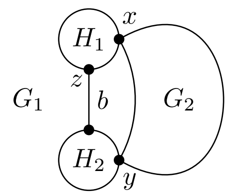

Figure 5: A sketch of the structure of the graph G from the proof of Lemma 6.4.

Theorems 2.2 and 3.1,

$$g(G-b) = g(\widehat{G_1 - b}) + g(\widehat{G_2^+}) = g(G_1) + g^+(G_2) < g^+(G_1) + g^+(G_2) = h_1(G) = g(G).$$

If $xy \notin E(G)$ and $\eta(G_1, G_2) = 1$, then $\theta(G_2) = 0$, and we obtain that

$$g(G-b) = g(\widehat{G_1 - b}) + g(\widehat{G_2}) = g(G_1) + g(G_2) < g(G_1) + g(G_2) + 1 = g(G).$$

In both cases $g(G-b) < g(G)$, and thus $g(\mu G) < g(G)$ for each $\mu \in \mathcal{M}(G_1)$. We conclude that G_1 is minor-tight in G.

Assume now that (ii) holds. Let $\mu \in \mathcal{M}(G_1)$ and assume first that μG_1 is connected. If $\mu = (b, /)$, then $(b, /) \in \Delta_1(g_a^+)$, since $\epsilon^+(G_1) = 0$ and $\epsilon^+(\mu G_1) = 1$ (and $g^+(\mu G_1) = g^+(G_1)$). Otherwise, $\mu \in \Delta_1(g)$, since $G_1 \in \mathcal{D}$. Since $xy \notin E(G)$, $\eta(G_1, G_2) = 1$, and $\epsilon^+(G_2) = 1$, Lemma 3.8 gives that $g(\mu G) < g(G)$.

The case when $\mu = (b, -)$ remains. By Theorems 2.2 and 3.1,

$$g(G-b) = g(\widehat{G_1 - b}) + g(\widehat{G_2}) = g(G_1) + g(G_2) < g(G_1) + g(G_2) + 1 = g(G).$$

We have that $g(\mu G) < g(G)$ for each $\mu \in \mathcal{M}(G_1)$. We conclude that G_1 is minor-tight. ■

We close this section by showing that in an obstruction of connectivity 2, there always exists a two-vertex-cut such that neither of the parts belongs to $\mathcal{D}$.

Lemma 6.4 *Let $G \in \mathrm{Forb}(\mathbb{S}_k)$ be of connectivity 2. Then there exists a 2-vertex-cut $\{x, y\}$ such that neither of the parts of G when viewed as an xy-sum of two graphs belongs to $\mathcal{D}$.*

Proof Let G be an xy-sum of $G_1, G_2 \in \mathcal{G}_{xy}^{\circ}$. Suppose that $G_1 \in \mathcal{D}$. Since G_1 is minor-tight in G and $\theta(G_1) = 0$, Lemma 6.3 gives $\epsilon^+(G_2) = 1$ and $\eta(G_1, G_2) = 1$. From the definition of $\eta(G_1, G_2)$ we conclude that $\theta(G_2) = 1$, as $\epsilon^+(G_1) = 0$. Let b be a bar of G_1 and let H_1 and H_2 be the components of $G_1 - b$. We may assume that H_1 contains at least one edge. Let x be the common vertex of H_1 and G_2 and let z be the endpoint of b incident with H_1. Let us view G as an xz-sum of H_1 and $G_2' = G_2 + H_2 + b$ (see Figure 5). We claim that neither H_1 nor G_2' belongs to $\mathcal{D}$.

By Lemma 6.2 applied to G_1, b is the unique cut-edge separating x and y and thus there is no cut-edge in H_1 separating x and z. Therefore, H_1 is not a dumbbell. We shall show that $\theta(G_2') = 1$ and hence $G_2' \notin \mathcal{D}$. The graph $G_2'^+$ can be viewed as an xy-sum of G_2 and the graph $G_1' = H_2 + b + zx$. The graph G_1' is a dumbbell and thus

$xy \in E(G)$	$\epsilon^+(G_2)$	$\eta(G_1, G_2)$	G_1 belongs to
yes	0	—	$\mathcal{C}^\circ(g^+)$
	1		$\mathcal{C}^\circ(g_a^+)$
no	0	0	$\mathcal{C}^\circ(g^+)$
		1	$\mathcal{C}^\circ(g)$ or $\mathcal{C}^\circ(g^+)$
		2	$\mathcal{C}^\circ(g)$
	1	-1	$\mathcal{C}^\circ(g_a^+)$
		0	$\mathcal{C}^\circ(g_a^+)$ or $\mathcal{H}^1$
		1	$\mathcal{C}^\circ(g)$, $\mathcal{C}^\circ(g_a)$, $\mathcal{C}^\circ(g_a^+)$, or $\mathcal{D}$
		2	$\mathcal{C}^\circ(g)$ or $\mathcal{H}^0$

Table 2: Classification of minor-tight parts of 2-sums of graphs.

$\epsilon^+(G_1') = 0$ by Lemma 6.1. By Theorem 2.2, $g(G_2') = g(H_2) + g(G_2)$. By Theorem 3.1, using $\epsilon^+(G_1) = 0$ and $\theta(G_2) = 1$,

$$g(G_2'^+) = \min\{g(G_1') + g(G_2) + 1, g(G_1'^+) + g(G_2^+)\} \geq g(H_2) + g(G_2) + 1.$$

Therefore $\theta(G_2') = 1$. We conclude that $G_2' \notin \mathcal{D}$. ∎

7 General Orientable Surfaces

In this section, we prove a general theorem that classifies minor-tight parts of a 2-sum of graphs. The classification that is given in Theorem 7.1 is also summarized in Table 2.

Theorem 7.1 *Let G be an xy-sum of connected graphs $G_1, G_2 \in \mathcal{G}_{xy}^\circ$. The graph G_1 is minor-tight if and only if the following statements hold (see Table 2).*

(i) *If $xy \in E(G)$, then $G_1 \in \mathcal{C}^\circ(g^+)$ if $\epsilon^+(G_2) = 0$ and $G_1 \in \mathcal{C}^\circ(g_a^+)$ otherwise.*
(ii) *If $xy \notin E(G)$ and $\eta(G_1, G_2) = -1$, then $G_1 \in \mathcal{C}^\circ(g_a^+)$.*
(iii) *If $xy \notin E(G)$ and $\eta(G_1, G_2) = 0$, then $G_1 \in \mathcal{C}^\circ(g^+)$ if $\epsilon^+(G_2) = 0$ and $G_1 \in \mathcal{C}^\circ(g_a^+) \cup \mathcal{H}^1$ otherwise.*
(iv) *If $xy \notin E(G)$ and $\eta(G_1, G_2) = 1$, then $G_1 \in \mathcal{C}^\circ(g) \cup \mathcal{C}^\circ(g^+)$ if $\epsilon^+(G_2) = 0$ and $G_1 \in \mathcal{C}^\circ(g) \cup \mathcal{C}^\circ(g_a) \cup \mathcal{C}^\circ(g_a^+) \cup \mathcal{D}$ otherwise.*
(v) *If $xy \notin E(G)$ and $\eta(G_1, G_2) = 2$, then $G_1 \in \mathcal{C}^\circ(g)$ if $\epsilon^+(G_2) = 0$ and $G_1 \in \mathcal{C}^\circ(g) \cup \mathcal{H}^0$ otherwise.*

