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The orbit of a Hölder continuous path under a hyperbolic toral automorphism

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Abstract. Let $f: T^3 \to T^3$ be a hyperbolic toral automorphism lifting to a linear automorphism with real eigenvalues. We prove that there is a Hölder continuous path in T^3 whose orbit-closure is 1-dimensional. This strengthens results of Hancock and Przytycki concerning continuous paths, and contrasts with results of Franks and Mañé concerning rectifiable paths.

1. *Introduction*

Let $f: T^n \to T^n$ be a hyperbolic toral automorphism. Franks proved in [**1**] that any compact invariant set in T^n that contains a non-constant C^1 path also contains a torus of dimension at least two which is a coset of a subgroup of T^n. Subsequently, Mañé [**5**] has proved the same result for rectifiable paths. On the other hand, Hancock [**2**] has shown in the case $n = 3$ how to construct, for all f, C^0-paths whose orbit-closures have dimension one, and Przytycki [**6**], using more delicate methods, has constructed such paths for all higher n. In his thesis, [**3**], Hancock asked whether the condition of Hölder continuity, which lies between continuity and rectifiability, takes paths into the Franks–Mañé or the Hancock–Przytycki camp. In this paper we prove that the latter is the case, at least for $n = 3$, for certain maps f and for certain values of the Hölder index. Our approach is a modification of Przytycki's. We shall deal with higher dimensional tori and invariant sets (and also more comprehensively with $n = 3$) in another paper.

2. *Definitions and statement of the theorem*

A *hyperbolic toral automorphism* of $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is a map

$$f: T^n \to T^n$$

that lifts to a linear automorphism

$$L: \mathbb{R}^n \to \mathbb{R}^n$$

with no eigenvalues of modulus 1. Thus

(i) the matrix A of L has integer entries and determinant ± 1, and

(ii) $\mathbb{R}^n$ splits as the direct sum $E^s \oplus E^u$ of L-invariant subspaces such that all eigenvalues of $L|E^s$ and $L|E^u$ have modulus respectively <1 and >1.

Neither the stable summand E^s nor the unstable summand E^u contains points of $\mathbb{Z}^n$ other than 0. The cosets of E^s and E^u project by the standard covering map

$$\pi: \mathbb{R}^n \to T^n$$

onto the stable and unstable manifolds of f.

A map $g: X \to Y$ of metric spaces is *Hölder continuous of index* α $(0 < \alpha \le 1)$ if, for some constant C,

$$d_Y(g(x), g(x')) \le C d_X(x, x')^\alpha$$

for all $x, x' \in X$. The case $\alpha = 1$ gives *Lipschitz* maps. Since Lipschitz paths are rectifiable, we are only interested in the case $\alpha < 1$.

When $n = 3$ and L is as above, the characteristic polynomial of L is irreducible over $\mathbb{Z}$ and so cannot have a repeated root. Thus either L or L^{-1} has one eigenvalue with modulus <1 and two with modulus >1. The latter two are either real and unequal or complex conjugate. Both cases can arise, for example

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 & 1 \\ -4 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

In this paper, we restrict ourselves to the first type of automorphism. We give T^3 the flat metric.

THEOREM *Let $f: T^3 \to T^3$ be a hyperbolic toral automorphism lifting to a linear automorphism L. Suppose that L has real eigenvalues. Then there is a Hölder continuous path*

$$\delta: [0, 1] \to T^3$$

whose orbit-closure is 1-dimensional.

3. *Proof of the theorem*

We may assume, replacing f by f^{-1} if necessary, that L has two eigenvalues λ and μ with

$$|\lambda| > |\mu| > 1.$$

In fact, we may even assume that

$$\lambda > \mu > 3,$$

for we can achieve this by replacing f by some power f^r; if a path has a 1-dimensional orbit closure Γ under f^r, then its orbit closure

$$\Gamma \cup f(\Gamma) \cup \cdots \cup f^{r-1}(\Gamma)$$

under f is 1-dimensional. Our technique is to start with a linear path γ_0 in E^u and, by an infinite sequence of modifications, to obtain from γ_0 a path γ in E^u such that the positive semi-orbit of $\delta = \pi\gamma$ under f does not intersect a certain neighbourhood U of the circle

$$\pi(\mathbb{Z}^2 \times \mathbb{R})$$

in T^3. The modified paths γ_r in the sequence are all Lipschitz and our main job is to keep tabs on Lipschitz constants to ensure that the limit γ of the sequence is

