

CORE-CONSISTENCY AND TOTAL INCLUSION FOR METHODS OF SUMMABILITY

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1. Introduction. We shall consider methods of summation A, B, \dots defined by matrices of real elements $(a_{mn}), (b_{mn}), (m, n = 1, 2, \dots)$ which are regular, that is, have the three well-known properties of Toeplitz (**4**, p. 43). A method A is said to be *core-consistent with the method B for bounded sequences* if the A -core (**3**, p. 137; and **4**, p. 55) of each real bounded sequence is contained in its B -core. B is *totally included* in A , $B \ll A$, if each real sequence which is B -summable to a definite limit (this limit may be finite or infinite of a definite sign) is also A -summable to the same limit. It will be shown in the present paper that if the matrix A is core-consistent with the positive matrix B , then A is "almost" divisible by B on the right. This statement is made precise in Theorem 1 below. The proof (§2) involves some elementary properties of convex sets in Banach spaces. In §3, the same method is used to prove a similar result for the relation $B \ll A$ (Theorem 2). Some simple corollaries are given in §4.

Let l_1 be the Banach space of elements $\mathbf{x} = (x_n)$, with norm

$$\|\mathbf{x}\| = \sum_{n=1}^{\infty} |x_n|,$$

so that the rows of the matrices A, B are elements $\mathbf{a}_m, \mathbf{b}_m$ of l_1 . Elements $\mathbf{x}, \mathbf{y} \in l_1$ are called disjoint if $x_n y_n = 0$ ($n = 1, 2, \dots$); an element $\mathbf{x} \in l_1$ is positive, $\mathbf{x} \geq 0$, if $x_n \geq 0$ ($n = 1, 2, \dots$). If $\mathbf{x} = (x_1, x_2, \dots, x_n, \dots) \in l_1$, we shall write

$$\mathbf{x}^q = (x_1, \dots, x_q, 0, 0, \dots), \quad \mathbf{x}_p = (0, \dots, 0, x_p, x_{p+1}, \dots),$$

$$\mathbf{x}_p^q = (0, \dots, 0, x_p, \dots, x_q, 0, \dots), \quad p \leq q.$$

We also use the same notation for sets $E \subset l_1$, for instance E_p^q is the set of all \mathbf{x}_p^q with $\mathbf{x} \in E$. A cone $K \subset l_1$ is a set such that

$$\sum_1^n c_k \mathbf{x}_k \in K$$

whenever $c_k \geq 0, \mathbf{x}_k \in K$. For instance, the set of all positive elements is a cone in l_1 .

We shall prove the following theorems:

THEOREM 1. *Let A, B be regular matrices and let A be core-consistent with B . If B is positive, that is if $\mathbf{b}_m \geq 0$ ($m = 1, 2, \dots$), there is a positive regular matrix C such that the norm of the m th row of $CB - A$ tends to zero for $m \rightarrow \infty$.*

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The case where the elements of the sequences, or of the matrices, are complex is not essentially different as will be shown in §2.

If $A = (a_{mn})$, we shall write A_p for the matrix obtained from A by replacing all a_{mn} with $n < p$ by zeros.

THEOREM 2. *If A, B are regular row-finite matrices, B positive and*

$$(i) \quad B \ll A,$$

there is an integer p and a regular positive row-finite matrix C such that

$$(1) \quad CB_p = A_p;$$

this remains true if (i) is replaced by the (formally weaker) hypothesis that

(ii) $\tau_n \rightarrow +\infty$ always implies $|\sigma_n| \rightarrow +\infty$, where σ_n and τ_n are the A - and the B -transforms of a sequence s_n , respectively.

If B is the unit matrix I , these results were known before; for the case of Theorem 1 see Agnew (1), also (3, p. 149); for Theorem 2, Hurwitz (5) or (4, p. 53).