Proof Let us start with the "only if" part of the theorem. Assume first that G_1 has no cut-edge that separates x and y. Lemma 3.8 classifies which graph parameters of G_1 are decreased by the minor-operations in $\mathcal{M}(G_1)$. If it is a single parameter, then G_1 belongs to the critical class corresponding to the parameter. For example, if $xy \in E(G)$ and $\epsilon^+(G_2) = 0$, then $\mathcal{M}(G_1) = \Delta_1(g^+)$ by Lemma 3.8(i), and thus $G_1 \in \mathcal{C}^\circ(g^+)$. The statements (i), (ii), (iii) for $\epsilon^+(G_2) = 0$, and (v) for $\epsilon^+(G_2) = 0$ are proved in this way, and we omit the details. Let us focus on the remaining cases. In all of them, we have that $xy \notin E(G)$.

Let us start with the case when $\eta(G_1, G_2) = 1$. If $\epsilon^+(G_2) = 0$, then $\mathcal{M}(G_1) = \Delta_1(g) \cup \Delta_1(g^+)$ by Lemma 3.8(iv). By Lemma 4.2, G_1 belongs to either $\mathcal{C}^\circ(g)$ or $\mathcal{C}^\circ(g^+)$. If $\epsilon^+(G_2) = 1$, then $\mathcal{M}(G_1) = \Delta_1(g) \cup \Delta_1(g_a^+)$ by Lemma 3.8(iv). By Lemma 4.5, G_1 belongs to either $\mathcal{C}^\circ(g)$, $\mathcal{C}^\circ(g_a)$, or $\mathcal{C}^\circ(g_a^+)$. This proves (iv).

If $\eta(G_1, G_2) = 0$ and $\epsilon^+(G_2) = 1$, then $\mathcal{M}(G_1) = \Delta_1(g_a^+) \cup \Delta_2(g)$ by Lemma 3.8(iii). By definition, G_1 belongs to either $\mathcal{C}^\circ(g_a^+)$ or $\mathcal{H}^1$. Thus, (iii) holds. If $\eta(G_1, G_2) = 2$ and $\epsilon^+(G_2) = 1$, then $\mathcal{M}(G_1) = \Delta_1(g) \cup \Delta_2(g_a^+)$ by Lemma 3.8(v). By definition, G_1 belongs to either $\mathcal{C}^\circ(g)$ or $\mathcal{H}^0$. Thus (v) is true.

Assume now that G_1 has a cut-edge that separates x and y, and thus G_1 is a dumbbell. Since G_1 is minor-tight, Lemma 6.3 gives that $\epsilon^+(G_2) = 1$ and that either $G_1 \in \mathcal{C}^\circ(g_a^+)$ and $xy \in E(G)$ or $\eta(G_1, G_2) = 1$, or $G_1 \in \mathcal{D}$, $xy \notin E(G)$, and $\eta(G_1, G_2) = 1$. The statements (ii), (iii), and (v) are vacuously true, since either $xy \in E(G)$ or $\eta(G_1, G_2) = 1$. The statement (i) is true, since $G_1 \in \mathcal{C}^\circ(g_a^+)$ if $xy \in E(G)$. The statement (iv) is true, since either $G_1 \in \mathcal{C}^\circ(g_a^+)$ or $G_1 \in \mathcal{D}$. This completes the "only if" part of the proof.

It remains to prove the "if" part. Lemma 3.8 is now used to prove that G_1 is minor-tight. Assume first that G_1 has no cut-edge separating x and y. If G_1 belongs to one of the classes $\mathcal{C}^\circ(g)$, $\mathcal{C}^\circ(g^+)$, or $\mathcal{C}^\circ(g_a^+)$, then it is straightforward to check that in each case Lemma 3.8 asserts that G_1 is minor-tight. We shall omit the proof here and do only the cases when $G_1 \in \mathcal{C}^\circ(g_a)$ or G_1 is a hopper.

If $G_1 \in \mathcal{C}^\circ(g_a)$, $xy \notin E(G)$, $\epsilon^+(G_2) = 1$, and $\eta(G_1, G_2) = 1$, then Corollary 3.7 asserts that $\mathcal{M}(G_1) = \Delta_1(g) \cup \Delta_1(g_a^+)$. Lemma 3.8 gives that G_1 is minor-tight. Finally, let us assume that G_1 is a hopper. If $G_1 \in \mathcal{H}^1$, $xy \notin E(G)$, $\epsilon^+(G_2) = 1$, and $\eta(G_1, G_2) = 0$, then $\mathcal{M}(G_1) = \Delta_2(g) \cup \Delta_1(g_a^+)$ by definition of $\mathcal{H}^1$. Lemma 3.8 gives that G_1 is minor-tight. If $G_1 \in \mathcal{H}^0$, $xy \notin E(G)$, $\epsilon^+(G_2) = 1$, and $\eta(G_1, G_2) = 2$, then $\mathcal{M}(G_1) = \Delta_1(g) \cup \Delta_2(g_a^+)$ by definition of $\mathcal{H}^0$. Lemma 3.8 gives that G_1 is minor-tight.

Assume now that G_1 is a dumbbell with bar b. If $G_1 \in \mathcal{D}$ (and $\epsilon^+(G_2) = 1$, $xy \notin E(G)$, and $\eta(G_1, G_2) = 1$), then G_1 is minor-tight by Lemma 6.3(ii). By Lemma 6.1, $G_1 \notin \mathcal{C}^\circ(g) \cup \mathcal{C}^\circ(g^+)$. Since $\mathcal{H}^0$ and $\mathcal{H}^1$ are subsets of $\mathcal{C}^\circ(g^+)$, we have that $G_1 \notin \mathcal{H}^0 \cup \mathcal{H}^1$ (Lemmas 5.1 and 5.2). Thus we may assume that $G_1 \in \mathcal{C}^\circ(g_a) \cup \mathcal{C}^\circ(g_a^+)$ and $\epsilon^+(G_2) = 1$. By Lemma 6.1, $\epsilon^+(G_1) = 0$. Since $\epsilon(G_1 - b) = \epsilon^+(G_1 - b) = 0$ and $g(G_1 - b) = g(G)$, we conclude that $G_1 \notin \mathcal{C}^\circ(g_a)$. Hence $G_1 \in \mathcal{C}^\circ(g_a^+) \setminus \mathcal{C}^\circ(g_a)$ and $(b, -) \in \Delta_1(g_a^+)$. Since $\epsilon^+(G_1 - b) = \epsilon^+(G_1) = 0$, $(b, -) \in \Delta_1(g^+)$. By property (S2) of Lemma 3.5, $\theta(G_1) = 1$. Since $\epsilon^+(G_1) = 0$ and $\eta(G_1, G_2) \leq 1$, we conclude that $\eta(G_1, G_2) = 1$. By Lemma 6.3(i), G_1 is minor-tight in G. This completes the proof of the theorem. ■

Note that a graph can belong to several critical classes at the same time. For example, if $G \in \mathcal{C}^\circ(g)$ such that $\theta(G) = 1$ and $\epsilon^+(G) = 0$, then G belongs to all four classes, $\mathcal{C}^\circ(g)$, $\mathcal{C}^\circ(g^+)$, $\mathcal{C}^\circ(g_a)$, and $\mathcal{C}^\circ(g_a^+)$.