Hölder continuous. The negative semi-orbit of δ does intersect U, but only near $\pi(0)$. Thus the orbit of δ is not dense in T^3. It would be quite feasible to prove directly, using Przytycki's methods, that the orbit closure of δ is 1-dimensional. Fortunately this is not necessary, since, in the $n=3$ case, f cannot have 2- or 3-dimensional closed invariant sets apart from T^3 itself (see [**4**, theorem 9]).

Let

$$G=(\mathbb{Z}^2\times\mathbb{R})\cap E^u.$$

We may identify E^u with $\mathbb{R}^2$ under a linear isomorphism such that, in the new coordinates (x, y) in $\mathbb{R}^2$, L is given by

$$L(x, y)=(\lambda x, \mu y).$$

We may, for convenience, suppose that the distance in $\mathbb{R}^2$ from 0 to the next nearest point of the lattice G is greater than $2\sqrt{2}$. It follows that if two points of G are joined by a line of slope m with $|m|\leq 1$ then the horizontal distance between the points is >2.

We now wish to describe the neighbourhood U mentioned above. We fix numbers a and b with

$$2a=b<(\lambda/\mu)-1, \tag{1}$$

and, for each point $(p, q)\in G$, we take a diamond-shaped neighbourhood $D_0(p, q)$ centred on (p, q) with width a and height b. Thus:

$$D_0(p, q)=\{(x, y)\in\mathbb{R}^2\colon 4|x-p|+2|y-q|<b\}.$$

For $r>0$, and for all $(p, q)\in f^{-r}(G)$, we define

$$D_r(p, q)=f^{-r}(D_0(f^r(p, q)))=\{(x, y)\in\mathbb{R}^2\colon 4\lambda^r|x-p|+2\mu^r|y-q|<b\}.$$

We call the intersection of $D_r(p, q)$ and the line $x=p$, the *vertical core* of $D_r(p, q)$. We further insist that b is small enough for the following condition to hold:

(2_0) if a line of slope m with $|m|\leq 1$ intersects distinct diamonds $D_0(p, q)$ and $D_0(p', q')$ then

$$|p-p'|>2.$$

Notice that this implies, for all $r\geq 0$,

(2_r) if a line of slope m with $|m|\leq\lambda^r/\mu^r$ intersects distinct diamonds $D_r(p, q)$ and $D_r(p', q')$ then

$$|p-p'|>2/\lambda^r.$$

Finally, let $\tilde{D}_r(p, q)$ be $D_r(p, q)$ with a and b replaced by fixed numbers $\tilde{a}$ and $\tilde{b}$ satisfying

$$2\tilde{a}=\tilde{b}<b(\mu-3)/(\mu-1) \tag{3}$$

and let D be the union of $\tilde{D}_0(p, q)$ for all $(p, q)\in G$. Identifying E^u with a subset of $\mathbb{R}^3$ once more, let V be the vertical cylinder generated by D (that is to say, $p^{-1}(p(D))$, where p is vertical projection of $\mathbb{R}^3$ onto the plane $z=0$) and let $U=\pi(V)$.