2. Core-consistency. If Theorem 1 is true for a given pair of matrices A, B , it is also true for any two matrices A', B' with rows $\mathbf{a}'_m, \mathbf{b}'_m$ satisfying

$$\|\mathbf{a}_m - \mathbf{a}'_m\| \rightarrow 0, \quad \|\mathbf{b}_m - \mathbf{b}'_m\| \rightarrow 0.$$

This and the regularity of A, B imply that we may assume A, B to be row-finite, and such that there is a sequence $n(m)$ increasing to $+\infty$ with $a_{mn} = b_{mn} = 0$ for $n < n(m)$.

LEMMA. *In the above conditions there exist two sequences $p = p(m) < q(m)$ such that $p(m) \rightarrow \infty$ for $m \rightarrow \infty$ and that*

$$(2) \quad \rho(\mathbf{a}_m, K) = \rho(\mathbf{a}_m, K_p^q);$$

here $\rho(\mathbf{a}_m, K)$ is the distance from \mathbf{a}_m to the cone K generated by the \mathbf{b}_λ ($\lambda = 1, 2, \dots$).

Proof. For a given m , let $m_1 \leq m_2$ be such that \mathbf{b}_μ is disjoint with \mathbf{a}_m if μ does not satisfy $m_1 \leq \mu \leq m_2$; we may assume that $m_1 \rightarrow \infty$ for $m \rightarrow \infty$. Let K' be the cone generated by the \mathbf{b}_μ , $m_1 \leq \mu \leq m_2$, let $p(m) = n(m_1)$ and let q be so large that $b_{\mu n} = 0$, $m_1 \leq \mu \leq m_2$, $a_{mn} = 0$ for $n > q$. Then $\mathbf{a}_{mp}^q = \mathbf{a}_m$, $K_p^q = K'$, and therefore

$$(3) \quad \rho(\mathbf{a}_m, K) \leq \rho(\mathbf{a}_m, K') = \rho(\mathbf{a}_m, K_p^q).$$

On the other hand, let $\mathbf{x} \in K$, then \mathbf{x} is a linear combination, with positive coefficients, of some of the \mathbf{b}_λ . If we omit from it all those \mathbf{b}_λ which are not \mathbf{b}_μ , we shall obtain another element $\mathbf{x}' \in K'$. The omitted \mathbf{b}_λ are disjoint with \mathbf{a}_m and all $b_{\lambda n}$ satisfy $b_{\lambda n} \geq 0$. This implies

$$\|\mathbf{a}_m - \mathbf{x}_p^q\| \geq \|\mathbf{a}_m - \mathbf{x}'_p^q\|.$$

Since $K_p^q \subset K_p^q$, it follows that $\rho(\mathbf{a}_m, K_p^q) = \rho(\mathbf{a}_m, K'_p^q)$ and using (3) we obtain $\rho(\mathbf{a}_m, K) \leq \rho(\mathbf{a}_m, K_p^q)$. The inverse inequality is obvious, and (2) follows.

Proof of Theorem 1. We shall show that

$$(4) \quad \rho(\mathbf{a}_m, K) \rightarrow 0.$$

If this is not true, there exists by the Lemma an $\epsilon > 0$, a sequence of disjoint \mathbf{a}_{m_i} , and a sequence of disjoint intervals $[p_i, q_i]$ with

$$\rho(\mathbf{a}_{m_i}, K_{p_i}^{q_i}) > \epsilon.$$

If

$$\mathbf{y} = \sum c_i \mathbf{a}_{m_i}$$

is a linear combination of the \mathbf{a}_{m_i} with $c_i > 0$, $\sum c_i = 1$ and if $\mathbf{x} \in K$, we can put

$$\mathbf{z}_i = c_i^{-1} \mathbf{x}_{p_i}^{q_i} \in K_{p_i}^{q_i}$$

and have

$$\|\mathbf{y} - \mathbf{x}\| = \|\sum c_i \mathbf{a}_{m_i} - \sum c_i \mathbf{z}_i\| = \sum c_i \|\mathbf{a}_{m_i} - \mathbf{z}_i\| > \epsilon.$$