We finish this section by the following corollary which shows that at least one part of a 2-sum is an "obstruction" for a surface.

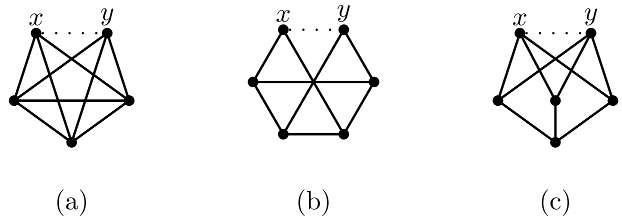

Figure 6: The family $\mathcal{C}_0^\circ(g^+)$. The third graph is the sole member of the family $\mathcal{C}_0^\circ(g)$.

Corollary 7.2 *Let G be an xy-sum of connected graphs $G_1, G_2 \in \mathcal{G}_{xy}^\circ$. If both G_1 and G_2 are minor-tight, then the following statements hold:*

(i) *G_1 and G_2 belong to $\mathcal{C}^\circ(g) \cup \mathcal{C}^\circ(g^+) \cup \mathcal{C}^\circ(g_a) \cup \mathcal{C}^\circ(g_a^+) \cup \mathcal{D}$;*
(ii) *if $\epsilon^+(G_2) = 0$, then $G_1 \in \mathcal{C}^\circ(g) \cup \mathcal{C}^\circ(g^+)$;*
(iii) *either G_1 or G_2 belongs to $\mathcal{C}^\circ(g) \cup \mathcal{C}^\circ(g^+)$.*

Proof By Lemma 5.1 and 5.2, $\mathcal{H}^0$ and $\mathcal{H}^1$ are subfamilies of $\mathcal{C}^\circ(g^+)$. Thus (i) and (ii) follow from Theorem 7.1 as it covers all possible combinations of the parameters describing G. We shall now prove (iii). Assume that G_2 does not belong to $\mathcal{C}^\circ(g) \cup \mathcal{C}^\circ(g^+)$. If G_2 is a dumbbell, then Lemma 6.1 gives that $\epsilon^+(G_2) = 0$, and thus $G_1 \in \mathcal{C}^\circ(g) \cup \mathcal{C}^\circ(g^+)$ by (ii). Thus we may assume that μG_2 is connected for each $\mu \in \mathcal{M}(G_2)$. Lemma 4.2 applied to G_2 gives that there exists a minor-operation $\mu \in \mathcal{M}(G_2)$ such that $\mu \notin \Delta_1(g) \cup \Delta_1(g^+)$. By Corollary 3.9, $\mu \in \Delta_1(g_a^+)$. Since $\mu \notin \Delta_1(g^+)$, we have that $\epsilon^+(G_2) = 0$ by (S1) (Lemma 3.5 applied to g_a^+ and g^+). Therefore, (ii) gives that $G_1 \in \mathcal{C}^\circ(g) \cup \mathcal{C}^\circ(g^+)$. ∎

8 Torus

In this section, we characterize obstructions for embedding graphs into the torus of connectivity 2. We first show that the classes $\mathcal{C}_0^\circ(g)$ and $\mathcal{C}_0^\circ(g^+)$ are related to Kuratowski graphs K_5 and $K_{3,3}$.

Lemma 8.1 *The class $\mathcal{C}_0^\circ(g)$ consists of a single graph, $K_{3,3}$ with non-adjacent terminals (Figure 6(c)). The class $\mathcal{C}_0^\circ(g^+)$ consists of the three graphs shown in Figure 6.*

Proof The obstructions $\mathrm{Forb}(\mathbb{S}_0)$ for the 2-sphere are $K_{3,3}$ and K_5. As we observed in Section 4, a graph G belongs to $\mathcal{C}_0^\circ(g)$ if only if $\widehat{G}$ is isomorphic to a graph in $\mathrm{Forb}(\mathbb{S}_0)$ with the terminals non-adjacent. Since $xy \notin E(G)$, $\widehat{G}$ cannot be isomorphic to K_5, and there is a unique 2-labeled graph isomorphic to $K_{3,3}$ with two non-adjacent terminals.

Let us show first that each graph in Figure 6 belongs to $\mathcal{C}_0^\circ(g^+)$. If $\widehat{G^+}$ is isomorphic to a Kuratowski graph, the lemma follows from the Kuratowski theorem. Otherwise $\widehat{G}$ is isomorphic to $K_{3,3}$ with x and y non-adjacent. It suffices to show that μG^+ is planar for each minor-operation $\mu \in \mathcal{M}(G)$, as G^+ clearly embeds into the torus.

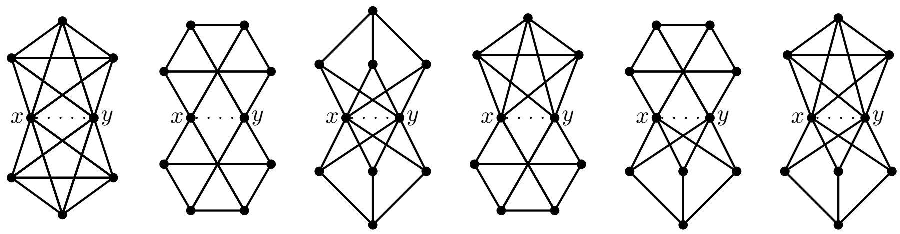

Figure 7: $\mathcal{T}_2$, the xy-sums of Kuratowski graphs that belong to $\mathcal{C}_0^\circ(g_a) \cap \mathcal{C}_0^\circ(g_a^+)$.

Pick an arbitrary edge $e \in E(G)$. The graph $G^+ - e$ has 9 edges and is not isomorphic to $K_{3,3}$ as it contains a triangle. The graph G^+/e has only 5 vertices and (at most) 9 edges. Since e was arbitrary, it follows that μG^+ is planar for every $\mu \in \mathcal{M}(G)$. We conclude that $G \in \mathcal{C}_0^\circ(g^+)$.