Let $I=[0, 1]$. We define maps

$$\gamma_r\colon I\to\mathbb{R}^2 \qquad (r\geq 0)$$

inductively, starting with $\gamma_0(t) = (t, 0)$. We assume that γ_{r-1} is defined $(r \geq 1)$ and is the graph of a piecewise linear function

$$g_{r-1}: I \to \mathbb{R}$$

with Lipschitz constant

$$\text{Lip}\, g_{r-1} \leq \lambda^{r-1}/\mu^{r-1}.$$

Thus if $\gamma_{r-1}(I)$ intersects any diamond $D_{r-1}(p, q)$, it intersects its vertical core. Also, by (2_{r-1}), if $\gamma_{r-1}(I)$ intersects two such vertical cores, the horizontal distance between them is greater than $2/\lambda^{r-1}$. Suppose that the intersections of $\gamma_{r-1}(I)$ with such vertical cores are at

$$\gamma_{r-1}(t_i), \qquad 1 \leq i \leq m,$$

and that $\gamma_{r-1}(t_i)$ is distance b_i below the top of the vertical core (so that $b_i < b/\mu^{r-1}$). We define γ_r by

$$\gamma_r(t) = \begin{cases} \gamma_{r-1}(t) & \text{if } |t - t_i| \geq 1/\lambda^{r-1} \quad \text{for all } i, \\ (t, g_{r-1}(t) + (1 - |u|)b_i) & \text{if } t = t_i + u/\lambda^{r-1}, -1 \leq u \leq 1, \quad \text{for some } i. \end{cases}$$

We write $\gamma_r(t) = (t, g_r(t))$. Notice that, for all $t \in I$,

$$0 \leq g_r(t) - g_{r-1}(t) \leq b/\mu^{r-1}. \tag{4}$$

Also

$$\begin{aligned} \text{Lip}\, g_r &\leq \text{Lip}\, g_{r-1} + \text{Lip}\,(g_r - g_{r-1}) \\ &\leq \lambda^{r-1}/\mu^{r-1} + b(\lambda^{r-1}/\mu^{r-1}) \\ &\leq \lambda^r/\mu^r \end{aligned} \tag{5}$$

by (1), as required for the induction. By (4), the sequence (γ_r) converges to a path γ which is the graph of a function $g: I \to \mathbb{R}$. Moreover, since by (4), for all $s \geq r$ and all $t \in I$,

$$0 \leq g_s(t) - g_r(t) \leq b/[\mu^{r-1}(\mu - 1)] \tag{6}$$

and by (3),

$$b/2\mu^{r-1} - \tilde{b}/2\mu^{r-1} > b/[\mu^{r-1}(\mu - 1)]$$

the fact that $\gamma_r(I)$ does not intersect any $D_{r-1}(p, q)$ implies that $\gamma_s(I)$ does not intersect any $\tilde{D}_r(p, q)$, for any $r \geq 0$.

Now note that, for all $r > 0$ and $1/\lambda^r < |t - t'| < 1/\lambda^{r-1}$,

$$\begin{aligned} |g(t) - g(t')| &\leq |g_r(t) - g_r(t')| + |(g(t) - g_r(t)) - (g(t') - g_r(t'))| \\ &\leq (\lambda^r/\mu^r)|t - t'| + b/[\mu^{r-1}(\mu - 1)] \qquad \text{(by (5) and (6))} \\ &\leq (\lambda^r/\mu^r)(1 + b\mu/(\mu - 1))|t - t'|. \end{aligned} \tag{7}$$

Write

$$C = (\lambda/\mu)(1 + b\mu/(\mu - 1)).$$

Then (7) says that

$$(|t - t'|, |g(t) - g(t')|)$$

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lies beneath the straight line segment joining $(1/\lambda^r, (\mu/\lambda)C/\mu^r)$ to $(1/\lambda^{r-1}, C/\mu^{r-1})$. This is below the segment of the curve

$$y = Cx^{\log \mu/\log \lambda}$$

joining $(1/\lambda^r, C/\mu^r)$ to $(1/\lambda^{r-1}, C/\mu^{r-1})$, since $\mu/\lambda < 1$ and the curve is concave downwards. This shows that g is Hölder continuous with constant C and index $\log \mu/\log \lambda$, and hence that γ is Hölder continuous with constant $\sqrt{1+C^2}$ and index $\log \mu/\log \lambda$.

Finally note that, by construction, images of γ under positive iterates of L avoid V entirely, while images under negative iterates intersect V in $\tilde{D}_0(0, 0)$ only. Thus, as explained above, the image of $\delta = \pi\gamma$ has 1-dimensional orbit-closure.

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