This shows that the convex set E generated by the \mathbf{a}_{m_i} is at a distance $\geq \epsilon$ from E , hence the ϵ -neighbourhood E_ϵ of E is disjoint with K . If K_ϵ is the cone generated by E_ϵ , K and K_ϵ are disjoint except for the origin. By a well-known theorem (7, Theorem 1.2), there is in l_1 a bounded linear functional $f(\mathbf{x})$ of norm one which is positive on K and negative on K_ϵ . Hence $f(\mathbf{y}) \leq -\epsilon$ on E (7, Lemma 1.2). This means that there is a bounded sequence s_ν with

$$\begin{aligned} \sum b_{m\nu} s_\nu &\geq 0 & (m = 1, 2, \dots), \\ \sum a_{m_i, \nu} s_\nu &\leq -\epsilon & (i = 1, 2, \dots), \end{aligned}$$

and contradicts the hypothesis of Theorem 1.

From (4) it follows that for some row-finite positive matrix $C = (c_{mn})$,

$$\|\mathbf{a}_m - \sum_n c_{mn} \mathbf{b}_n\| \rightarrow 0, \quad m \rightarrow \infty.$$

Finally, this C will be necessarily regular, provided we agree to take $c_{mn} = 0$ whenever $\mathbf{b}_n = 0$. For

$$c_{mn} b_{n\nu} \leq \sum_{n=1}^{\infty} c_{mn} b_{n\nu} = a_{m\nu} + o(1) = o(1), \quad m \rightarrow \infty$$

implies that $c_{mn} \rightarrow 0$ for $m \rightarrow \infty$ and each n . On the other hand,

$$\begin{aligned} \sum_{\nu=1}^{\infty} a_{m\nu} &= \sum_{\nu} \sum_n c_{mn} b_{n\nu} + o(1) \\ &= \sum_n c_{mn} \sum_{\nu} b_{n\nu} + o(1) \end{aligned}$$

together with

$$\sum_{\nu} a_{m\nu} = 1 + o(1), \quad \sum_{\nu} b_{n\nu} = 1 + o(1)$$

imply that $\sum_n c_{mn} \rightarrow 1$ for $m \rightarrow \infty$. This completes the proof.

The concept of the core is defined also for sequences of complex numbers (3, p. 137). Accordingly, we may introduce the concept of core-consistency

as well for matrices and sequences with complex elements. With this new definition, Theorem 1 holds literally as before.

For the proof assume that A is a regular matrix with complex elements, B is positive and that A is core-consistent with B for bounded sequences. Hence, by Knopp's core theorem (6, p. 115) or (4, p. 55), the core of the B -transform of any bounded sequence s_n is included in the core of s_n . But A is core-consistent with B , and so the core of the A -transform of s_n also is included in the core of s_n . This implies (3, p. 149) that $A = A' + V$ where A is a positive regular matrix, and the norm of the m th row of the matrix V tends to zero as $m \rightarrow \infty$. Clearly, A' also is core-consistent with B , for complex or more particularly, for real sequences. It then follows from the original Theorem 1 that there exists a positive regular matrix C such that the norm of the m th row of $CB - A'$ tends to zero from $m \rightarrow \infty$. Consequently, the norm of the m th row of

$$CB - A = CB - A' - V$$

also tends to zero for $m \rightarrow \infty$. This proves our assertion.

The converse of this (as well as the converse of Theorem 1) is a direct consequence of Knopp's core theorem. Thus, let A, B, C , be three regular matrices, C positive, such that the norm of the m th row of $CB - A$ tends to zero for $m \rightarrow \infty$. Then the core of the transform of any bounded complex sequence s_n by CB coincides with the core of the transform of s_n by A . The transform of s_n by CB is the transform by C of the transform of s_n by B . Hence the core of the transform of s_n by CB is included in the core of the transform of s_n by B , by virtue of Knopp's core theorem. In other words, CB , and hence A , are core-consistent with B for bounded sequences.