We shall show now that there are no other graphs in $\mathcal{C}_0^\circ(g^+)$. Let $G \in \mathcal{C}_0^\circ(g^+)$. By Lemma 4.1, there is a graph $H \in \mathrm{Forb}^*(\mathbb{S}_0)$ such that either $\widehat{G}$ is isomorphic to H or G is isomorphic to the graph obtained from H by deleting an edge and making the ends terminals. It is not hard to see that this yields precisely the graphs in Figure 6. ∎

Note that the first two graphs in Figure 6 have θ equal to 1 and the last one has θ equal to 0. We summarize the properties of graphs in $\mathcal{C}_0^\circ(g^+)$ in the following lemma.

Lemma 8.2 *For each graph $G \in \mathcal{C}_0^\circ(g^+)$, the graph G^+ is xy-alternating on the torus, G/xy is planar, and $\theta(G) = 1$ if and only if $G \notin \mathcal{C}_0^\circ(g)$.*

Proof By Lemma 8.1, $\widehat{G}$ or $\widehat{G^+}$ is isomorphic to a Kuratowski graph. The xy-alternating embeddings of Kuratowski graphs are depicted in Figure 2. Since each Kuratowski graph G is xy-alternating for each pair of vertices of G, the graph G^+ is also xy-alternating for each pair of vertices of G by Lemma 3.3. For each Kuratowski graph G, the graph G/xy has at most 5 vertices and at most 9 edges. Thus G/xy contains no Kuratowski graph as a minor and is therefore planar. ∎

Mohar and Škoda [7] presented the complete list of graphs in $\mathcal{C}_0(g_a)$. We describe them using six subclasses $\mathcal{T}_1, \ldots, \mathcal{T}_6$ of $\mathcal{G}_{xy}^\circ$. Let $\mathcal{T}_1$ be the class of graphs that contains each $G \in \mathcal{G}_{xy}^\circ$ such that $\widehat{G}$ is isomorphic to a Kuratowski graph plus one or two isolated vertices that are terminals in G, $\mathcal{T}_2$ the class of graphs shown in Figure 7, $\mathcal{T}_3$ the class of graphs corresponding to the graphs in Figure 8, $\mathcal{T}_4$ the class of graphs corresponding to the graphs in Figure 12, $\mathcal{T}_5$ the class of graphs depicted in Figure 13, and $\mathcal{T}_6$ the class of graphs corresponding to the graphs in Figure 14.

Theorem 8.3 (Mohar and Škoda [7]) *A graph $G \in \mathcal{G}_{xy}$ belongs to $\mathcal{C}_0(g_a)$ if and only if one of the following holds:*

(i) *$xy \notin E(G)$ and $G \in \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_4$;*
(ii) *$xy \in E(G)$ and $G - xy \in \mathcal{T}_5 \cup \mathcal{T}_6$.*

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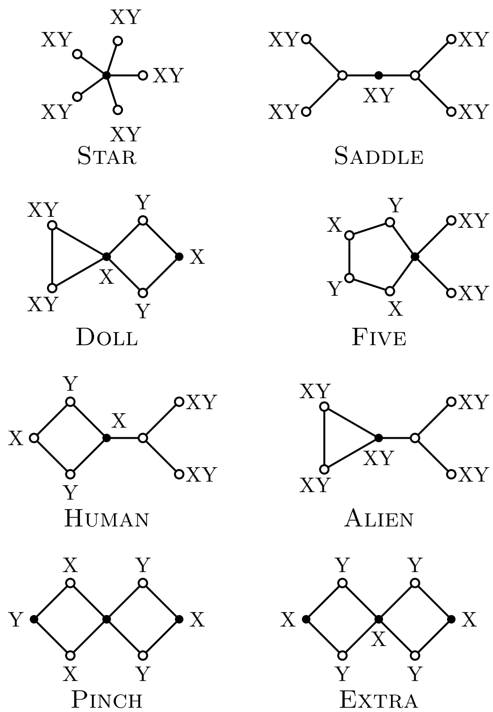

Figure 8: The XY-labelled representation of $\mathcal{T}_3 \subseteq \mathcal{C}_0^\circ(g_a) \cap \mathcal{C}_0^\circ(g_a^+)$. For each white vertex $v \in V(G)$, we have $g(G - v) = 1$.

The graphs in $\mathcal{T}_1$ are disconnected, and hence they do not appear in an xy-sum of connectivity 2. We will use the following facts about the class $\mathcal{C}_0(g_a)$.

Lemma 8.4 *For each graph $G \in \mathcal{C}_0(g_a)$, we have $g^+(G) = g(G) = 1$ and hence $\epsilon(G) = \epsilon^+(G) = \theta(G) = 0$.*

Proof Observe that each graph in $\mathcal{C}_0(g_a)$ is nonplanar. We shall prove that $g^+(G) \leq 1$ for each $G \in \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_6$, which implies that $g^+(G) = g(G) = 1$ for each $G \in \mathcal{C}_0(g_a)$ by Theorem 8.3. For a graph $G \in \mathcal{T}_1$, $\widehat{G^+}$ has two blocks, one isomorphic to a Kuratowski graph and the other consisting of a single edge. Thus $g^+(G) = g(\widehat{G^+}) = 1$. Each graph G in $\mathcal{T}_2$ can be obtained as an xy-sum of two Kuratowski graphs. Theorem 3.1 gives that $g(G) = 1$ and $\theta(G) = 0$, since both parts of G are xy-alternating. Hence $g^+(G) = 1$.

To prove that a graph $G \in \mathcal{T}_3 \cup \mathcal{T}_4$ has $g^+(G) = 1$, it is sufficient to provide an embedding of G^+ in the torus. Figures 8 and 12 show that $G - x - y$ has a drawing in the plane with all neighbors of x and y on the outer face. Thus G/xy is a planar graph. Moreover, the edges in the local rotation around the identified vertex in G/xy can be written as $S_1 S_2 \cdots S_6$ where edges in S_1, S_3, S_5 are those incident with x in G and

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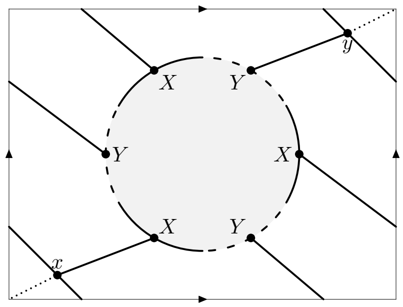

Figure 9: An embedding of G^+ in the torus for a graph G such that G/xy is planar and $G - e$ is xy-alternating in the torus for some edge e incident with x or y.

S_2, S_4, S_6 are incident with y in G. Therefore G^+ admits an embedding in the torus as shown in Figure 9. In the figure, a single edge is drawn from x to the boundary of the planar patch for all the consecutive edges that connect x and the planar patch.

We shall show that this structure of graphs in $\mathcal{T}_3 \cup \mathcal{T}_4$ is not accidental. Let $e \in E(G)$ be an edge incident with x or y, say $e = xv$. If $G - e$ is nonplanar, then $G - e$ has an xy-alternating embedding Π into the torus. The two Π-angles at x of the xy-alternating face divide the edges in the local rotation around x into two sets, S_1 and S_3. Similarly, the edges incident with y form sets S_2 and S_4. It is not hard to see that since G/xy is planar, we can pick Π so that v is Π-cofacial with y (it is not Π-cofacial with x since G is not xy-alternating). We can assume that v lies in the region of edges in S_4. Thus G/xy has the structure described above with $S_5 = \{e\}$ and S_4 split into sets S_4' and S_6. It is thus enough to show that there exists an edge e incident with x or y such that $G - e$ is nonplanar. For $G \in \mathcal{T}_3$ and an edge $e \in E(G)$ incident with a white vertex in Figure 8, $G - e$ is nonplanar. For $G \in \mathcal{T}_4$, the edges e such that $G - e$ is nonplanar are depicted in Figure 12 as underlined labels.

Each graph G in $\mathcal{T}_5 \cup \mathcal{T}_6$ is planar. Thus $g^+(G) = g(G^+) \le 1$. ■

We suspect that $\epsilon^+(G) = \epsilon(G) = \theta(G) = 0$ for all graphs in $\mathcal{C}(g_a)$, but the proof seems out of reach. See [7] for more details. Lemmas 5.7 and 8.4 classify when a graph in $\mathcal{C}_0(g_a) \cup \mathcal{C}_0(g_a^+)$ has θ equal to 1. We have the following corollary.

Corollary 8.5 *Let G be a graph in $\mathcal{C}_0^\circ(g_a) \cup \mathcal{C}_0^\circ(g_a^+)$. Then $g^+(G) = 1$ and $\epsilon^+(G) = 0$. Moreover, $\theta(G) = 1$ if and only if $G \in \mathcal{C}_0^\circ(g_a^+) \setminus \mathcal{C}_0^\circ(g_a)$.*

Proof Let $G \in \mathcal{C}_0^\circ(g_a^+) \setminus \mathcal{C}_0^\circ(g_a)$. By Lemma 4.3, $G^+ \in \mathcal{C}_0(g_a)$. By Lemma 5.7, $\theta(G) = 1$ and $\epsilon^+(G) = 0$. Since $g_a^+(G) = 1$, $g^+(G) = g_a^+(G) - \epsilon^+(G) = 1$.

If $G \in \mathcal{C}_0^\circ(g_a)$, then $G \in \mathcal{C}_0(g_a)$ and thus $\theta(G) = \epsilon^+(G) = 0$ and $g^+(G) = 1$ by Lemma 8.4. ■

The classes $\mathcal{T}_1, \dots, \mathcal{T}_6$ lie in $\mathcal{C}_0^\circ(g_a^+) \cup \mathcal{C}_0^\circ(g_a)$. More precise membership as depicted in Figure 10 is proved below. We shall use the following observation.

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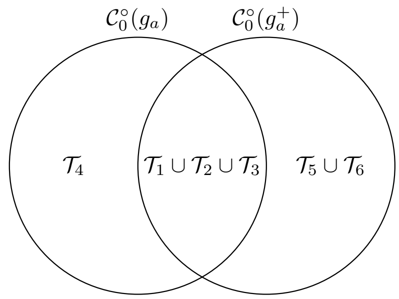

Figure 10: Venn diagram of critical classes (for alternating genus) for the torus.

Lemma 8.6 *Let $G \in \mathcal{G}_{xy}$, $\mathcal{P}$ a minor-monotone graph parameter, and*

$$v \in V(G) \setminus \{x, y\}.$$

If $\mathcal{P}(G - v) = \mathcal{P}(G)$, then $\mathcal{P}(\mu G) = \mathcal{P}(G)$ for each $\mu = (uv, \cdot) \in \mathcal{M}(G)$.

Proof Let $\mu = (uv, \cdot) \in \mathcal{M}(G)$. Since $G - v$ is a minor of μG and $\mathcal{P}$ is minor-monotone, $\mathcal{P}(G) \geq \mathcal{P}(\mu G) \geq \mathcal{P}(G - v) = \mathcal{P}(G)$. ■

Lemma 8.6 can be used to prove that $\Delta_1(g) = \varnothing$ if we can find a vertex cover U of G such that $g(G - v) = g(G)$ for each $v \in U$. We shall use this idea to prove that $\mathcal{T}_3 \subseteq \mathcal{C}_0^\circ(g_a^+)$. The following lemma will be also used.

Lemma 8.7 ([7, Lemma 19]) *Let $G \in \mathcal{G}_{xy}^\circ$ be a graph such that G/xy is planar. If $g_a^+(G) \geq 1$, then either x and y have at least five common neighbors or there are six distinct non-terminal vertices $v_1, \ldots, v_6$ such that v_1, v_2, v_3 are adjacent to x, and v_4, v_5, v_6 are adjacent to y.*

In order to determine if a graph $G \in \mathcal{C}_0^\circ(g_a)$ also belongs to $\mathcal{C}_0^\circ(g_a^+)$ we can either use Lemma 4.4 or note that, since $g_a^+(G) \geq g_a(G)$ and g_a^+ is minor-monotone by Lemma 3.2, each graph $G \in \mathcal{C}_0^\circ(g_a)$ contains a graph in $\mathcal{C}_0^\circ(g_a^+)$ as a minor.

Lemma 8.8 $\mathcal{C}_0^\circ(g_a) \cap \mathcal{C}_0^\circ(g_a^+) = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, $\mathcal{C}_0^\circ(g_a) \setminus \mathcal{C}_0^\circ(g_a^+) = \mathcal{T}_4$, *and* $\mathcal{C}_0^\circ(g_a^+) \setminus \mathcal{C}_0^\circ(g_a) = \mathcal{T}_5 \cup \mathcal{T}_6$.

Proof By Theorem 8.3, $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4 \subseteq \mathcal{C}_0^\circ(g_a)$. Let us start by proving that $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \subseteq \mathcal{C}_0^\circ(g_a^+)$. Suppose that $G \in \mathcal{T}_1$. Then it is not difficult to see that $G \in \mathcal{C}_0^\circ(g_a^+)$, since $\widehat{G^+}$ has two blocks, one isomorphic to a Kuratowski graph and the other consisting of a single edge.

Let $G \in \mathcal{T}_2$ and $\mu \in \mathcal{M}(G)$. Since G is an xy-sum of two graphs in $\mathcal{C}_0^\circ(g^+)$, neither contraction nor deletion of an edge on one side destroys the Kuratowski graph on the other side. Thus $g(\mu G) = 1$ and $\mathcal{M}(G) \cap \Delta_1(g) = \varnothing$. By Lemma 4.4, $G \in \mathcal{C}_0^\circ(g_a^+)$.