3. Total inclusion. We shall now prove Theorem 2, deducing (1) from the hypothesis (ii). Let $\rho_{mp} = \rho(\mathbf{a}_{mp}, K_p)$; we first show that

$$(5) \quad \rho_{mp} = 0 \text{ for all } p \text{ sufficiently large and } m = 1, 2, \dots$$

Let (5) be false. Since the ρ_{mp} decrease for m fixed and increasing p and finally become zero, we deduce that for each p , $\rho_{mp} > 0$ for an infinity of m . Now $\rho_{mp} > 0$ implies the existence of $\delta > 0$, $\epsilon > 0$ such that the sphere S in l_1 with center \mathbf{a}_{mp} and radius δ does not have common points with the cone K' generated by the points $\mathbf{b}_{\lambda p}$ ($\lambda = 1, 2, \dots$), and by the spheres with radii ϵ around those of the $\mathbf{b}_{\mu p}$ ($\mu = 1, 2, \dots, m$) which are not zero. Hence, there is a functional

$$f(\mathbf{x}) = \sum x_n s_n, \quad \|f\| = 1 \text{ in } l_1,$$

generated by a bounded sequence s_n with $s_n = 0$ for $n < p$, such that the hyperplane $f(\mathbf{x}) = 0$ separates S and K' and supports S (by Eidelheit's theorem, (7, Theorem 1.6)). If $f(\mathbf{x}) \geq 0$ on K' , we have

$$\tau_\mu = f(\mathbf{b}_{\mu p}) \geq \epsilon \quad \text{for } \mathbf{b}_{\mu p} \neq 0 \quad (\mu = 1, 2, \dots, m)$$

(7, Lemma 1.2) and

$$0 > \sigma_m = f(\mathbf{a}_{mp}) \geq -\|f\|\delta = -\delta.$$

By fixing $\epsilon > 0$, taking $\delta > 0$ sufficiently small, and then multiplying the s_n with a sufficiently large positive number, we obtain the following statement:

(*) For each m, p with $\rho_{mp} > 0$ and for any two positive numbers M, η , there is a bounded sequence s_n with $s_n = 0$ for $n < p$ such that

$$\begin{aligned} \sigma_m &= \sum_n a_{mn} s_n = -\eta, & \tau_\lambda &= \sum_n b_{\lambda n} s_n \geq 0 & (\lambda = 1, 2, \dots), \\ \tau_\mu &> M \text{ if } 1 \leq \mu \leq m \text{ and } \mathbf{b}_{\mu p} \neq 0. \end{aligned}$$

We now define inductively increasing sequences of integers $p_1, p_2, \dots, m_1, m_2, \dots$ and bounded sequences $\mathbf{s}^{(i)}$ satisfying $s_n^{(i)} = 0$ for $n < p_i$. If

$$p_1, \dots, p_{i-1}; m_1, \dots, m_{i-1}; \mathbf{s}^{(1)}, \dots, \mathbf{s}^{(i-1)}$$

are already defined, take p_i so large that $a_{\mu n} = 0$ for $n \geq p_i, \mu = m_1, \dots, m_{i-1}$, then find an $m_i > m_{i-1}$ with

$$|\sum a_{m_i, n} (s_n^{(1)} + \dots + s_n^{(i-1)})| < \frac{1}{2}, \rho(\mathbf{a}_{m_i p_i}, K_{p_i}) > 0.$$