Let us prove now that $\mathcal{T}_3 \subseteq \mathcal{C}_0^\circ(g_a^+)$. Consider a graph $G \in \mathcal{T}_3$. By Lemma 4.4, it is enough to show that $\Delta_1(g) \setminus \Delta_1(g_a^+) = \varnothing$. Let $\mu \in \mathcal{M}(G)$. Let U be the set of

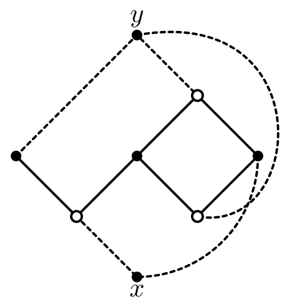

Figure 11: The graph Pinch minus a vertex. The white vertices form one part of the $K_{3,3}$-subdivision.

white vertices of G as depicted in Figure 8. It is not hard to show that for each $v \in U$, $G - v$ is nonplanar. We omit the detailed proof of this fact and only demonstrate the proof technique on the graph Pinch. Since U is an orbit of the isomorphism group of Pinch, it is enough to show that $G - u$ is nonplanar for one of the vertices $u \in U$. Indeed, $G - u$ is isomorphic to a subdivision of $K_{3,3}$ as is exhibited in Figure 11. Thus $G - u$ is nonplanar for each $u \in U$ as required.

By Lemma 8.6, we can assume that the edge e of μ is not covered by a vertex in U. This proves that the graphs Star, Ribbon, Five, and Four are in $\mathcal{C}_0^\circ(g_a^+)$, since U is a vertex cover. For the other graphs, observe that the vertices in U cover all the edges not incident with a terminal. Thus e corresponds to a label on a black vertex of G in Figure 8. Assume that $\mu = (e, -)$. By inspection, the conclusion of Lemma 8.7 is violated for $G - e$. Hence $g_a^+(\mu G) = 0$ and $\mu \in \Delta_1(g_a^+)$. We can assume now that $\mu = (e, /)$. When G is one of the graphs Saddle, Human, Alien, or Bowtie, when G is Extra with e incident with the non-terminal vertex of degree 5, and when G is Doll with e incident with the non-terminal vertex of degree 5, the graph μG^+ is an xy-sum of two graphs G_1 and G_2. We observe that in all cases, the graphs G_1^+ and G_2^+ are planar, and thus μG^+ is planar by Theorem 3.1. We conclude that $\mu \in \Delta_1(g_a^+)$. If G is Pinch, then μG is a proper minor of Four. Since we already showed that Four $\in \mathcal{C}_0^\circ(g_a^+)$, we have that $\mu \in \Delta_1(g_a^+)$ in this case as well. If G is Doll and e is incident with the black vertex of degree 3, then μG is a proper minor of Four. The remaining case is that G is Extra and e is incident with a non-terminal black vertex of degree 3. Again, μG is a proper minor of Five and thus $\mu \in \Delta_1(g_a^+)$.

By Lemma 4.3, the class $\mathcal{C}_0^\circ(g_a^+) \setminus \mathcal{C}_0^\circ(g_a)$ contains precisely the graphs G such that $G^+ \in \mathcal{C}_0(g_a)$. Theorem 8.3 gives that the graphs in $\mathcal{T}_5 \cup \mathcal{T}_6$ (and only those) have that property.

We prove that $\mathcal{C}_0^\circ(g_a) \setminus \mathcal{C}_0^\circ(g_a^+) = \mathcal{T}_4$ by showing that $\mathcal{T}_4 \cap \mathcal{C}_0^\circ(g_a^+) = \varnothing$. Since each $G \in \mathcal{T}_4$ has a proper minor in $\mathcal{T}_6 \subseteq \mathcal{C}_0^\circ(g_a^+)$, G does not belong to $\mathcal{C}_0^\circ(g_a^+)$. Pentagon is a minor of Rocket and Lollipop, while Hexagon is a minor of Bullet, Frog, and Hive. Hence, $\mathcal{T}_4 \subseteq \mathcal{C}_0^\circ(g_a) \setminus \mathcal{C}_0^\circ(g_a^+)$. We have shown that the classes $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_5, \mathcal{T}_6$ are subclasses of $\mathcal{C}_0^\circ(g_a^+)$. Hence, $\mathcal{C}_0^\circ(g_a) \setminus \mathcal{C}_0^\circ(g_a^+) \subseteq \mathcal{T}_4$ by Theorem 8.3. ■

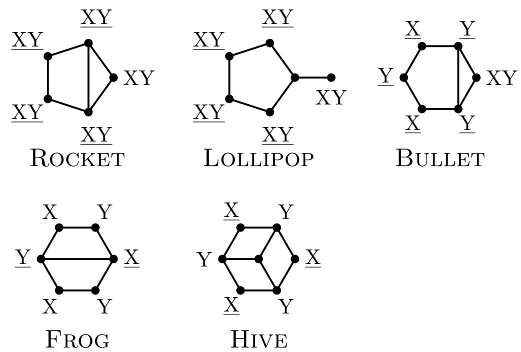

Figure 12: The XY-labelled representation of $\mathcal{T}_4 = \mathcal{C}_0^\circ(g_a) \setminus \mathcal{C}_0^\circ(g_a^+)$.

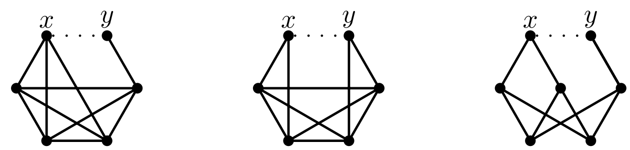

Figure 13: $\mathcal{T}_5$, splits of Kuratowski graphs that belong to $\mathcal{C}_0^\circ(g_a^+) \setminus \mathcal{C}_0^\circ(g_a)$

Let us present some restrictions on an xy-sum that is an obstruction for the torus.

Lemma 8.9 *Let G be an xy-sum of connected graphs $G_1, G_2 \in \mathcal{G}_{xy}^\circ$. If $G \in \mathrm{Forb}(\mathbb{S}_1)$, then*

(i) $g^+(G_1) = g^+(G_2) = 1$,
(ii) $\epsilon^+(G_1)\epsilon^+(G_2) = 0$, *and*
(iii) $\eta(G_1, G_2) = 2$ *if and only if* $xy \in E(G)$.

Proof Suppose that $G \in \mathrm{Forb}(\mathbb{S}_1)$. If $g^+(G_1) \geq 2$, then G_1^+ contains a toroidal obstruction. Since G_1^+ is a proper minor of G, this contradicts the fact that $G \in \mathrm{Forb}(\mathbb{S}_1)$. Thus $g^+(G_1) \leq 1$ and $g^+(G_2) \leq 1$ by symmetry. If $g^+(G_1) = 0$, then $g^+(G) \leq 1$ by Theorem 3.1, a contradiction.[5] We conclude that $g^+(G_1) = 1$ and also $g^+(G_2) = 1$ by symmetry. This shows (i).

If $\epsilon^+(G_1)\epsilon^+(G_2) = 1$, then it follows from Theorem 3.1 that

$$g(G) \leq g^+(G_1) + g^+(G_2) - \epsilon^+(G_1)\epsilon^+(G_2) = 1,$$

a contradiction. Thus $\epsilon^+(G_1)\epsilon^+(G_2) = 0$ and (ii) holds.

To show (iii), suppose that $xy \notin E(G)$ and $\eta(G_1, G_2) = 2$. By (i) and (ii), this is only possible if $g(G_1) = g(G_2) = 0$. By Theorem 3.1, $g(G) \leq g(G_1) + g(G_2) + 1 \leq 1$, a contradiction. The other implication follows from Lemma 3.10. ■

[5]The fact that $g^+(G_1)$ and $g^+(G_2)$ are at least 1 is a simple observation; see for example [6].

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$xy \in E(G)$	$\epsilon^+(G_2)$	$\eta(G_1, G_2)$	G_1
yes	0	—	$\mathcal{C}_0^\circ(g^+)$
	1		$\mathcal{C}_0^\circ(g_a^+)$
no	0	0	$\mathcal{C}_0^\circ(g^+)$
		1	$\mathcal{C}_0^\circ(g)$ or $\mathcal{C}_0^\circ(g^+)$
	1	0	$\mathcal{C}_0^\circ(g_a^+)$
		1	$\mathcal{C}_0^\circ(g_a)$ or $\mathcal{C}_0^\circ(g_a^+)$

Table 3: Classification of the parts of obstructions for the torus.

It is time to present the main theorem of this section that derives a full characterization of the obstructions for the torus of connectivity 2. It can be viewed as an application of Theorem 7.1 with the outcome summarized in Table 3.

Theorem 8.10 *Suppose that G is an xy-sum of connected graphs $G_1, G_2 \in \mathcal{G}_{xy}^\circ$ and that the following statements hold:*

(i) $G_1 \in \mathcal{C}_0^\circ(g^+)$,
(ii) $G_2 \in \mathcal{C}_0^\circ(g_a) \cup \mathcal{C}_0^\circ(g_a^+)$,
(iii) *$xy \in E(G)$ if and only if $G_1 \notin \mathcal{C}_0^\circ(g)$ and $G_2 \notin \mathcal{C}_0^\circ(g_a)$, and*
(iv) *if $\theta(G_1) = \theta(G_2) = 0$, then $G_2 \in \mathcal{C}_0^\circ(g_a^+)$.*

Then $G \in \mathrm{Forb}(\mathbb{S}_1)$. Furthermore, every obstruction for the torus of connectivity 2 can be obtained this way.

Proof The proof consists of two parts. In the first part, we prove that each graph satisfying conditions (i)–(iv) is an obstruction for the torus. In the second part, all obstructions of connectivity 2 are shown to be constructed this way.

Let us assume that (i)–(iv) holds. To show that G is an obstruction for the torus, we need to prove that G_1, G_2, and xy (when $xy \in E(G)$) are minor-tight and that $g(G) = 2$. By (i) and Lemma 8.2, $\epsilon^+(G_1) = 1$ and $g^+(G_1) = 1$. By (ii) and Corollary 8.5, $\epsilon^+(G_2) = 0$ and $g^+(G_2) = 1$. Hence, $h_1(G) = 2$. If $\eta(G_1, G_2) = 2$, then $\theta(G_1) = \theta(G_2) = 1$. Thus $G_1 \in \mathcal{C}_0^\circ(g^+) \setminus \mathcal{C}_0^\circ(g)$ by Lemma 8.2 and $G_2 \in \mathcal{C}_0^\circ(g_a^+) \setminus \mathcal{C}_0^\circ(g_a)$ by Corollary 8.5. By (iii), $xy \in E(G)$. Consequently, we have either $\eta(G_1, G_2) \le 1$ or $xy \in E(G)$. This excludes the case where $\eta(G_1, G_2) = 2$ and $xy \notin E(G)$ and we shall use it below. If $xy \in E(G)$, then by Theorem 3.1, $g(G) = h_1(G) = 2$ as required. Similarly, if $xy \notin E(G)$ and $\eta(G_1, G_2) \le 1$, then $h_1(G) \le h_0(G)$ by (3.5). Hence $g(G) = h_1(G) = 2$ by Theorem 3.1.

It remains to prove minor-tightness. Since $\epsilon^+(G_2) = 0$ and $G_1 \in \mathcal{C}_0^\circ(g^+)$, Theorem 7.1 gives that G_1 is minor-tight. If $G_2 \in \mathcal{C}_0^\circ(g_a^+)$, then G_2 is minor-tight by Theorem 7.1, since $\epsilon^+(G_2) = 1$. Otherwise, $G_2 \in \mathcal{C}_0^\circ(g_a) \setminus \mathcal{C}_0^\circ(g_a^+)$ and $\theta(G_2) = 0$ by Corollary 8.5. Thus $\theta(G_1) = 1$ by (iv) and we have that $\eta(G_1, G_2) = 1$. We conclude that G_2 is minor-tight by Theorem 7.1.

If $xy \in E(G)$, then (iii) implies that $G_1 \in \mathcal{C}_0^\circ(g^+) \setminus \mathcal{C}_0^\circ(g)$ and $G_2 \in \mathcal{C}_0^\circ(g_a^+) \setminus \mathcal{C}_0^\circ(g_a)$. Therefore, $\theta(G_1) = 1$ by Lemma 8.2 and $\theta(G_2) = 1$ by Corollary 8.5. Hence

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$\eta(G_1, G_2) = 2$. Lemma 8.2 applied to G_1 implies that $g(G_1/xy) < g^+(G_1)$. Thus xy is minor-tight in G by Lemma 3.10. We conclude that G is an obstruction for the torus by Lemma 2.1.

Let us now prove that, for a graph $G \in \text{Forb}(\mathbb{S}_1)$ of connectivity 2, there exists a 2-vertex-cut $\{x, y\}$ such that when G is viewed as an xy-sum of graphs G_1 and G_2, statements (i)–(iv) hold. We pick x and y as guaranteed by Lemma 6.4 so that $G_1, G_2 \notin \mathcal{D}$. Since G is an obstruction, the subgraphs G_1, G_2, and xy (if present) are minor-tight. By Lemma 8.9, $g^+(G_1) = g^+(G_2) = 1$ and $\epsilon^+(G_1)\epsilon^+(G_2) = 0$. We may assume by symmetry that $\epsilon^+(G_2) = 0$. By Corollary 7.2(ii), the graph G_1 belongs to $\mathcal{C}_0^\circ(g) \cup \mathcal{C}_0^\circ(g^+) = \mathcal{C}_0^\circ(g^+)$, since $g(G_1) \le g^+(G_1) = 1$. Hence (i) holds. By Lemma 8.2, $\epsilon^+(G_1) = 1$.