By (*), there is a bounded sequence $\mathbf{s}^{(i)}$ with $s_n^{(i)} = 0$ for $n < p_i$ such that

$$(6) \quad \sum_{n=1}^{\infty} a_{m_i, n} (s_n^{(1)} + \dots + s_n^{(i)}) = -1,$$

$$(7) \quad \sum_n b_{\lambda n} s_n^{(i)} \geq 0 \quad (\lambda = 1, 2, \dots),$$

$$(8) \quad \sum_n b_{\mu n} s_n^{(i)} > i \text{ if } b_{\mu p_i} \neq 0 \quad (\mu = 1, \dots, m).$$

Let $\mathbf{s} = (s_n)$ be the sequence defined by $s_n = \sum_i s_n^{(i)}$; for each n this sum has only a finite number of terms. Since \mathbf{a}_{m_i} and $\mathbf{s}^{(j)}$ are disjoint for $j > i$, we have by (6), $\sigma_{m_i} = -1$, and by (7) and (8), $\tau_\lambda \rightarrow \infty$, which contradicts the hypothesis and proves (5).

Fixing a p for which (5) holds, we consider an arbitrary m . For each $\epsilon > 0$ there is an \mathbf{x} in K such that

$$(9) \quad \|\mathbf{a}_{mp} - \mathbf{x}_p\| < \epsilon, \quad \mathbf{x} = \sum c_\mu \mathbf{b}_\mu, c_\mu \geq 0.$$

Let q be the last index n with $a_{mn} \neq 0$. If we omit from the last sum all \mathbf{b}_μ for which

$$\sum_{n>q} b_{\mu n} > \sum_{n \leq q} b_{\mu n},$$

we shall obtain an element $\mathbf{x}' \in K$ with

$$\|\mathbf{a}_{mp} - \mathbf{x}'_p\| \leq \|\mathbf{a}_{mp} - \mathbf{x}_p\|.$$

It follows that μ in (9) may be assumed bounded for all ϵ . Then we must have

$$(10) \quad \mathbf{a}_{mp} = \sum_{n=1}^N c_{mn} \mathbf{b}_{np}, \quad c_{mn} \geq 0.$$

This proves the theorem, for the argument used in the proof of Theorem 1 shows that $C = (c_{mn})$ is regular, provided in (10) we take $c_{mn} = 0$ whenever $\mathbf{b}_{np} = 0$.

We give some corollaries to Theorem 2, assuming that the matrices A , B are regular and row-finite and that B is positive. We compare the following relations (for the definition of the core of a possibly unbounded sequence see (4, p. 55)):

- (i) $B \ll A$.
- (ii) For each sequence s_n , $\tau_n \rightarrow +\infty$ implies that $|\sigma_n| \rightarrow +\infty$.
- (iii) $A_p = CB_p$ for some p with $C \geq 0$.
- (iv) A is core-consistent with B for all real sequences.
- (v) A is core-consistent with B for all complex sequences.

Then we have:

THEOREM 3. *Conditions (i)-(v) are equivalent.*

Proof. Clearly, (i) \rightarrow (ii). Theorem 2 shows that (ii) implies (iii) and it is easy to see that (iii) \rightarrow (i). From the definitions of the properties concerned we have (v) \rightarrow (iv) \rightarrow (ii). Finally, Knopp's core theorem states that (iii) \rightarrow (v). This completes the proof.

4. Applications. For further illustration of Theorems 1 and 2 we shall give some applications to totally equivalent and core equivalent methods. Two methods A , B are *totally equivalent*, if $A \ll B$ and $B \ll A$; they are *core-equivalent for bounded sequences* if the A -core of each bounded sequence coincides with its B -core. In what follows, V is a matrix such that the norm of the m th row tend to zero for $m \rightarrow \infty$, and I is the unit matrix.

THEOREM 4. (i) *A method A is core-equivalent with I for bounded sequences if and only if A has a representation*

$$(11) \quad A = A' + V$$

with positive A' , where A' contains a sequence of rows of the form

$$(12) \quad \mathbf{a}'_{m_n} = (0, \dots, 0, a_{m_n, n}, 0, \dots), \quad n = 1, 2, \dots$$

(then necessarily $m_n \rightarrow \infty$, $a_{m_n, n} \rightarrow 1$ for $n \rightarrow \infty$.)