Since $\epsilon^+(G_2) = 0$, Lemma 8.2 gives that $G_2 \notin \mathcal{C}_0^\circ(g^+)$. By Corollary 7.2(i), the graph G_2 belongs to $\mathcal{C}_0^\circ(g_a) \cup \mathcal{C}_0^\circ(g_a^+)$ since $G_2 \notin \mathcal{D}$, $g^+(G_2) = 1$, and g^+ bounds all the other parameters. Thus (ii) is true.

We prove equivalence in (iii) at once. By Lemma 8.9(iii), we have that $xy \in E(G)$ if and only if $\eta(G_1, G_2) = 2$. Since $\epsilon^+(G_2) = 0$, $\eta(G_1, G_2) = 2$ if and only if $\theta(G_1) = \theta(G_2) = 1$. By Lemma 8.2 and Corollary 8.5, $\theta(G_1) = \theta(G_2) = 1$ if and only if $G_1 \in \mathcal{C}_0^\circ(g^+) \setminus \mathcal{C}_0^\circ(g)$ and $G_2 \in \mathcal{C}_0^\circ(g_a^+) \setminus \mathcal{C}_0^\circ(g_a)$. We conclude that (iii) holds.

For (iv), suppose that $G_2 \notin \mathcal{C}_0^\circ(g_a^+)$. Since $G_2 \notin \mathcal{C}_0^\circ(g^+)$ and G_2 is minor-tight, Theorem 7.1 gives that $\eta(G_1, G_2) = 1$ (as $\mathcal{H}_0^1 \subseteq \mathcal{C}_0^\circ(g^+)$ by Lemma 5.2). We conclude that either $\theta(G_1) = 1$ or $\theta(G_2) = 1$, and thus (iv) holds. This finishes the proof of the theorem. ∎

Corollary 8.11 *There are 68 obstructions for the torus of connectivity 2.*

Proof By Theorem 8.10, for each $G \in \text{Forb}(\mathbb{S}_1)$ of connectivity 2, there exists a 2-vertex-cut $\{x, y\}$ such that G is an xy-sum of parts G_1 and G_2 satisfying (i)–(iv). Let us count the number of graphs in $\text{Forb}(\mathbb{S}_1)$ of connectivity 2 by counting the number of non-isomorphic xy-sums satisfying (i)–(iv).

Let us first count the number of pairs G_1 and G_2 for which (i), (ii), and (iv) of Theorem 8.10 hold. The graphs in $\mathcal{T}_1$ are disconnected so their 2-sum with G_1 is not 2-connected. The number of connected graphs in $\mathcal{C}_0^\circ(g_a) \cup \mathcal{C}_0^\circ(g_a^+)$ is $|\mathcal{T}_2 \cup \cdots \cup \mathcal{T}_6| = 27$ and the number of graphs in $\mathcal{C}_0^\circ(g^+)$ is 3. Thus we have precisely 81 pairs satisfying (i) and (ii). However, some of them do not satisfy (iv).

Let us consider property (iv). There is only a single graph in $\mathcal{C}_0^\circ(g^+)$ that has θ equal to 0 (Figure 6(c)). By Lemma 8.8, there are precisely $|\mathcal{T}_4| = 5$ graphs in

$$\mathcal{C}_0^\circ(g_a) \setminus \mathcal{C}_0^\circ(g_a^+);$$

they all have θ equal to 0 by Corollary 8.5. Thus 5 pairs out of the total of 81 do not satisfy (iv) of Theorem 8.10 giving the total of 76 pairs satisfying (i), (ii), and (iv).

For fixed graphs G_1 and G_2 in $\mathcal{G}_{xy}^\circ$, there are four different xy-sums with parts G_1 and G_2; there are two ways to identify two graphs on two vertices, and the edge xy is either present or not. Precisely two of those xy-sums satisfy (iii) as the presence of xy depends only on G_1 and G_2. Since for each graph in $\mathcal{C}_0^\circ(g^+)$ there is an automorphism exchanging the terminals, there is precisely one xy-sum with parts G_1 and G_2 that satisfies (i) and (iii).

Figure 14: The XY-labelled representation of $\mathcal{T}_6 \subseteq \mathcal{C}_0^\circ(g_a^+) \setminus \mathcal{C}_0^\circ(g_a)$.

Therefore, for each of the 76 pairs, there is a unique xy-sum satisfying (i)–(iv). By Theorem 8.10, each such xy-sum is an obstruction for the torus. Some of the obtained obstructions are isomorphic though. Let G be an xy-sum of G_1 and G_2 and G' be an $x'y'$-sum of G_1' and G_2' such that both G and G' satisfy (i)–(iv), and there is an isomorphism ψ of $\widehat{G}$ and $\widehat{G}'$. If $\psi(\{x,y\}) \neq \{x',y'\}$, then $\psi(\{x,y\})$ is a 2-vertex-cut in G'. It is not hard to see that G' has another 2-vertex cut only if $G_2' \in \mathcal{T}_5$. We can see that the preimage of ψ of one side of $\psi(\{x,y\})$ is a graph in $\mathcal{C}_0^\circ(g^+) \setminus \mathcal{C}_0^\circ(g)$. Therefore, $G_1 \in \mathcal{C}_0^\circ(g^+) \setminus \mathcal{C}_0^\circ(g)$ and $G_2 \in \mathcal{T}_5 \subseteq \mathcal{C}_0^\circ(g_a^+) \setminus \mathcal{C}_0^\circ(g_a)$. By (iii), $xy \in E(G)$. But $\psi(x)$ is not adjacent to $\psi(y)$, a contradiction.

We may assume now that $\psi(\{x,y\}) = \{x',y'\}$. If $\psi(V(G_1)) = V(G_1')$, then $G_1 \cong G_1'$ and $G_2 \cong G_2'$, as argued above. Thus $\psi(V(G_1)) \neq V(G_1')$. It is not hard to check that only the graphs in $\mathcal{T}_2$ have a subgraph isomorphic to a graph in $\mathcal{C}_0^\circ(g^+)$. There are 18 pairs G_1, G_2 such that $G_1 \in \mathcal{C}_0^\circ(g^+)$ and $G_2 \in \mathcal{T}_2$, but there are precisely 10 non-isomorphic obstructions for the torus obtained from these 18 pairs. We conclude that there are 68 non-isomorphic obstructions for the torus of connectivity 2. ■

9 Open Problems

The following questions remain unanswered:

(a) Do hoppers exist? If they do, what is the smallest genus k such that the class $\mathcal{H}_k^0$ ($\mathcal{H}_k^1$, or $\mathcal{H}_k^2$) is non-empty?

(b) Is it possible that there exists a graph $G \in \mathcal{C}^\circ(g)$ with $\theta(G) = 1$? In other words, can each of the graphs G and G^+ be an obstruction for an orientable surface? What is the smallest k such that this is the case for a graph of genus k?

(c) What is the smallest $m(r)$ such that there exists an r-connected obstruction G of genus k with a pair of vertices x, y such that G is *not* xy-alternating. For example, $m(0) = 2$. We do not know the value $m(r)$ for any $r > 0$.

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