(ii) *A regular row-finite method A is totally equivalent with I if and only if for some p , A_p is positive and contains a sequence of rows of the form (12).*

Proof. (i) The conditions are clearly sufficient. It follows from Theorem 1 that (11) with a positive A' is necessary. Again by Theorem 1, there is a positive regular matrix C and a V' with $CA' = I + V'$. For each n we have

$$(13) \quad \sum_{m=1}^{\infty} c_{nm} \mathbf{a}'_m = \mathbf{e}_n$$

with $e_{nl} \geq 0$, $\mathbf{e}_n = (e_{nl})$, $e_{nn} \rightarrow 1$ and

$$\sum_{l \neq n} e_{nl} \rightarrow 0$$

for $n \rightarrow \infty$. Let

$$\epsilon_n = \sum_{l \neq n} \frac{e_{nl}}{e_{nn}}.$$

Then $\epsilon_n \rightarrow 0$ for $n \rightarrow \infty$. Since the c_{nm} are all positive, it follows from (13) that there is at least one $m = m_n$ such that

$$\sum_{l \neq n} \frac{a'_{ml}}{a'_{mn}} \leq \epsilon_n.$$

For otherwise, multiplying the relations

$$\sum_{l \neq n} a'_{ml} > \epsilon_n a'_{mn} \quad (m = 1, 2, \dots)$$

with c_{nm} and adding we would obtain by means of (13) that

$$\sum_{l \neq n} e_{nl} > \epsilon_n e_{nn},$$

which contradicts the definition of ϵ_n .

We now replace by zero the elements a_{ml} of the rows of A' with $m = m_n$, $l \neq n$ ($n = 1, 2, \dots$). Denoting the matrix thus obtained again by A' , we see that (11) and (12) are satisfied. This proves (i); the proof of (ii) is similar.

Theorem 4(i) may serve to show, for instance, that if a regular Hausdorff method H_ρ is core-equivalent with I for bounded sequences, then H_ρ is identical with I .

A method A is normal if $a_{mn} = 0$ for $n > m$ and $a_{nn} \neq 0$ ($n = 1, 2, \dots$). In this case A has an inverse A^{-1} . If A, B are normal, there is a triangular matrix C with $A = CB$.

THEOREM 5. *Let the regular normal methods A, B be totally equivalent. Then there exists a sequence $c_m \rightarrow 1$ such that for some p ,*

$$(14) \quad a_{mn} = c_m b_{mn}, \quad m = 1, 2, \dots; n = p, p + 1, \dots$$

Proof. Let $A = CB, B = DA$, then the matrices C, D are triangular, regular and totally equivalent with I . We have

$$a_{mn} = c_{mm} b_{mn}, \quad b_{mn} = d_{mm} a_{mn},$$

hence

$$c_{mm} d_{mm} = 1,$$

and we obtain $c_{mm} \rightarrow 1$. From Theorem 4 (ii) it follows that for all sufficiently large n , $c_{mn} = 0$ if $n \neq m$. Putting $c_m = c_{mm}$, we obtain (14).

It should be added that sometimes it is even possible to prove that A, B are identical if they are totally equivalent. Let $A = H_\rho, B = H_\sigma$, be two regular and normal Hausdorff methods. Then

$$\sum_{n=0}^m |a_{mn}|$$

converges for $m \rightarrow \infty$ to the "essential" total variation of $g(x)$. From (14) it follows that

$$\sum_{n=0}^m |a_{mn} - b_{mn}| \rightarrow 0,$$

hence g and g_1 are essentially identical. Thus we obtain a remark of Bosanquet (2, p. 452) that H_ρ, H_{ρ_1} are identical if they are totally equivalent.

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